Government Spending and Industrialization in a Schumpeterian Economy

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Abstract

The goal of this study is to contribute to the debate on the role of government spending in shaping the growth process. We take the analysis in three new directions. First, we investigate the role of government spending in a scale-invariant Schumpeterian model of endogenous innovation. Second, we allow public spending to be the catalyst that precipitates the takeoff of the economy. Third, we postulate a production structure that violates the conventional condition for endogenous growth, namely, that the economy's reduced-form production function must be linear in the accumulated factor. With non-distortionary taxation, increasing productive government spending causes an earlier industrial takeoff and faster economic growth. With distortionary labor-income tax under elastic labor supply, instead, increasing productive government spending has a U-shaped effect on the timing of the industrial takeoff and an inverted-U effect on economic growth. Using cross-country panel data, we document an inverted-U relationship between productive government spending and economic growth. Calibrating the model to US data, we find that raising productive government spending from its historical value to to its growth-maximizing value causes an earlier industrial takeoff by over six decades and an increase in the long-run level of output by 129%. We also explore the robustness of our results under consumption tax and corporate income tax.

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1 Introduction

The goal of this study is to contribute to the debate on the role of government spending in shaping the growth process. We take the seminal study by Barro (1990) as our point of departure and focus on the government's provision of productive public services to the private sector. We then take the analysis in three new directions. First, we investigate the role of government spending in a scale-invariant Schumpeterian model of endogenous innovation. Second, we allow public spending to be the catalyst that precipitates the takeoff of the economy. This is a question never raised before in the theoretical work on the growth effects of public spending. Third, we postulate a production structure that violates the condition for endogenous growth stressed in Barro (1990), namely, that the economy's reduced-form production function must be linear in the accumulated factor. Consequently, our results expand significantly the conditions under which general equilibrium models produce constant exponential growth in steady state, in particular innovation-driven growth that starts at a specific date, accelerates throughout the secular transition, and in steady state is scale invariant and subject to policy action.

The Barro (1990) framework is an excellent point of departure for our analysis because it is analytically transparent and delivers a convincing fundamental insight about the positive contribution that government expenditure can make to economic growth. Specifically, the model produces a hump-shaped relation between economic growth and government spending as a fraction of GDP, a relation that reflects the competing effects of productive public services and distortionary taxation. Since its development more than three decades ago, this insight has stood the test of time and has received strong empirical support.

The empirical literature on the growth effects of public spending is quite large, see for example, Kormendi and Meguire (1985), Ahmed (1986), Aschauer (1989), Levine and Renelt (1992), Evans and Karras (1994), Andres *et al.* (1996), Devarajan *et al.* (1996), Kneller *et al.* (1999) and Folster and Henrekson (2001). Several of these studies identify either a positive, a negative or even an insignificant effect of government spending on economic growth. For a recent exhaustive review of this literature, see Arawatari *et al.* (2023).

For our purposes, Kneller *et al.* (1999) stands out because, first, it tests explicitly the Barro (1990) model and, second, it pays close attention to the financing of public spending. That is, it incorporates the government's budget constraint as an explicit element of the regression analysis. Its main conclusion validates the core prediction of Barro (1990): productive public spending financed with non-distortionary taxes boosts growth, unproductive public spending financed with distortionary taxes lowers growth. Since in the real world most governments engage in a mixture of productive and non-productive spending and of distortionary and non-distortionary taxes, it is not surprising that many studies that do not check explicitly for the government's budget constraint do not reach clear conclusions on the sign of the growth effect of public spending. Kneller *et al.* (1999), in contrast, accounts for the government's budget constraint and as a consequence identifies cleanly the tradeoff driving the Barro (1990) insight.

Given our emphasis on the role of the provision of productive public services as the catalyst of the takeoff, a natural interpretation of the government spending driving our model is that it represents the allocation of resources to building and maintaining a public infrastructure that produces a flow of productive services that enhances the productivity of the private sector. The evidence on the importance of infrastructure is extensive and motivates a similarly extensive literature that discusses the desirability of governments taking an active role in building and maintaining infrastructure. This literature emphasizes that history provides several examples of infrastructure building that arguably triggered dramatic growth accelerations. Examples of empirical studies in this vein are Fernald (1999) and Agrawal *et al.* (2017). They both consider road-building investment in the 20th century and provide evidence that such public infrastructure has significant positive effects on productivity and innovation in the US economy. Similarly, Donaldson and Hornbeck (2016) consider the expansion of the railroad network in the 19th century and find significant effects on the agricultural sector of the US economy. As an example of the policy arguments that build on this literature, consider one of the United Nations Sustainable Development Goals: "to build resilient infrastructure, promote sustainable industrialization and foster innovation", the rationale being that "sustained growth must include industrialization that [...] is supported by innovation and resilient infrastructure".[1] This quote describes quite well the main results produced by our theoretical model, which we now outline.

