Online Appendix for "Public R&D, Private R&D and Growth: A Schumpeterian Approach" by Huang, Lai and Peretto

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The Appendix collects all the proofs and derivations using both simple (flow view) public and private R&D interaction model and the generalized (stock view) of cross-knowledge fertilization model presented in the manuscript.

A Household problem: derivation of (5) and (6)

The current value Hamiltonian to the household problem solved by a representative individual is

$$\mathcal{L} = \ln c + \iota\left[(r-\lambda)a + w - P_C c - w s_G\right],$$

where $(r-\lambda)a + w - P_C c - s_G$ is the budget constraint per capita, s_G is lump-sum tax per capita and ι is the dynamic multiplier. The necessary conditions for the maximization problem are

$$\mathcal{L}_c = 0 \qquad \rightarrow \qquad \frac{1}{c} = \iota P_C, \tag{A.1}$$

$$\mathcal{L}_a = (\rho - \lambda)\iota - i \qquad \rightarrow \qquad \iota(r - \lambda) = (\rho - \lambda)\iota - i, \tag{A.2}$$

and

$$\lim_{t - \infty} \iota \, a \, e^{-(\rho - \lambda)t} = 0, \tag{A.3}$$

where the last equation is the standard transversality condition. Time-differentiating (A.1) and substituting the result into (A.2) yields (5).

Next, household minimizes the cost expenditure per capita,

$$\min_{X_j} P_C c - \int_0^N P_j \frac{X_i}{L} dt,$$

subject to

$$c = N^{\omega} \left[\left(\frac{1}{N} \right)^{\frac{1}{\epsilon}} \int_0^N \left(\frac{X_i}{L} \right)^{\frac{\epsilon - 1}{\epsilon}} dt \right]^{\frac{\epsilon}{\epsilon - 1}}.$$

Given P_c and P_j for all j, the F.O.C. with respect to X_j yields equation (6).

B A flow view model of the private and public R&D interaction

B.1 Proof of Lemma 1: derivation of (11), (12), and (13)

In the following, we derive a simple model with a flow view of private and public R&D interaction by setting κ to zero. The two knowledge accumulation processes in equations (8) and (9) thus become

$$\dot{Z}_i = \alpha f(s_G) K_i L_{Z_i}, \tag{B.1}$$

and

$$\dot{D}_i = B_i L_{G_i}. \tag{B.2}$$

The interaction is only captured by the factor $f(s_G)$ which measures the knowledge spillover from public R&D employment to private R&D technology.

B.1.1 Intermediate firm's profit maximization problem and returns to in-house and entry R&D

The typical intermediate firm maximizes its present value,

$$\max_{\{L_{Z_i},P_i\}} V_i(t) = \int_t^\infty \Pi_i \, e^{-\int_t^\tau (r(s)+\sigma)ds} d\tau, \quad \sigma > 0, \tag{B.3}$$

where $\Pi_i \equiv P_i X_i - wL_{X_i} - wL_{Z_i}$ is the instantaneous profit flow, r is the real interest rate and σ is an exogenous death shock. The firm chooses the time path of the price, P_i, and R&D, L_{Z_i}, subject to the demand curve in (6) and the production function in (7) and the R&D technology (B.1) in (8), taking public R&D policy s_G as given. Moreover, we define q_i as the co-state variable that represents the value of the marginal unit of knowledge, The above optimization problem becomes to maximize the following current-value Hamiltonian,

$$CVH_i = P_i X_i - Z_i^{-\theta} D_i^{-\gamma} X_i - \phi - L_{Z_i} + q_i \dot{Z}_i,$$

s.t. the demand curve in (6) and the R&D technology (B.1) in (8). By taking the first-order derivative with respect to P_i, we yield the rule of optimal price (11),

$$P_i = \frac{\epsilon}{\epsilon - 1} Z_i^{-\theta} D_i^{-\gamma}. \tag{B.4}$$

Moreover, the derivative of CVH_i with respect to L_{Z_i} in the linear profit function yields

$$L_{Z_i} = \begin{cases} 0 & for \, 1 > q_i \alpha f(s_G) K \\ L_Z/N & for \, 1 = q_i \alpha f(s_G) K \cdot \\ \infty & for \, 1 < q_i \alpha f(s_G) K \end{cases}$$

The interior solution is determined under the condition that the marginal cost of R&D equals its marginal benefit. Moreover, the F.O.C. for state variable Z_i is

$$\frac{\partial CVH_i}{\partial Z_i} = rq_i - \dot{q}_i.$$

Rearranging it yields the return to in-house R&D,

$$r^Z + \sigma \equiv \frac{\partial \Pi_i / \partial Z_i}{q_i} + \frac{\dot{q}_i}{q_i}, \tag{B.5}$$

Next, considering the interior solution and takes logarithm and time derivatives on $1 = q_j \alpha f(s_G) K$ yields $\dot{q}_i / q_i = -\dot{K}/K$. Secondly, we substitute the demand curve (6), the manufacturing production (7) and the pricing rule (B.4) into profit flow and yields

$$\Pi_i = \frac{1}{\epsilon} LE \frac{Z_i^{\theta(\epsilon-1)} D_i^{\gamma(\epsilon-1)}}{\int_0^N Z_j^{\theta(\epsilon-1)} D_j^{\gamma(\epsilon-1)} dj} - \phi - L_{Z_i}.$$

Taking the derivative of Π_i with respect to Z_i yields

$$\partial \Pi_i / \partial Z_i = \frac{1}{\epsilon} LE \frac{\theta(\epsilon - 1)}{Z_i} \frac{Z_i^{\theta(\epsilon-1)} D_i^{\gamma(\epsilon-1)}}{\int_0^N Z_j^{\theta(\epsilon-1)} D_j^{\gamma(\epsilon-1)} dj}.$$

Substitute the resulting expression of the derivative, $\partial \Pi_i / \partial Z_i$, and the condition, $\dot{q}_i / q_i = -\dot{K}/K$, into (B.5) along with the fact that $\dot{K}/K = \alpha f(s_G) L_{Z_i}$ from (B.1). Further imposing a symmetry and combining no arbitrage condition with the return to riskless loan yield the return to in-house R&D in (12),

$$r = r^Z \equiv \alpha f(s_G) \left[\frac{\theta(\epsilon - 1)}{\epsilon} \frac{LE}{N} - \frac{L_Z}{N} \right] - \sigma. \tag{B.6}$$

B.1.2 Net entry/exit

The expression for the rate of return to entry is

$$r^N + \sigma \equiv \frac{\Pi_i}{V_i} + \frac{\dot{V}_i}{V_i}. \tag{B.7}$$

Taking logarithm and time derivative with respect to the free entry condition, $V_i = LE/\beta N$, yields $\dot{V}_i/V_i = \dot{E}/E + \lambda - \dot{N}/N$. Substituting $\dot{V}_i/V_i = \dot{E}/E + \lambda - \dot{N}/N$ and the equilibrium profit, $\frac{1}{\epsilon}LE/N - \phi - L_{Z_i}$, into above and imposing symmetry yield the return to entry innovation in (13),

$$r = r^N \equiv \left[\frac{1}{\epsilon}\frac{LE}{N} - \phi - \frac{L_Z}{N}\right]\frac{\beta N}{LE} + \frac{\dot{E}}{E} + \lambda - \frac{\dot{N}}{N} - \sigma. \tag{B.8}$$

B.2 Proof of Lemma 2: derivation of (14) and (15)

Substituting the demand curve from (6) into the intermediate production in (7) with a symmetry implied by the pricing rule in (B.4), we can obtain

$$L_X = \frac{(\epsilon - 1)LE}{\epsilon} + N\phi. \tag{B.9}$$

Second, we plug the above expression and L_N from (**??**) into resource constraint, $L = L_G + L_X + L_N + L_Z$, to get

$$L = L_G + \frac{(\epsilon - 1)LE}{\epsilon} + N\phi + (\dot{N} + \sigma N)\frac{LE}{\beta N} + L_Z.$$

Rearranging it yields the expression for L_Z/N,

$$\frac{L_Z}{N} = \frac{L - L_G}{N} - \frac{(\epsilon - 1)LE}{\epsilon N} - \phi - (\frac{\dot{N}}{N} + \sigma)\frac{LE}{\beta N}.$$

Further substituting it into rate of return to entry in (B.8) and rearranging it yield

$$r^N = \beta\left[1 - \frac{(L - L_G)}{LE}\right] + \frac{\dot{E}}{E} + \lambda.$$

By applying the no arbitrage condition across r^N and the riskless return rate r from the Euler equation in (5), we, thus, can obtain equation (14),

$$E = E^* \equiv \frac{\beta(1-s_G)}{\beta-\rho+\lambda}, \tag{B.10}$$

where $s_G = L_G/L$.

Substitutes the pricing rule into P_C in (3) and combines the E^* solved above, we can get the real GDP pe capita in (15).

B.3 Firm-level innovation

Substituting $r = \rho$ and $E = E^*$ from Lemma 2 into (B.6) yields

$$\frac{L_Z}{N} = \max\left\{\frac{\theta(\epsilon-1)}{\epsilon}\frac{LE^*}{N} - \frac{\sigma+\rho}{\alpha f(s_G)}, 0\right\}, \tag{B.11}$$

where the threshold,

$$\bar{n} = \frac{\alpha f(s_G)\theta(\epsilon-1)}{\epsilon(\sigma+\rho)}E^*,$$

is obtained by solving $\frac{\theta(\epsilon-1)}{\epsilon}\frac{E^*}{n} - \frac{\sigma+\rho}{\alpha f(s_G)} = 0$. Substituting (B.11) into (B.1) yields equation (16),

$$\hat{Z} \equiv \frac{\dot{Z}_i}{Z} = \max\left\{f(s_G)(1-s_G)\frac{\beta\alpha\theta(\epsilon-1)}{\epsilon\,(\beta-\rho+\lambda)}\frac{1}{n} - \sigma - \rho, \quad 0\right\}. \tag{B.12}$$

B.4 Market structure dynamics

B.4.1 Proof of Proposition 1: derivations of (CG), (17), (18) and (19)

By plugging L_N from (**??**), L_X from (B.9), and L_Z from (B.11) into the resource constraint and rearranging it, we obtain

$$\frac{\dot{N}}{N} + \sigma = \frac{\beta}{LE^*}\left[L - L_G - \frac{(\epsilon-1)LE^*}{\epsilon} - N\phi - \frac{\theta(\epsilon-1)}{\epsilon}LE^* + N\frac{\sigma+\rho}{\alpha f(s_G)}\right].$$

Replacing $L - L_G$ with $(1 - (\rho - \lambda)/\beta)\, LE$ derived from lemma 2 into the above expression and rearranging it yield

$$\frac{\dot{n}}{n} = \frac{\beta\,[1 - \theta(\epsilon - 1)]}{\epsilon} - (\rho + \sigma) - n\frac{\beta}{E^*}\left(\phi - \frac{\sigma + \rho}{\alpha f(s_G)}\right). \tag{B.13}$$

Setting $\frac{\dot{n}}{n} = 0$ and defining $v \equiv \frac{\beta[1-\theta(\epsilon-1)]}{\epsilon} - (\rho + \sigma) > 0$ (i.e., the first condition in CG), we can obtain

$$n^* = \frac{\frac{v(1-s_G)}{\beta-\rho+\lambda}}{\left(\phi - \frac{\sigma+\rho}{\alpha f(s_G)}\right)}, \tag{B.14}$$

which is equation (18). The boundary condition that $n^* < \bar{n}$ which ensures the in-house R&D being active in steady state yields the second inequality in CG,

$$\phi - \frac{\rho + \sigma}{f(s_G)\alpha} - \left[1 + \frac{v\epsilon}{\beta\theta(\epsilon - 1)}\right] > 0.$$

Moreover, we can rewrite (B.13) as

$$\frac{\dot{n}}{n} = v\left(1 - \frac{n}{n^*}\right), \tag{B.15}$$

which is the logistic differential equation in (17). The analytical solution for it is

$$n(t) = \frac{n^*}{1 + e^{-vt}\left(\frac{n^*}{n_0} - 1\right)}, \tag{B.16}$$

which is equation (19).

B.4.2 Proof of Proposition 2: derivation of (20)

Combining steady state mass of firm per capita (B.14) and the consumption expenditure E^*, we obtain the steady state firm size in equation (20),

$$\frac{E^*}{n^*} = \left(\frac{LE}{N}\right)^* = \frac{\beta(1-s_G)}{\beta-\rho+\lambda} / \left(\frac{\frac{v(1-s_G)}{\beta-\rho+\lambda}}{\phi - \frac{\sigma+\rho}{\alpha f(s_G)}} \right) = \left(\phi - \frac{\sigma+\rho}{\alpha f(s_G)}\right) \frac{\beta}{v}, \tag{B.17}$$

and consequently the steady state in-house R&D per firm in equation (21) is

$$\left(\frac{L_Z}{N}\right)^* = \frac{\theta(\epsilon-1)}{\epsilon} \left(\frac{LE}{N}\right)^* - \frac{\sigma+\rho}{\alpha f(s_G)}.$$

and the steady state private knowledge growth in equation (22) is

$$\hat{Z}^* = \alpha f(s_G) \frac{\theta(\epsilon-1)}{\epsilon} \left(\frac{LE}{N}\right)^* - \rho - \sigma. \tag{B.18}$$

B.4.3 Proof of Proposition 3

When the knowledge-base and personnel-interaction effects are absent (i.e., $\gamma = \xi = 0$), the steady state consumption expenditure, E^*, from (B.10) and the firm size per capita, n^*, from (B.14) become

$$E^* = \frac{\beta(1-s_G)}{\beta-\rho+\lambda},$$

and

$$n^* = \frac{\frac{v(1-s_G)}{\beta-\rho+\lambda}}{\left(\phi - \frac{\sigma+\rho}{\alpha}\right)}.$$

Both expressions are decreasing in s_G. Moreover, the expressions for $\left(\frac{LE}{N}\right)^*$, $\left(\frac{L_Z}{N}\right)^*$ and $\hat{Z}^*$ above become

$$\left(\frac{LE}{N}\right)^* = \left(\phi - \frac{\sigma+\rho}{\alpha}\right) \frac{\beta}{v},$$

$$\left(\frac{L_Z}{N}\right)^* = \frac{\theta(\epsilon-1)}{\epsilon}\left(\frac{LE}{N}\right)^* - \frac{\sigma+\rho}{\alpha},$$

and

$$\hat{Z}^* = \alpha\frac{\theta(\epsilon-1)}{\epsilon}\left(\frac{LE}{N}\right)^* - \rho - \sigma,$$

respectively. We can see that s_G has no effect on all three expressions. Moreover, TFP, T, is defined as

$$T \equiv N^\omega Z^\theta D^\gamma,$$

and thus the steady state growth of TFP, which is also the growth of output as well as consumption per capita is

$$\begin{aligned}\hat{T}^* = \hat{y}^* = \hat{c}^* &= \omega\hat{N}^* + \theta\hat{Z}^* + \gamma\hat{D}^* \\ &= \omega\lambda + \theta\left[\alpha\frac{\theta(\epsilon-1)}{\epsilon}\left(\frac{LE}{N}\right)^* - \rho - \sigma\right],\end{aligned}$$

where the second equality is implied by applying $\gamma = 0$. We can see that s_G has no impact on $\hat{T}^*$, $\hat{y}^*$ and $\hat{c}^*$. We complete our proofs for Proposition 3.

B.4.4 Proof of Proposition 4

When the knowledge-base and personnel-interaction effects are present (i.e., $\gamma, \xi > 0$), the steady state consumption expenditure E^* from (B.10) and the firm size n^* from (B.14) are

$$E^* = \frac{\beta(1-s_G)}{\beta-\rho+\lambda},$$

and

$$n^* = \frac{\frac{v(1-s_G)}{\beta-\rho+\lambda}}{\left(\phi - \frac{\sigma+\rho}{\alpha f(s_G)}\right)},$$

which both remain decreasing in s_G, while the expressions for $\left(\frac{LE}{N}\right)^*$, $\left(\frac{L_Z}{N}\right)^*$ and $\hat{Z}^*$ above become

$$\left(\frac{LE}{N}\right)^* = \left(\phi - \frac{\sigma+\rho}{\alpha f(s_G)}\right)\frac{\beta}{v}, \tag{B.19}$$

$$\left(\frac{L_Z}{N}\right)^* = \frac{\theta(\epsilon-1)}{\epsilon}\left(\frac{LE}{N}\right)^* - \frac{\sigma+\rho}{\alpha f(s_G)},$$

and

$$\hat{Z}^* = \alpha f(s_G)\frac{\theta(\epsilon-1)}{\epsilon}\left(\frac{LE}{N}\right)^* - \rho - \sigma. \tag{B.20}$$

All expressions are increasing in s_G. Besides, the steady state growth of TFP as well as the growth of output and consumption per capita becomes

$$\begin{aligned}
\hat{T}^* &= \hat{y}^* = \hat{c}^* = \omega\hat{N}^* + \theta\hat{Z}^* + \gamma\hat{D}^* \\
&= \omega\lambda + \theta\left[\alpha f(s_G)\frac{\theta(\epsilon-1)}{\epsilon}\left(\frac{LE}{N}\right)^* - \rho - \sigma\right] + \gamma\frac{L_G}{L}\left(\frac{L}{N}\right)^* \\
&= \omega\lambda + \theta\left[\alpha f(s_G)\frac{\theta(\epsilon-1)}{\epsilon}\left(\frac{LE}{N}\right)^* - \rho - \sigma\right] + \gamma\left(s_G/\frac{\frac{v(1-s_G)}{\beta-\rho+\lambda}}{\left(\phi - \frac{\sigma+\rho}{\alpha f(s_G)}\right)}\right),
\end{aligned}$$

which is clearly increasing in s_G. We complete the proof of Proposition 4.