As mentioned, to study the effects of public spending on the industrialization of an economy and the emergence of innovation, we develop a scale-invariant Schumpeterian growth model that features an endogenous takeoff and a productive structure that is not linear in the accumulated factor (firm knowledge).[2] The provision of productive government services makes the economy more productive and increases the level of output. As in Barro (1990), in our growth-theoretic framework the magnitude of this effect is determined by the elasticity of output with respect to government spending in the production function for a final homogenous good. Differently from Barro (1990), however, our Schumpeterian model generates endogenous growth even when this elasticity is zero.[3] We make this elasticity positive to enable productive government spending to have a positive effect on firm size and economic growth, not as a necessary condition for endogenous growth. We then investigate how public spending and its financing affect the timing of the takeoff and the overall shape of the secular growth process.

Given that government needs to collect tax revenue to finance its spending, the modelling of taxation plays an important role in our growth-theoretic analysis. We first consider a simple case in which government spending is financed by a non-distortionary labor income tax under perfectly inelastic labor supply. In this case, increasing productive government spending leads to an earlier industrial takeoff and a higher transitional growth rate by increasing firm size in the short run. It does not, however, affect economic growth in the long run (steady state) due to the absence of the scale effect in our Schumpeterian growth model with endogenous market structure.[4]

Then, we consider a more realistic case in which government spending is financed by

[1] https://www.un.org/sustainabledevelopment/infrastructure-industrialization/

[2] The model builds on ideas first explored in Peretto (2007b) to study the effects of distrortionary taxes that finance productive public spending. That study considers only local dynamics around the steady state and follows the literature's conventional wisdom in postulating a productive structure that is linear in firm knowledge.

[3] This is due to the property of robust endogenous growth first reported in Peretto (2018). We discuss this property in detail when we introduce the formal model.

[4] See Laincz and Peretto (2006) for a discussion of the scale effect in the Schumpeterian growth model.

a distortionary labor income tax that reduces employment under elastic labor supply. In this case, although increasing productive government spending continues to have no effect on the steady-state growth rate due to the scale invariance of our Schumpeterian model, it has a U-shaped effect on the timing of industrial takeoff and an inverted-U effect on the transitional growth rate. To link this theoretical result to the data, we revisit the cross-country evidence and document with the most recent panel data an inverted-U relationship between productive government spending and economic growth. We also calibrate our model to US data and find that raising productive government spending from its historical value to its growth-maximizing value causes an earlier industrial takeoff by over six decades and an increase in the long-run level of output by 129%.

We explore the robustness of our results under other tax instruments: consumption tax and corporate income tax. We find that when government spending is financed by a consumption tax, increasing it leads to an earlier industrial takeoff and a higher transitional growth rate but does not affect economic growth in the long run. These results are in line with the abovementioned non-distortionary labor income tax because the consumption tax does not affect the equilibrium level of employment, so that only the positive effect of productive government spending on firm size remains in the short run.

When government spending is financed by a corporate income tax, increasing it has a U-shaped effect on the timing of industrial takeoff. This result is also in line with the abovementioned distortionary labor income tax, except that in this case the negative effect now arises from the reduction of firm entry (rather than employment) caused by the corporate income tax. The more important difference is that under the corporate income tax, the effect of productive government spending on the steady-state growth rate in the industrial era becomes positive because the corporate income tax increases firm size in the long run by discouraging firm entry (see Peretto 2007a for an extensive discussion of this property of this class of models).