B.4.5 Proof of Proposition 5: derivation of (23), (24) and (25)

Since n^* is decreasing in s_G, a decrease in s_G from s_G^0 increases n^* such that

$$\frac{n^*}{n_0} - 1 \equiv \frac{\frac{(1-s_G)}{\left(\phi - \frac{\sigma+\rho}{\alpha f(s_G)}\right)}}{\frac{(1-s_G^0)}{\left(\phi - \frac{\sigma+\rho}{\alpha f(s_G^0)}\right)}} - 1 \equiv \Delta > 0.$$

We can obtain $n^* = n\left(1 + e^{-vt}\Delta\right)$ from (B.16) and substituting it into (B.15) yields transitional path of net entry rate per capita,

$$\frac{\dot{n}}{n} = v\left(1 - \frac{n}{n\left(1+e^{-vt}\Delta\right)}\right) = \left(1 - \frac{1}{\left(1+e^{-vt}\Delta\right)}\right) = \frac{e^{-vt}\Delta}{\left(1+e^{-vt}\Delta\right)}.$$

and thus the path of net entry rate in equation (23),

$$\hat{N} \equiv \frac{\dot{N}}{N} = \frac{e^{-vt}\Delta}{(1+e^{-vt}\Delta)} + \lambda. \tag{B.21}$$

We next substitute $n^* = n\left(1+e^{-vt}\Delta\right)$ into (B.17) yields the transitional path for firm size,

$$\frac{E^*}{n} = \left(1+e^{-vt}\Delta\right)\left(\phi - \frac{\sigma+\rho}{\alpha f(s_G)}\right)\frac{\beta}{v}. \tag{B.22}$$

Plugging it back to (B.12) yields transitional path of private knowledge growth in equation (24),

$$\hat{Z} = \left(1+e^{-vt}\Delta\right)\left(\alpha f(s_G)\phi - \sigma - \rho\right)\frac{\beta\theta(\epsilon-1)}{v\epsilon} - \sigma - \rho. \tag{B.23}$$

Finally, the transitional path of TFP growth rate is obtained by taking logarithm and time derivative of $T \equiv N^{\omega} Z_i^{\theta} D_i^{\gamma}$ with respect to time and yields

$$\hat{T} = \omega\hat{N} + \theta\hat{Z} + \gamma\hat{D}.$$

Substituting $\hat{N}$ and $\hat{Z}$ from the above expressions and $\hat{D}$ from (B.2) (where $\dot{D}_i = D_i\frac{L_G}{N} = D_i\frac{s_G}{n}$) and $n^* = n\left(1+e^{-vt}\Delta\right)$ into above yield the expression (25),

$$\hat{T} = \frac{1}{\epsilon-1}\left(\frac{e^{-vt}\Delta}{(1+e^{-vt}\Delta)} + \lambda\right) + \left(1+e^{-vt}\Delta\right)\left(\alpha f(s_G)\phi - \sigma - \rho\right)\frac{\beta\theta^2(\epsilon-1)}{v\epsilon} - \sigma - \rho$$
$$+\gamma\frac{(1+e^{-vt}\Delta)}{n^*}s_G.$$

We complete the proof for Proposition (5).

B.5 The dynamic relation between public and private R&D

B.5.1 The derivation for the share of labor force employed in R&D sector

To derive the transitional path of in-house R&D per firm, we substitute (B.22) into (B.11) and yields

$$\frac{L_Z(t)}{N(t)} = \left(1+e^{-vt}\Delta\right)\left(\phi - \frac{\sigma+\rho}{\alpha f(s_G)}\right)\frac{\beta\theta(\epsilon-1)}{\epsilon v} - \frac{\sigma+\rho}{\alpha f(s_G)}.$$

Multiplying both sides of the above expression by the mass of firm per capita and further substituting n^* from (B.14) into it yield the share of labor force employed in in-house R&D as

$$\frac{L_Z(t)}{L(t)} = \frac{(1-s_G)}{\beta-\rho+\lambda}\left[\frac{\beta\theta(\epsilon-1)}{\epsilon} - \frac{v(\sigma+\rho)}{\phi\alpha f(s_G)-\sigma-\rho}\frac{1}{(1+e^{-vt}\Delta)}\right] \tag{B.24}$$

Moreover, rearranging (**??**) yields

$$\frac{\dot{N}}{N} = \frac{\beta}{E}\frac{L_N}{L} - \sigma \Rightarrow \frac{L_N}{L} = \frac{E}{\beta}\left(\frac{\dot{N}}{N}+\sigma\right).$$

Substituting (B.21) and $E^* = \frac{\beta(1-s_G)}{\beta-\rho+\lambda}$ into above, we get the transitional path of employment share of entry R&D

$$\frac{L_N(t)}{L(t)} = \frac{(1-s_G)}{\beta-\rho+\lambda}\left(\frac{e^{-vt}\Delta}{(1+e^{-vt}\Delta)}+\lambda+\sigma\right). \tag{B.25}$$

Finally, summing up (B.24) and (B.25), we obtain the transitional path of labor share of employment in private R&D.

$$\frac{L_Z(t)+L_N(t)}{L(t)} = \frac{(1-s_G)}{\beta-\rho+\lambda}\left[\frac{\beta\theta(\epsilon-1)}{\epsilon} - \frac{v(\sigma+\rho)}{\phi\alpha f(s_G)-\sigma-\rho}\frac{1}{(1+e^{-vt}\Delta)} + \frac{e^{-vt}\Delta}{(1+e^{-vt}\Delta)}+\lambda+\sigma\right].$$

B.5.2 The derivation for the share of R&D expenditure to GDP ratio

Now we are in a position to derive the transitional path of R&D expenditure to GDP ratio. First, the public R&D expenditure to GDP ratio is

$$\frac{w(t)L_G(t)}{P_C(t)Y(t)} \equiv \frac{L_G}{LE^*} = \frac{s_G}{E^*}.$$

Next, the in-house R&D expenditure to GDP share is

$$\frac{w(t)L_Z(t)}{P_C(t)Y(t)} \equiv \frac{L_Z}{LE^*} = \frac{L_Z/L}{E^*} = \frac{(1-s_G)}{E^*(\beta-\rho+\lambda)}\left[\frac{\beta\theta(\epsilon-1)}{\epsilon} - \frac{v(\sigma+\rho)}{\phi\alpha f(s_G)-\sigma-\rho}\frac{1}{(1+e^{-vt}\Delta)}\right],$$

and the total private R&D expenditure to GDP share is

$$\frac{w(t)(L_Z(t)+L_N(t))}{P_C(t)Y(t)} \equiv \frac{L_Z+L_N}{LE^*} = \frac{(L_Z+L_N)/L}{E^*} =$$
$$\frac{(1-s_G)}{E^*(\beta-\rho+\lambda)}\left[\frac{\beta\theta(\epsilon-1)}{\epsilon} - \frac{v(\sigma+\rho)}{\phi\alpha f(s_G)-\sigma-\rho}\frac{1}{(1+e^{-vt}\Delta)} + \frac{e^{-vt}\Delta}{(1+e^{-vt}\Delta)} + \lambda + \sigma\right].$$

B.6 Welfare

Consider the utility,

$$U = \int_0^\infty e^{-\rho t} L(t) \ln c(t)\, dt, \tag{B.26}$$

where the $c(t)$ is the aggregator of intermediate goods with social return to variety,

$$c(t) = N^\omega \left[\left(\frac{1}{N}\right)^{\frac{1}{\epsilon}} \int_0^N \left(\frac{X_i}{L}\right)^{\frac{\epsilon-1}{\epsilon}} dt\right]^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1 \quad \omega > 0.$$

Substituting the demand from (6), using the pricing rule and the symmetry assumption, the above expression becomes

$$c(t) = T_\omega E, \tag{B.27}$$

where $T_\omega \equiv \frac{\epsilon-1}{\epsilon}N^\omega Z^\theta D^\gamma$ and $Z_i = Z$ and $D_i = D$ for all i. Taking logarithm on T_ω yields

$$
\begin{aligned}
\ln T_\omega &= \ln\frac{\epsilon-1}{\epsilon} + \omega \ln N + \theta \ln Z_i + \gamma \ln D_i \\
&= \ln\frac{\epsilon-1}{\epsilon} + \omega\left(\ln L + \ln n\right) + \theta\left(\ln Z_{i,0} + \int_0^t \hat{Z}_t dt\right) + \gamma\left(\ln D_{i,0} + \int_0^t \hat{D}_t dt\right) \\
&= \ln\frac{\epsilon-1}{\epsilon} + \omega\left(\ln L_0 + \int_0^t \lambda dt + \ln n\right) + \theta \ln Z_{i,0} + \theta \hat{Z}^* t + \theta \int_0^t \left(\hat{Z}_t - \hat{Z}^*\right) dt \\
&+ \gamma \ln D_{i,0} + \gamma \hat{D}^* t + \gamma \int_0^t \hat{D}_t - \hat{D}^* dt \\
&= \ln\frac{\epsilon-1}{\epsilon} + +\omega\left(\ln L_0 + \lambda t + \ln n\right) + \theta \ln Z_{i,0} + \gamma \ln D_{i,0} + \theta \hat{Z}^* t + \theta \int_0^t \left(\hat{Z}_t - \hat{Z}^*\right) dt \\
&+ \gamma \hat{D}^* t + \gamma \int_0^t \hat{D}_t - \hat{D}^* dt \\
&= \ln\frac{\epsilon-1}{\epsilon} + +\omega\left(\ln L_0 + \lambda t + \ln n\right). + \theta \ln Z_{i,0} + \gamma \ln D_{i,0} + \theta \hat{Z}^* t + \theta \int_0^t \left(\hat{Z}_t - \hat{Z}^*\right) dt \\
&+ \gamma \frac{s_G}{n^*} t + \gamma \int_0^t \left(\frac{s_G}{n} - \frac{s_G}{n^*}\right) dt.
\end{aligned}
$$