Our study relates to the literature that examines how government spending affects economic growth. As stated, the seminal study by Barro (1990) introduces government spending as a way to obtain an AK endogenous growth model that features as an equilibrium outcome constant returns to scale to physical capital. Subsequent studies, such as Barro and Sala-i-Martin (1992), Futagami *et al.* (1993), Glomm and Ravikumar (1994), Futagami and Mino (1995), Turnovsky (1996, 2000), Futagami *et al.* (2008), Chatterjee and Turnovsky (2012) and Maebayashi *et al.* (2017), explore different ways to model productive government spending in capital-based growth models. Our study complements these contributions by introducing productive government spending in a scale-invariant Schumpeterian innovation-driven growth model with endogenous industrial takeoff.

Our study also relates to the literature on innovation and economic growth. The seminal study in this literature is Romer (1990), who develops the first R&D-based growth model driven by the development of new products (horizontal innovation). The roughly contemporaneous study by Aghion and Howitt (1992) develops the quality-ladder Schumpeterian growth model in which innovation is driven by the improvement of the quality of products (vertical innovation); see also Grossman and Helpman (1991) and Segerstrom *et al.* (1990). Subsequent studies in this literature combine the two dimensions of innovation to develop the Schumpeterian growth model with endogenous market structure that removes the scale effect; see Smulders and van de Klundert (1995), Peretto (1998, 1999) and Howitt

(1999). Peretto (2015) builds on these contributions and develops a model that features the endogenous activation of the two dimensions of innovation when the economy crosses dimension-specific thresholds or market size, a property that produces an endogenous take-off. This study expands the scope of this literature by exploring the effects of productive government spending in the scale-invariant Schumpeterian growth model with endogenous market structure and endogenous takeoff.

Finally, our study relates to the branch of the literature that examines the effects of fiscal policy on innovation and economic growth. For example, Lin and Russo (1999), Zeng and Zhang (2002), Peretto (2003, 2007a, 2007b, 2011), Haruyama and Itaya (2006), Chen *et al.* (2017) and Arawatari *et al.* (2023) also explore the effects of various fiscal policy instruments in different variants of the innovation-driven growth model. Some of these studies also consider the scale-invariant Schumpeterian growth model with endogenous market structure. Our study contributes to this literature by exploring the effects of productive government spending and various tax instruments on the endogenous transition of an economy from pre-industrial stagnation to innovation-driven growth.[5]

The rest of this study is organized as follows. Section 2 provides some stylized facts. Section 3 presents the Schumpeterian growth model. Section 4 explores the effects of productive government spending. Section 5 considers two extensions with different tax instruments. The final section concludes.

2 Stylized facts

In this section, we revisit the stylized facts from cross-country panel data that motivate our study. As mentioned, there exists an established empirical literature that examines the relationship between government spending and economic growth. While many of the early studies found either a positive, a negative or even an insignificant effect of government spending on economic growth, the more recent studies that account for the government's budget constraint find an inverted-U relationship between government spending and economic growth; see Coayla (2021) for a recent review.[6] As stated, this result validates the Barro (1990) insight. It is useful to review the existing evidence with the most recent data and establish firmly the inverted-U relation between public spending and economic growth. This relation provides one of the main motivations for our theoretical investigation.[7]

Our theoretical model also predicts that increasing productive government spending has

[5] See also Iacopetta and Peretto (2021), Chu, Fan and Wang (2020), Chu, Kou and Wang (2020), Chu, Furukawa and Wang (2022), Chu, Peretto and Wang (2022) and Chu, Peretto and Xu (2023), who explore other mechanisms, such as corporate governance, status-seeking culture, intellectual property rights, rent-seeking government, agricultural revolution and international trade, of endogenous takeoff in the Schumpeterian growth model.

[6] Most of these studies are based on a single country or a small number of countries. A notable exception is Asimakopoulos and Karavias (2016), who also consider cross-country panel data.

[7] In this study, we are interested in additional aspects of the growth path that the literature has not examined before, most prominently the timing of the transition from stagnation to growth and the potential role of public spending in explaining large difference in income across countries as the result of the secular cumulation of differences in growth rates. Ideally, we would examine these other aspects with data on the timing of the takoff in various countries. Such data, unfortunately, is not available.