Substituting the solution for n from (B.16), the growth paths of in-house and public R&D technology from (B.12) and B.2 and their steady state values into above and defining $\ln T_{\omega,0} \equiv \ln\frac{\epsilon-1}{\epsilon} + \omega \ln L_0 + \theta \ln Z_{i,0} + \gamma \ln D_{i,0}$, we obtain

$$
\begin{aligned}
\ln T_\omega &= \ln T_{\omega,0} + \omega\lambda t + \omega\left(\ln\frac{\mathrm{n}^*}{1+\mathrm{e}^{-\mathrm{vt}}\left(\frac{\mathrm{n}^*}{\mathrm{n}_0}-1\right)}\right) + \left[\theta\hat{Z}^* + \gamma\frac{s_G}{n^*}\right] t \\
&+ \theta\int_0^t \left[\left(f(s_G)\frac{\alpha\theta(\epsilon-1)}{\epsilon}\frac{E}{n} - \sigma - \rho\right) - \left(f(s_G)\frac{\alpha\theta(\epsilon-1)}{\epsilon}\frac{E}{n^*} - \sigma - \rho\right)\right] dt + \gamma\int_0^t \left(\frac{s_G}{n} - \frac{s_G}{n^*}\right) dt \\
&= \ln T_{\omega,0} + \left[\theta\hat{Z}^* + \gamma\frac{s_G}{n^*} + \omega\lambda\right] t + \omega\left(\ln \mathrm{n}_0 \frac{\frac{\mathrm{n}^*}{\mathrm{n}_0}}{1+\mathrm{e}^{-\mathrm{vt}}\Delta}\right) + \theta f(s_G)\frac{\alpha\theta(\epsilon-1)}{\epsilon}\frac{E}{n^*}\int_0^t \left(\frac{n^*}{n} - 1\right) dt \\
&+ \gamma\int_0^t \left(\frac{s_G}{n} - \frac{s_G}{n^*}\right) dt \\
&= \ln T_{\omega,0} + \left[\theta\hat{Z}^* + \gamma\frac{s_G}{n^*} + \omega\lambda\right] t + \omega\left(\ln \mathrm{n}_0 + \ln\frac{1+\Delta}{1+\mathrm{e}^{-\mathrm{vt}}\Delta}\right) \\
&+ \left[\gamma\frac{s_G}{n^*} + \theta\left(\hat{Z}^* + \sigma + \rho\right)\right]\int_0^t \left(e^{-vt}\Delta\right) dt,
\end{aligned}
$$

where the last two two terms using the fact that $\hat{Z}^* = f(s_G)\frac{\alpha\theta(\epsilon-1)}{\epsilon}\frac{E}{n^*} - \sigma - \rho$, the solution for

$n = n(t) = \frac{n^*}{1+e^{-vt}\left(\frac{n^*}{n_0}-1\right)}$ in (B.16), and the definition, $\Delta \equiv \frac{n^*}{n_0} - 1$.

We further solve $\int_0^t (e^{-vt}\Delta)\, dt = -\frac{1}{v}e^{-vt}\Delta + \frac{1}{v}\Delta = \frac{\Delta}{v}(1-e^{-vt})$ and substitute it back to the above expression and yield

$$\ln T_\omega = \ln T_{\omega,0} + \omega \ln n_0 + \left[\theta \hat{Z}^* + \gamma \frac{s_G}{n^*} + \omega\lambda\right] t + \left[\gamma \frac{s_G}{n^*} + \theta\left(\hat{Z}^* + \sigma + \rho\right)\right] \frac{\Delta}{v}(1-e^{-vt})$$
$$+\,\omega\left(\ln \frac{1+\Delta}{1+\mathrm{e}^{-\mathrm{vt}}\Delta}\right).$$

Taking logarithm on (B.27) and substituting $\ln T_\omega$ back to it yield

$$\ln c(t) = \ln E + \ln T_{\omega,0} + \omega \ln n_0 + \left[\theta \hat{Z}^* + \gamma \frac{s_G}{n^*} + \omega\lambda\right] t + \left[\gamma \frac{s_G}{n^*} + \theta\left(\hat{Z}^* + \sigma + \rho\right)\right] \frac{\Delta}{v}(1-e^{-vt})$$
$$+\,\omega\left(\ln \frac{1+\Delta}{1+\mathrm{e}^{-\mathrm{vt}}\Delta}\right).$$

We further substitute the above expression back to the life time utility (B.26) and set $F \equiv \ln T_{\omega,0} + \omega \ln n_0 = 0$, we get

$$U = \int_0^\infty e^{-(\rho-\lambda)t}\left[F + \ln E + \left[\theta \hat{Z}^* + \gamma \frac{s_G}{n^*} + \omega\lambda\right] t + \left[\gamma \frac{s_G}{n^*} + \theta\left(\hat{Z}^* + \sigma + \rho\right)\right] \frac{\Delta}{v}(1-e^{-vt})\right.$$
$$\left. +\,\omega\left(\ln \frac{1+\Delta}{1+\mathrm{e}^{-\mathrm{vt}}\Delta}\right)\right] dt$$
$$= \frac{\ln E}{\rho-\lambda} + \underbrace{\left[\theta \hat{Z}^* + \gamma \frac{s_G}{n^*} + \omega\lambda\right] \int_0^\infty e^{-(\rho-\lambda)t} t dt}_{(a)} + \left[\gamma \frac{s_G}{n^*} + \left(\hat{Z}^* + \sigma + \rho\right)\right] \underbrace{\frac{\Delta}{v}\int_0^\infty e^{-(\rho-\lambda)t}(1-e^{-vt})dt}_{(b)}$$
$$+ \underbrace{\int_0^\infty e^{-(\rho-\lambda)t}\omega\left(\ln \frac{1+\Delta}{1+\mathrm{e}^{-\mathrm{vt}}\Delta}\right) dt}_{(c)}.$$

Next, we obtain the closed form solution for (a), (b) and (c) as follows:

By setting $a = t$ and $db = e^{-(\rho-\lambda)t}dt$, we get the expression for (a) with integration by part,

$$\begin{aligned}
&\left[\theta\hat{Z}^* + \gamma\frac{s_G}{n^*} + \omega\lambda\right]\int_0^\infty e^{-(\rho-\lambda)t}tdt \\
&= \left[\theta\hat{Z}^* + \gamma\frac{s_G}{n^*} + \omega\lambda\right]\left[\left.-\frac{t}{\rho-\lambda}e^{-(\rho-\lambda)t}\right]_0^\infty - \int_0^\infty -\frac{1}{\rho-\lambda}e^{-(\rho-\lambda)t}dt\right] \\
&= \left[\theta\hat{Z}^* + \gamma\frac{s_G}{n^*} + \omega\lambda\right]\left\{\left.-\frac{t}{\rho-\lambda}e^{-(\rho-\lambda)t}\right]_0^\infty \left.-\frac{1}{(\rho-\lambda)^2}e^{-(\rho-\lambda)t}\right]_0^\infty\right\} \\
&= \frac{\theta\hat{Z}^* + \gamma\frac{s_G}{n^*} + \omega\lambda}{(\rho-\lambda)^2} = \frac{\theta\hat{Z}^* + \gamma\hat{D}^* + \omega\lambda}{(\rho-\lambda)^2}.
\end{aligned}$$

The integration for (b) is

$$\begin{aligned}
&\left[\gamma\frac{s_G}{n^*} + \theta\left(\hat{Z}^* + \sigma + \rho\right)\right]\frac{\Delta}{v}\int_0^\infty e^{-(\rho-\lambda)t}(1 - e^{-vt})dt \\
&= \left[\gamma\frac{s_G}{n^*} + \theta\left(\hat{Z}^* + \sigma + \rho\right)\right]\frac{\Delta}{v}\int_0^\infty \left(e^{-(\rho-\lambda)t} - e^{(-\rho+\lambda-v)t}\right)dt \\
&= \frac{\left[\gamma\frac{s_G}{n^*} + \theta\left(\hat{Z}^* + \sigma + \rho\right)\right]\frac{\Delta}{v}}{\rho-\lambda} - \frac{\left[\gamma\frac{s_G}{n^*} + \theta\left(\hat{Z}^* + \sigma + \rho\right)\right]\frac{\Delta}{v}}{\rho-\lambda+v} \\
&= \frac{\left[\gamma\frac{s_G}{n^*} + \theta\left(\hat{Z}^* + \sigma + \rho\right)\right]\Delta}{(\rho-\lambda)\left((\rho-\lambda)+v\right)} = \frac{\left[\gamma\hat{D}^* + \theta\left(\hat{Z}^* + \sigma + \rho\right)\right]\Delta}{(\rho-\lambda)\left((\rho-\lambda)+v\right)}.
\end{aligned}$$

Finally, integration of (c) with certain approximation yields

$$\begin{aligned}
\int_0^\infty e^{-(\rho-\lambda)t}\omega\left(\ln\frac{1+\Delta}{1+\mathrm{e}^{-\mathrm{vt}}\Delta}\right)dt &= \int_0^\infty e^{-(\rho-\lambda)t}\omega\left[\ln(1+\Delta) - \ln\left(1+e^{-vt}\Delta\right)\right]dt \\
&\simeq \int_0^\infty e^{-(\rho-\lambda)t}\omega\left[\Delta - e^{-vt}\Delta\right]dt \\
&= \omega\Delta\left(\frac{1}{(\rho-\lambda)} - \frac{1}{(\rho-\lambda)+v}\right) = \frac{\omega v\Delta}{(\rho-\lambda)\left[(\rho-\lambda)+v\right]}.
\end{aligned}$$

C The general model of knowledge cross fertilization (the stock view)

We recover the general cross-fertilization knowledge spillover function with two knowledge stocks from (8) and (9) which are

$$\dot{Z}_i = \alpha f(s_G) K_i \left[\frac{1+\kappa\left(\frac{D_i}{K_i}\right)^{\eta}}{1+\kappa} \right]^{\frac{1}{\eta}} L_{Z_i}, \tag{C.1}$$

and

$$\dot{D}_i = D_i \left[\frac{1+\kappa\left(\frac{D_i}{K_i}\right)^{-\delta}}{1+\kappa} \right]^{\frac{1}{\delta}} L_{G_i}. \tag{C.2}$$

C.1 Proof of Lemma 2

Before we proceed, we adopt the same procedure as we prove for Lemma 2 in subsection 2.1, we find that Lemma 2 also holds in this general version of the model with

$$E = E^* \equiv \frac{\beta(1-s_G)}{\beta-\rho+\lambda},$$

and $r = \rho$.

C.2 Innovation behavior

The intermediate firm's profit maximization yields the derivative of profit function as in the flow version under symmetry,

$$\partial \Pi_i / \partial Z_i = \frac{1}{\epsilon} L \frac{E}{N} \frac{\theta(\epsilon-1)}{Z},$$

while the F.O.C. of current-value Hamiltonian function with respect to L_{Z_i} yields

$$\frac{1}{q_i} = \alpha f(s_G) K \left[\frac{1 + \kappa (k)^{\eta}}{1 + \kappa} \right]^{\frac{1}{\eta}},$$

where $K = K_i = Z_i = Z$ and $k \equiv D/K = D_i/K_i$ under symmetry.

Taking the logarithm of $1/q$ and differentiating it with respect time yields

$$\frac{\dot{q}}{q} = -\frac{\kappa (k)^{\eta}}{[1 + \kappa (k)^{\eta}]} \frac{\dot{k}}{k} - \hat{Z}. \tag{C.3}$$

Plugging (**??**) and (C.3) back to (B.6) and using the fact that $r = \rho$ and $E = E^*$ yield the key equation for private R&D behavior:

$$\rho + \sigma = \frac{\theta(\epsilon - 1)}{\epsilon} \frac{LE}{N} \alpha f(s_G) \left[\frac{1 + \kappa (k)^{\eta}}{1 + \kappa} \right]^{\frac{1}{\eta}} - \frac{\kappa (k)^{\eta}}{[1 + \kappa (k)^{\eta}]} \frac{\dot{k}}{k} - \hat{Z}. \tag{C.4}$$

Next, with some manipulation, the return to entry in symmetric equilibrium becomes

$$\begin{aligned} \rho + \sigma = \frac{\Pi}{V} + \frac{\dot{V}}{V} &= \left[\frac{1}{\epsilon} \frac{E^*}{n} - \phi - \frac{L_Z}{N} \right] \frac{\beta n}{E^*} - \frac{\dot{n}}{n} \\ &= \left[\frac{1}{\epsilon} \frac{E^*}{n} - \phi - \frac{\hat{Z}}{\alpha f(s_G) K_i \left[\frac{1 + \kappa \left(\frac{D_i}{K_i} \right)^{\eta}}{1 + \kappa} \right]^{\frac{1}{\eta}}} \right] \frac{\beta n}{E^*} - \frac{\dot{n}}{n}, \end{aligned} \tag{C.5}$$

where the second equality is applied by using (C.1).

C.3 The Firm innovation

Noting that $\dot{k}/k = \hat{D} - \hat{Z}$, we substitute it into (C.4) and obtain

$$\rho + \sigma = \frac{\theta(\epsilon - 1)}{\epsilon} \frac{LE}{N} \alpha f(s_G) (1 + \kappa)^{-\frac{1}{\eta}} [1 + \kappa (k)^{\eta}]^{\frac{1}{\eta}} - \frac{\kappa (k)^{\eta}}{[1 + \kappa (k)^{\eta}]} \hat{D} - \frac{1}{[1 + \kappa (k)^{\eta}]} \hat{Z}.$$

Rearrange it and replace $\hat{D}$ with (C.2). Using $r = \rho$ and $E = E^*$ implied in Lemma 2, we obtain

$$\hat{Z} = \frac{\left(\frac{\theta(\epsilon-1)}{\epsilon} E^* \alpha f(s_G) \left[\frac{1+\kappa(k)^\eta}{1+\kappa}\right]^{\frac{1}{\eta}} - \frac{\kappa(k)^\eta}{[1+\kappa(k)^\eta]} \left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} s_G\right) \frac{1}{n} - \sigma - \rho}{\frac{1}{1+\kappa(k)^\eta}}. \tag{C.6}$$

This function identifies the boundary of the region with $\hat{Z} = 0$ (or $L_Z/N > 0$), that is,

$$n > n_{\hat{Z}=0}(k) \equiv \frac{\left(\frac{\theta(\epsilon-1)}{\epsilon} E^* \alpha f(s_G) \left[\frac{1+\kappa(k)^\eta}{1+\kappa}\right]^{\frac{1}{\eta}} - \frac{\kappa(k)^\eta}{[1+\kappa(k)^\eta]} \left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} s_G\right)}{\sigma + \rho}. \tag{C.7}$$

Rewriting the mass of firm in per capita term n from the entry process in (**??**) yields

$$\frac{\dot{n}}{n} = \frac{\beta}{LE} L_N - \sigma - \lambda. \tag{C.8}$$

We further replace $\frac{\dot{n}}{n}$ in (C.5) with the above expression and rearranging it yields

$$\frac{L_N}{N} = \left[\frac{1}{\epsilon} - \frac{(\rho - \lambda)}{\beta}\right] \frac{E^*}{n} - \left[\phi + \frac{\hat{Z}}{\alpha f(s_G) \left[\frac{1+\kappa(k)^\eta}{1+\kappa}\right]^{\frac{1}{\eta}}}\right]. \tag{C.9}$$

We further substitute $\hat{Z}$ from (C.6) into above and rearranging it yields

$$\begin{aligned} \frac{L_N}{N} = & \left[\frac{1}{\epsilon} - \frac{\theta(\epsilon - 1)}{\epsilon} (1 + \kappa(k)^\eta) - \frac{(\rho - \lambda)}{\beta}\right] \frac{E^*}{n} - \phi \\ & + \frac{\frac{\kappa(k)^\eta}{[1+\kappa(k)^\eta]} \left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} \frac{s_G}{n} + \sigma + \rho}{\alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}} \left[1 + \kappa(k)^\eta\right]^{\frac{1}{\eta} - 1}}. \end{aligned}$$

Similarly, this function can identify the boundary of the region with $\frac{L_N}{N} > 0$. By solving $\frac{L_N}{N} = 0$ for the threshold, $n_{L_N=0,L_Z>0}(k)$, we obtain

$$n \geq n_{L_N=0,L_Z>0}(k) \equiv \frac{\left[\frac{1}{\epsilon} - \frac{\theta(\epsilon-1)}{\epsilon}\left(1+\kappa\left(k\right)^{\eta}\right) - \frac{(\rho-\lambda)}{\beta}\right] E^{*} + \frac{\kappa(k)^{\eta}\left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} s_G}{\alpha f(s_G)\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}}}}{\phi - \frac{\sigma+\rho[1+\kappa(k)^{\eta}]}{\alpha f(s_G)\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}}}} \tag{C.10}$$

for the region when $\frac{L_N}{N} = 0$ and $\frac{L_Z}{N} > 0$. Moreover, let both $\hat{Z} = 0$ and $\frac{L_N}{N} = 0$ in (C.9), we can solve the boundary

$$n \geq n_{L_N=0,L_Z=0} \equiv \left[\frac{1}{\epsilon} - \frac{(\rho-\lambda)}{\beta}\right]\frac{E^{*}}{\phi}$$

for the region when both $\frac{L_N}{N} = 0$ and $\frac{L_Z}{N} = 0$. Combining the two boundaries derived above, we can identify the region above the curve of $L_N = 0$ shown in figure 1.

C.4 Cross-fertilization global dynamics (a "substitute" scenario i.e., $0 < \eta \leq 1$ and $0 < \delta \leq 1$)

Proof of Proposition 8 and the phase diagram in Figure 1.

Global dynamics of this general model can be characterized by the activation of in-house and entry R&D into four regions:

Region 1: $L_Z > 0$ and $L_N > 0$.

Substitute (C.6) into (C.5). With some manipulation, we obtain the expression for the firm size dynamics,

$$\frac{\dot{n}}{n} = \left[\frac{1}{\epsilon} - \frac{\theta(\epsilon-1)}{\epsilon}(1+\kappa\,(k)^{\eta}) - \frac{(\rho+\sigma)}{\beta}\right]\beta + \frac{\kappa\,(k)^{\eta}\left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} s_G}{\alpha f(s_G)\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}}}\frac{\beta}{E^*}$$
$$- \left[\phi - \frac{(\sigma+\rho)(1+\kappa\,(k)^{\eta})}{\alpha f(s_G)\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}}}\right]\frac{\beta n}{E^*}. \tag{C.11}$$

Next, using (C.2) to subtract (C.6) and rearranging it yield the expression for the dynamics of knowledge stock ratio k,

$$\frac{\dot{k}}{k} = (1+\kappa\,(k)^{\eta})\left[\frac{\left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} s_G - \frac{\theta(\epsilon-1)}{\epsilon}E^*\alpha f(s_G)\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}}}{n} + (\sigma+\rho)\right] \tag{C.12}$$

This dynamics system is governed by the following two loci. Setting $\frac{\dot{n}}{n} = 0$ in (C.11) yields

$$n_{\dot{n}=0}(k) \equiv \frac{\left[\frac{1}{\epsilon} - \frac{\theta(\epsilon-1)}{\epsilon}(1+\kappa\,(k)^{\eta}) - \frac{(\rho+\sigma)}{\beta}\right]E^* + \frac{\kappa(k)^{\eta}\left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} s_G}{\alpha f(s_G)\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}}}}{\left[\phi - \frac{(\sigma+\rho)(1+\kappa(k)^{\eta})}{\alpha f(s_G)\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}}}\right]},$$

where $\dot{n} \geq 0$ when $n \leq n_{\dot{n}=0}(k)$.

Setting $\frac{\dot{k}}{k} = 0$ in (C.12) yields

$$n_{\dot{k}=0}(k) \equiv \frac{\frac{\theta(\epsilon-1)}{\epsilon}E^*\alpha f(s_G)\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}} - \left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} s_G}{\sigma+\rho}, \tag{C.13}$$

where $\dot{k} \geq 0$ when $n \geq n_{\dot{k}=0}(k)$, We obtain k_1 by solving $n_{\dot{n}=0}(k) = 0$ and yield

$$k_1 = \text{argsolve}\left\{\left[\frac{1}{\epsilon} - \frac{\theta(\epsilon-1)}{\epsilon}(1+\kappa\,(k)^\eta) - \frac{(\rho+\sigma)}{\beta}\right]E^* + \frac{\kappa\,(k)^\eta\left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} s_G}{\alpha f(s_G)\left[\frac{1+\kappa(k)^\eta}{1+\kappa}\right]^{\frac{1}{\eta}}} = 0\right\}. \tag{C.14}$$

Next, we obtain k_2 by solving $n_{\dot{k}=0}(k) = 0$ and yield

$$k_2 = \text{argsolve}\left\{\frac{\theta(\epsilon-1)}{\epsilon}E^*\alpha f(s_G)\left[\frac{1+\kappa\,(k)^\eta}{1+\kappa}\right]^{\frac{1}{\eta}} = \left[\frac{1+\kappa\,(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} s_G\right\} \tag{C.15}$$

To characterize properly the phase diagram for this region involving the following **three steps**:

In the first step,

we prove that (i) $\left[\phi - \frac{(\sigma+\rho)(1+\kappa(k)^\eta)}{\alpha f(s_G)\left[\frac{1+\kappa(k)^\eta}{1+\kappa}\right]^{\frac{1}{\eta}}}\right]$ is increasing in k and converges to ϕ from below; (ii) $n_{\dot{n}=0}(k)$ is decreasing in k with k greater than a threshold value k_3; (iii) $\lim_{\mathrm{k}\to 0^+} \mathrm{n}_{\dot{\mathrm{n}}=0}(\mathrm{k}) = +\infty$.

Proof:

(i) $\left[\phi - \frac{(\sigma+\rho)(1+\kappa(k)^\eta)}{\alpha f(s_G)\left[\frac{1+\kappa(k)^\eta}{1+\kappa}\right]^{\frac{1}{\eta}}}\right]$ **is increasing in k and converges to ϕ from below.**

Under the assumption that $\phi - \frac{(\sigma+\rho)}{\alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}}} > 0$, when $0 < \eta \leq 1$, the denominator in $n_{\dot{n}=0}(k)$, $\left[\phi - \frac{(\sigma+\rho)}{\alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}}[1+\kappa(k)^\eta]^{\frac{1}{\eta}-1}}\right]$, is always positive and increasing in k because $\frac{[1+\kappa(k)^\eta]^{\frac{1}{\eta}}}{(1+\kappa(k)^\eta)} = (1+\kappa\,(k)^\eta)^{\frac{1}{\eta}-1} > 1$ for all $k > 0$ under $0 < \eta \leq 1$ and itself is increasing in k. Moreover, since $\lim_{k\to\infty} [1+\kappa\,(k)^\eta]^{\frac{1}{\eta}-1} = \infty$, it converges to ϕ from below as $k \to \infty$.

(ii) $\lim_{\mathrm{k}\to 0^+} \mathrm{n}_{\dot{\mathrm{n}}=0}(\mathrm{k}) = +\infty$**.**

When $0 < \eta \leq 1$, we obtain

$$\lim_{k\to 0^+}(1+\kappa\,(k)^\eta) = 1,$$

and

$$\lim_{k\to0^+}\left[1+\kappa\,(k)^\eta\right]^{\frac{1}{\eta}}=\left[1+\lim_{k\to0^+}\kappa\,(k)^\eta\right]^{\frac{1}{\eta}}=\left[1+0\right]^{\frac{1}{\eta}}=1.$$

Both equations imply that

$$\lim_{k\to0^+}\frac{(1+\kappa\,(k)^\eta)}{\left[1+\kappa\,(k)^\eta\right]^{\frac{1}{\eta}}}=1.$$

Moreover,

$$\lim_{k\to0^+}\frac{\kappa\,(k)^\eta}{\left[1+\kappa\,(k)^{-\delta}\right]^{-\frac{1}{\delta}}}=\frac{\kappa\,(k)^\eta}{\left[1+\kappa\frac{1}{(k)^\delta}\right]^{-\frac{1}{\delta}}}=\frac{\kappa\,(k)^\eta}{\left[\frac{(k)^\delta+\kappa}{(k)^\delta}\right]^{-\frac{1}{\delta}}}=\frac{\kappa\,(k)^\eta}{\frac{\left[(k)^\delta+\kappa\right]^{-\frac{1}{\delta}}}{(k)^{-1}}}=\frac{\kappa\,(k)^{\eta-1}}{\left[(k)^\delta+\kappa\right]^{-\frac{1}{\delta}}}=\infty.$$

Once we have the above results in hand, we can find that when $0<\eta\leq1$,

$$\lim_{k\to0^+}\frac{\left[\frac{1}{\epsilon}-\frac{\theta(\epsilon-1)}{\epsilon}(1+\kappa\,(k)^\eta)-\frac{(\rho+\sigma)}{\beta}\right]E^*+\frac{\kappa(k)^\eta(1+\kappa)^{-\frac{1}{\delta}}\left[1+\kappa(k)^{-\delta}\right]^{\frac{1}{\delta}}s_G}{\alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}}\left[1+\kappa(k)^\eta\right]^{\frac{1}{\eta}}}}{\left[\phi-\frac{(\sigma+\rho)(1+\kappa(k)^\eta)}{\alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}}\left[1+\kappa(k)^\eta\right]^{\frac{1}{\eta}}}\right]}$$

$$=\frac{\lim_{k\to0^+}\left[\frac{1}{\epsilon}-\frac{\theta(\epsilon-1)}{\epsilon}(1+\kappa\,(k)^\eta)-\frac{(\rho+\sigma)}{\beta}\right]E^*+\frac{(1+\kappa)^{-\frac{1}{\delta}}s_G}{\alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}}}\lim_{k\to0^+}\frac{1}{\left[1+\kappa(k)^\eta\right]^{\frac{1}{\eta}}}\lim_{k\to0^+}\frac{\kappa(k)^\eta}{\left[1+\kappa(k)^{-\delta}\right]^{-\frac{1}{\delta}}}}{\lim_{k\to0^+}\left[\phi-\frac{(\sigma+\rho)(1+\kappa(k)^\eta)}{\alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}}\left[1+\kappa(k)^\eta\right]^{\frac{1}{\eta}}}\right]}=\infty.$$

(iii) $n_{\dot{n}=0}(k)$ is decreasing in k with a sufficient condition that k is greater than a threshold value k_3.

Next, we know that the first term in the numerator of $n_{\dot{n}=0}(k)$ is $\left[\frac{1}{\epsilon}-\frac{\theta(\epsilon-1)}{\epsilon}(1+\kappa\,(k)^\eta)-\frac{(\rho+\sigma)}{\beta}\right]E^*$ which is decreasing in k and reaches $-\infty$ when k goest to ∞. Moreover, with some manipulation, the second term in the numerator becomes

$$\frac{\kappa\,(k)^\eta\left[1+\kappa\,(k)^{-\delta}\right]^{\frac{1}{\delta}}s_G}{\alpha f(s_G)\left[\frac{1+\kappa(k)^\eta}{1+\kappa}\right]^{\frac{1}{\eta}}}=\frac{\kappa\left[1+\kappa\,(k)^{-\delta}\right]^{\frac{1}{\delta}}s_G}{\alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}}\left[(k)^{-\eta^2}+\kappa\,(k)^{\eta(1-\eta)}\right]^{\frac{1}{\eta}}},$$

in which the term $\left[1+\kappa\left(k\right)^{-\delta}\right]^{\frac{1}{\delta}}$ on the top is decreasing in k with $\lim_{k\to\infty}\left[1+\kappa\left(k\right)^{-\delta}\right]^{\frac{1}{\delta}}=1$ and the term $\left[\left(k\right)^{-\eta^2}+\kappa\left(k\right)^{\eta(1-\eta)}\right]^{\frac{1}{\eta}}$ on the bottom is increasing in k when $k>\left(\frac{\eta}{(1-\eta)\kappa}\right)^{\frac{1}{\eta}}$, where the proof is shown below:

$$\partial\frac{\left[\left(k\right)^{-\eta^2}+\kappa\left(k\right)^{\eta(1-\eta)}\right]^{\frac{1}{\eta}}}{\partial k}=\frac{1}{\eta}\left[\left(k\right)^{-\eta^2}+\kappa\left(k\right)^{\eta(1-\eta)}\right]^{\frac{1}{\eta}-1}\eta\left[\left(-\eta+\kappa\left(k\right)^{\eta}\left(1-\eta\right)\right)\left(k\right)^{-\eta^2-1}\right]>0$$

$$=>-\eta+\kappa\left(k\right)^{\eta}\left(1-\eta\right)>0=>\kappa\left(k\right)^{\eta}>\frac{\eta}{(1-\eta)}=>k>\left(\frac{\eta}{(1-\eta)\kappa}\right)^{\frac{1}{\eta}}.$$

This implies the entire term, $\frac{\kappa(k)^{\eta}\left[1+\kappa(k)^{-\delta}\right]^{\frac{1}{\delta}}s_G}{\alpha f(s_G)\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}}}$, is decreasing k when $k>\left(\frac{\eta}{(1-\eta)\kappa}\right)^{\frac{1}{\eta}}$ and converges to 0. With all the information above indicates that $n_{\dot{n}=0}(k)$ is decreasing in k and converges to $-\infty$ when $k>\left(\frac{\eta}{(1-\eta)\kappa}\right)^{\frac{1}{\eta}}$ and it crosses horizontal axis in k_1 from above as we have obtained previously.

Next, we can also easily see that $n_{\dot{n}=0}(k)$ has the same shape as the $L_N=0$ boundary in (C.10), but is everywhere below it. Besides, we will prove later that the $L_Z=0$ ($\hat{Z}=0$) boundary in (C.7) starts out from a positive k_z from the horizontal axis and is increasing in k and since $\lim_{\mathrm{k}\to 0^+}\mathrm{n}_{\dot{\mathrm{n}}=0}(\mathrm{k})=+\infty$ is proved in (ii), there exists a intersection between $L_Z=0$ and $n_{\dot{n}=0}(k)$, where the intersection in the dimension of n is $\bar{n}^*=\left[\frac{1}{\epsilon}-\frac{\rho+\sigma}{\beta}\right]\frac{E^*}{\phi}$.[1]

We further substituting $\bar{n}^*$ into $n_{\dot{n}=0}(k)$, we can obtain k_3 that solves

$$\left[\frac{1}{\epsilon}-\frac{\rho+\sigma}{\beta}\right]\frac{E^*}{\phi}=\frac{\left[\frac{1}{\epsilon}-\frac{\theta(\epsilon-1)}{\epsilon}\left(1+\kappa\left(k\right)^{\eta}\right)-\frac{(\rho+\sigma)}{\beta}\right]E^*+\frac{\kappa(k)^{\eta}\left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}}s_G}{\alpha f(s_G)\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}}}}{\left[\phi-\frac{(\sigma+\rho)(1+\kappa(k)^{\eta})}{\alpha f(s_G)\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}}}\right]}.$$

If we specify the condition that $k>k_3>\left(\frac{\eta}{(1-\eta)\kappa}\right)^{\frac{1}{\eta}}$, Then we can guarantee that $n_{\dot{n}=0}(k)$ is monotonically decreasing in k for all $k>k_3$. Therefore, the proofs for (i), (ii) and (iii) are

[1]Specifically by substituting $\hat{Z}=0$ into (C.5) yields $\frac{\dot{n}}{n}=\left[\frac{1}{\epsilon}\frac{E^*}{n}-\phi\right]\frac{\beta n}{E^*}-(\rho+\sigma)$ and solving $\frac{\dot{n}}{n}=0$, we can obtain $\bar{n}^*=\left[\frac{1}{\epsilon}-\frac{\rho+\sigma}{\beta}\right]\frac{E^*}{\phi}$.

complete.

In the second step,

It is easy to verify that $n_{\dot{k}=0}(k)$ from (C.13) starts out zero at k_2 and is monotonically increasing in k.

In the third step,

we show that $k_1 > k_2$ as follows,

Proof of $k_1 > k_2$:

Rewrite equations (C.14) and (C.15) of the solutions k_1 and k_2 as:

$$\frac{\left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} s_G}{\alpha f(s_G)\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}}} = -\left[\frac{1}{\epsilon} - \frac{\theta(\epsilon-1)}{\epsilon}(1+\kappa(k)^{\eta}) - \frac{(\rho+\sigma)}{\beta}\right]\frac{E^*}{\kappa(k)^{\eta}} => k_1,$$

and

$$\frac{\left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} s_G}{\alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}}\left[\frac{1+\kappa(k)^{\eta}}{1+\kappa}\right]^{\frac{1}{\eta}}} = \frac{\theta(\epsilon-1)}{\epsilon}E^* => k_2.$$

The assumption that $v > 0$ in baseline model implies that right-hand side of the top equation is always less than the right-hand side of the bottom equation as shown below:

$$\begin{aligned} &-\left[\frac{1}{\epsilon} - \frac{\theta(\epsilon-1)}{\epsilon}(1+\kappa(k)^{\eta}) - \frac{(\rho+\sigma)}{\beta}\right]\frac{E^*}{\kappa(k)^{\eta}} \\ &\qquad = -\left[\frac{1}{\epsilon} - \frac{\theta(\epsilon-1)}{\epsilon} - \frac{(\rho+\sigma)}{\beta}\right]\frac{E^*}{\kappa(k)^{\eta}} + \frac{\theta(\epsilon-1)}{\epsilon}E^* < \frac{\theta(\epsilon-1)}{\epsilon}E^*, \end{aligned}$$

where $-\left[\frac{1}{\epsilon} - \frac{\theta(\epsilon-1)}{\epsilon} - \frac{(\rho+\sigma)}{\beta}\right] = -\frac{v}{\beta} < 0$.

Since the left hand side of the two equations is decreasing in k, it follows that $k_1 > k_2$.

Besides, the $L_Z = 0$ ($\hat{Z} = 0$) boundary in (C.7) starts out with positive k_z because $\lim_{k \to 0^+} n_{\hat{Z}=0}(k) = -\infty$ and $n_{\hat{Z}=0}(k)$ is increasing in k. This can be verified by showing that the limits of the first term and the second term are

$$\lim_{k \to 0^+} \frac{\theta(\epsilon - 1)}{\epsilon} E^* \alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}} \left[1 + \kappa\,(k)^{\eta}\right]^{\frac{1}{\eta}} = \frac{\theta(\epsilon - 1)}{\epsilon} E^* \alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}},$$

and

$$\lim_{k \to 0^+} \frac{\kappa\,(k)^{\eta}}{[1 + \kappa\,(k)^{\eta}]} (1+\kappa)^{-\frac{1}{\delta}} \left[1 + \kappa\,(k)^{-\delta}\right]^{\frac{1}{\delta}} s_G = \infty.$$

Moreover, the derivative of $n_{\hat{Z}=0}(k)$ with respect to k, after some manipulation, becomes

$$\begin{aligned}
&\frac{\partial n_{\hat{Z}=0}(k)}{\partial k} \\
&= \frac{\kappa\,(k)^{\eta-1}}{[1 + \kappa\,(k)^{\eta}]} \left[\frac{\theta(\epsilon - 1)}{\epsilon} E^* \alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}} \left[1 + \kappa\,(k)^{\eta}\right]^{\frac{1}{\eta}} - \frac{\eta}{[1 + \kappa\,(k)^{\eta}]} \left[1 + \kappa\,(k)^{-\delta}\right]^{\frac{1}{\delta}} (1+\kappa)^{-\frac{1}{\delta}} s_G \right] \\
&+ \frac{\kappa\,(k)^{\eta}}{[1 + \kappa\,(k)^{\eta}]} \kappa\,(k)^{-\delta-1} \left[1 + \kappa\,(k)^{-\delta}\right]^{\frac{1}{\delta}} (1+\kappa)^{-\frac{1}{\delta}} s_G.
\end{aligned}$$

Using $\hat{Z} = 0$, i.e.,

$$n(\sigma + \rho) = \frac{\theta(\epsilon - 1)}{\epsilon} E^* \alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}} \left[1 + \kappa\,(k)^{\eta}\right]^{\frac{1}{\eta}} - \frac{\kappa\,(k)^{\eta}}{[1 + \kappa\,(k)^{\eta}]} (1+\kappa)^{-\frac{1}{\delta}} \left[1 + \kappa\,(k)^{-\delta}\right]^{\frac{1}{\delta}} s_G,$$

and substituting it into above and rearrange it, we get

$$\begin{aligned}
\frac{\partial n_{\hat{Z}=0}(k)}{\partial k} &= \frac{\kappa\,(k)^{\eta-1}}{[1 + \kappa\,(k)^{\eta}]} \left[\left(\frac{n(\sigma + \rho)}{\left[\frac{1+\kappa(k)^{-\delta}}{1+\kappa}\right]^{\frac{1}{\delta}} s_G} + \frac{\kappa\,(k)^{\eta}}{[1 + \kappa\,(k)^{\eta}]} - \frac{\eta}{[1 + \kappa\,(k)^{\eta}]} + \kappa\,(k)^{-\delta} \right) \left[\frac{1 + \kappa\,(k)^{-\delta}}{1+\kappa} \right]^{\frac{1}{\delta}} s_G \right] \\
&> 0.
\end{aligned}$$

To guarantee the above inequality to hold, we need

$$\frac{n(\sigma+\rho)}{(1+\kappa)^{-\frac{1}{\delta}}\left[1+\kappa\left(k\right)^{-\delta}\right]^{\frac{1}{\delta}}s_G}+\frac{\kappa\left(k\right)^{\eta}}{\left[1+\kappa\left(k\right)^{\eta}\right]}+\kappa\left(k\right)^{-\delta}>\frac{\eta}{\left[1+\kappa\left(k\right)^{\eta}\right]}$$
$$=\frac{n(\sigma+\rho)}{(1+\kappa)^{-\frac{1}{\delta}}\left[1+\kappa\left(k\right)^{-\delta}\right]^{\frac{1}{\delta}}s_G}+1+\kappa\left(k\right)^{-\delta}>1>\frac{1+\eta}{\left[1+\kappa\left(k\right)^{\eta}\right]}.$$

As a result, we can find a sufficient condition,

$$\kappa\left(k\right)^{\eta}>\eta.$$

This can be further guaranteed by the restriction that $\kappa\left(k\right)^{\eta}>\frac{\eta}{1-\eta}>\eta$ which is the same restriction we make to ensure $n_{\dot{n}=0}(k)$ is decreasing in k. Therefore, $n_{\hat{Z}=0}(k)$ is increasing in k and $\lim_{k\to 0^+} n_{\hat{Z}=0}(k)=-\infty$. This guarantees that $L_Z=0$ ($\hat{Z}=0$) boundary in (C.7) starts out with positive k_z which solves $L_Z=0$ when $n=0$, that is,

$$0=\frac{\theta(\epsilon-1)}{\epsilon}E^*\alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}}\left[1+\kappa\left(k_z\right)^{\eta}\right]^{\frac{1}{\eta}}-\frac{\kappa\left(k_z\right)^{\eta}}{\left[1+\kappa\left(k_z\right)^{\eta}\right]}(1+\kappa)^{-\frac{1}{\delta}}\left[1+\kappa\left(k_z\right)^{-\delta}\right]^{\frac{1}{\delta}}s_G.$$

Also note that since

$$\frac{s_G}{f(s_G)E^*}=\frac{1}{\frac{f(s_G)}{s_G}E^*}=\frac{1}{\left(\frac{1}{s_G}+\xi\right)E^*}$$

is increasing in s_G, Both k_1 and k_2 are increasing in s_G. This suggests that the $\dot{n}=0$ locus shifts up with s_G while the $\dot{k}=0$ locus shifts down. With all the above information allows us to characterize the phase diagram for the system dynamics of region 1. The boundaries (ie., $L_Z=0$ and $L_N=0$) separates this region with others.

C.5 Derivation for Equations (30) and (31)

Steady state requires (C.4) to become

$$\rho + \sigma = \frac{\theta(\epsilon - 1)}{\epsilon}\frac{E^*}{n}\alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}}\left[1+\kappa\,(k)^{\eta}\right]^{\frac{1}{\eta}} - \hat{Z}, \tag{C.16}$$

and (C.5) to become

$$\rho + \sigma = \left[\frac{1}{\epsilon}\frac{E^*}{n} - \phi - \frac{\hat{Z}}{\alpha f(s_G)K_i(1+\kappa)^{-\frac{1}{\eta}}\left[1+\kappa\,(k)^{\eta}\right]^{\frac{1}{\eta}}}\right]\frac{\beta n}{E^*}. \tag{C.17}$$

We replace E^*/n by substituting (C.17) into (C.16). After some manipulation yields equation (31),

$$\hat{Z} = \left[\phi\alpha f(s_G)(1+\kappa)^{-\frac{1}{\eta}}\left[1+\kappa\,(k)^{\eta}\right]^{\frac{1}{\eta}} - (\rho+\sigma)\right]\frac{\theta\beta(\epsilon-1)}{\epsilon v} - (\rho+\sigma)\,.$$

Moreover, we know that the two knowledge growth rates are equal in steady state, implying that

$$\hat{Z}^* = \hat{D}^* = (1+\kappa)^{-\frac{1}{\delta}}\left[1+\kappa\,(k)^{-\delta}\right]^{\frac{1}{\delta}}\frac{s_G}{n}. \tag{C.18}$$

We rearrange (C.18) and substitute (C.16) for n, which yields equation (30),

$$\hat{Z} = \frac{\rho+\sigma}{E^*\frac{\alpha\frac{\theta(\epsilon-1)}{\epsilon}f(s_G)(1+\kappa)^{-\frac{1}{\eta}}\left[1+\kappa(k)^{\eta}\right]^{\frac{1}{\eta}}}{(1+\kappa)^{-\frac{1}{\delta}}\left[1+\kappa(k)^{-\delta}\right]^{\frac{1}{\delta}}s_G} - 1}.$$

C.6 Some interesting properties

For $s_G = 0$, the system dynamics in (C.11) and (C.12) can be degenerated to

$$\frac{\dot{n}}{n} = \left[\frac{1}{\epsilon} - \frac{\theta(\epsilon-1)}{\epsilon}(1+\kappa\,(k)^{\eta}) - \frac{(\rho+\sigma)}{\beta}\right]\beta - \left[\phi - \frac{(\sigma+\rho)(1+\kappa\,(k)^{\eta})}{\alpha(1+\kappa)^{-\frac{1}{\eta}}\left[1+\kappa\,(k)^{\eta}\right]^{\frac{1}{\eta}}}\right]\frac{\beta n}{E^*}, \tag{C.19}$$

and

$$\frac{\dot{k}}{k} = (1+\kappa\,(k)^{\eta}) \left[\frac{-\frac{\theta(\epsilon-1)}{\epsilon} E^{*} \alpha (1+\kappa)^{-\frac{1}{\eta}} \left[1+\kappa\,(k)^{\eta}\right]^{\frac{1}{\eta}}}{n} + (\sigma+\rho) \right], \tag{C.20}$$

where the two loci governing the dynamics are

$$\dot{n} \geq 0: \qquad n \leq n_{\dot{n}=0}(k) = \frac{\left[\frac{1}{\epsilon} - \frac{\theta(\epsilon-1)}{\epsilon}(1+\kappa\,(k)^{\eta}) - \frac{(\rho+\sigma)}{\beta}\right]\beta}{\left[\phi - \frac{(\sigma+\rho)(1+\kappa(k)^{\eta})}{\alpha(1+\kappa)^{-\frac{1}{\eta}}\left[1+\kappa(k)^{\eta}\right]^{\frac{1}{\eta}}}\right]\frac{\beta}{E^{*}}},$$

and

$$\dot{k} \geq 0: \qquad n \geq n_{\dot{k}=0}(k) \equiv \frac{\frac{\theta(\epsilon-1)}{\epsilon} E^{*} \alpha (1+\kappa)^{-\frac{1}{\eta}} \left[1+\kappa\,(k)^{\eta}\right]^{\frac{1}{\eta}}}{(\sigma+\rho)}.$$

The condition for $L_Z > 0$ (i.e.,$\hat{Z}_{s_G=0} > 0$) is

$$\hat{Z}_{s_G=0} = \frac{\left(\frac{\theta(\epsilon-1)}{\epsilon} E^{*} \alpha (1+\kappa)^{-\frac{1}{\eta}} \left[1+\kappa\,(k)^{\eta}\right]^{\frac{1}{\eta}}\right)\frac{1}{n} - \sigma - \rho}{\frac{1}{1+\kappa(k)^{\eta}}} > 0$$
$$=> n < \frac{\frac{\theta(\epsilon-1)}{\epsilon} E^{*} \alpha (1+\kappa)^{-\frac{1}{\eta}} \left[1+\kappa\,(k)^{\eta}\right]^{\frac{1}{\eta}}}{\sigma+\rho},$$

which identifies the region of phrase space where $\dot{k} < 0$. The non-negativity constraint on L_Z implies that we have $\dot{k} = 0$ whenever

$$n \geq n_{\hat{Z}_{s_G=0}}(k) \equiv \frac{\frac{\theta(\epsilon-1)}{\epsilon} E^{*} \alpha (1+\kappa)^{-\frac{1}{\eta}} \left[1+\kappa\,(k)^{\eta}\right]^{\frac{1}{\eta}}}{\sigma+\rho}.$$

The $n_{\hat{Z}_{s_G=0}}(k)$ locus has intercept,

$$n_{\hat{Z}_{s_G=0}}(0) = \frac{\frac{\theta(\epsilon-1)}{\epsilon} E^{*} \alpha (1+\kappa)^{-\frac{1}{\eta}}}{\sigma+\rho}.$$

The phase diagram we obtained can be distinguished into two main cases:

Case 1: for

$$n_{\dot{n}=0}(0) \leq n_{\hat{Z}_{s_{G=0}}}(0): \qquad \frac{\left[\frac{1}{\epsilon} - \frac{\theta(\epsilon-1)}{\epsilon} - \frac{(\rho+\sigma)}{\beta}\right]}{\left[\phi - \frac{(\sigma+\rho)}{\alpha(1+\kappa)^{-\frac{1}{\eta}}}\right]} \leq \frac{\frac{\theta(\epsilon-1)}{\epsilon}\alpha(1+\kappa)^{-\frac{1}{\eta}}}{\sigma+\rho}$$

$$\left[\frac{1}{\epsilon} - \frac{(\rho+\sigma)}{\beta}\right] \leq \phi\frac{\frac{\theta(\epsilon-1)}{\epsilon}\alpha(1+\kappa)^{-\frac{1}{\eta}}}{\sigma+\rho},$$

all initial conditions (k_0, n_0) yield paths that converge to the unique steady state $(0, n^*)$, which is the steady state endogenous growth driven by private R&D activity of the baseline Schumpeterian model with no government.

Case 2: for

$$n_{\dot{n}=0}(0) > n_{\hat{Z}_{s_{G=0}}}(0): \qquad \left[\frac{1}{\epsilon} - \frac{(\rho+\sigma)}{\beta}\right] > \phi\frac{\frac{\theta(\epsilon-1)}{\epsilon}\alpha(1+\kappa)^{-\frac{1}{\eta}}}{\sigma+\rho},$$

there is a set of zero growth steady state, the union of the point $(\bar{k}^*, \bar{n}^*)$ and the points $(\widetilde{k}^*, \bar{n}^*)$ for $\widetilde{k} \in (0, \bar{k}^*)$. All initial conditions (k_0, n_0) yield paths that converge to a point in this set. The value $\bar{k}^*$ is uniquely determined by the parameters (we find $\bar{k}^*$ and $\bar{n}^*$ by solving (C.19) and (C.20) at $\frac{\dot{n}}{n} = \frac{\dot{k}}{k} = 0$). In contrast, the value $\widetilde{k}^*$ depends on the specific path dictated by the initial condition and the law of motion of the system.