an inverted-U effect on *transitional* economic growth when the government finances spending with some form of distortionary taxation. Since we are not necessarily interested in the specific tax instrument that induces the non-monotonic relationship, we use the following empirical specification to establish the inverted-U relation between public spending and growth:

$$g_{it} = \vartheta_1 \gamma_{it} + \vartheta_2 \gamma_{it}^2 + \vartheta_3 y_{it-1} + \varphi_i + \varrho_t + \epsilon_{it},$$

where g_{it} denotes the average annual growth rate (measured by real GDP, real GDP per capita or real GDP per worker) in country i at period t, γ_{it} denotes the average value of productive government spending (defined as government spending on education, health, defence, and economic affairs) as a share of GDP in country i at period t, γ_{it}^2 denotes the quadratic term of γ_{it} in country i at period t, and y_{it-1} is the log value of per capita GDP in country i at the beginning of period t to capture the country's initial income level. φ_i is the country fixed effect, ϱ_t is the period fixed effect, and ϵ_{it} is the error term. The data is from 1975 to 2015, and we consider five years as a period (to remove cyclical fluctuations in the data), so we have 8 periods. After merging data from the IMF Government Finance Statistics and the Penn World Table, we have a sample of 189 observations covering 59 countries. We report the summary statistics of the variables in Appendix B.

Our theory predicts an inverted-U relationship between productive government spending and economic growth, which is captured by $\vartheta_1 > 0$ and $\vartheta_2 < 0$ in our regression model. Table 1 reports our baseline regression results. The dependent variable in column (1) is the average annual growth rate of real GDP, capturing output growth. The dependent variable in column (2) is the average annual growth rate of real GDP per capita, capturing income growth. The dependent variable in column (3) is the average annual growth rate of real GDP per worker, capturing labor productivity growth. In all columns, the coefficients on productive government spending are significantly positive, whereas the coefficients on the quadratic term are significantly negative. Therefore, the empirical relationship between productive government spending and economic growth follows an inverted-U pattern.

Table 1: Effects of productive government spending on economic growth

	GDP growth	per capita GDP growth	per worker GDP growth
	(1)	(2)	(3)
γ_{it}	1.748***	1.547***	1.284**
	(0.610)	(0.482)	(0.595)
γ_{it}^2	-5.599***	-5.035***	-3.773**
	(1.744)	(1.432)	(1.642)
y_{it-1}	-0.082***	-0.085***	-0.071***
	(0.013)	(0.013)	(0.010)
Country fixed effects	Yes	Yes	Yes
Period fixed effects	Yes	Yes	Yes
Observations	189	189	189
R-squared	0.706	0.687	0.717

Note: *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. Robust standard errors are in parentheses. The dependent variable in column (1) is the average annual growth rate of real GDP. The dependent variable in column

(2) is the average annual growth rate of real GDP per capita. The dependent variable in column (3) is the average annual growth rate of real GDP per worker.

In Table 1, we do not control for other explanatory variables. To mitigate omitted variable bias, we introduce a vector of control variables $\varkappa_{it}$ to our regression model as follows:

$$g_{it} = \vartheta_1 \gamma_{it} + \vartheta_2 \gamma_{it}^2 + \vartheta_3 y_{it-1} + \Gamma \varkappa_{it} + \varphi_i + \varrho_t + \epsilon_{it}.$$

Specifically, we control for the log value of population size, the log value of capital stock, and the degree of trade openness (measured by the average ratio of export plus import to GDP).[8] Population size captures the scale effect, whereas capital stock captures the effect of capital accumulation. Trade openness captures the effect of international trade. Table 2 reports the regression results. As before, the dependent variables in the three columns are the average annual growth rates of real GDP, real GDP per capita, and real GDP per worker, respectively. In all columns, the coefficients on productive government spending are significantly positive, whereas the coefficients on the quadratic term are significantly negative. Therefore, the inverted-U relationship between productive government spending and economic growth remains robust to controlling for additional explanatory variables $\varkappa_{it}$.

Table 2: Effects of productive government spending on economic growth (with controls)

	GDP growth	per capita GDP growth	per worker GDP growth
	(1)	(2)	(3)
γ_{it}	1.435**	1.263**	1.159*
	(0.604)	(0.496)	(0.595)
γ_{it}^2	-4.761***	-4.293***	-3.464**
	(1.758)	(1.504)	(1.663)
y_{it-1}	-0.106***	-0.113***	-0.090***
	(0.016)	(0.016)	(0.015)
Control variables	Yes	Yes	Yes
Country fixed effects	Yes	Yes	Yes
Period fixed effects	Yes	Yes	Yes
Observations	189	189	189
R-squared	0.754	0.744	0.746

Note: *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$. Robust standard errors are in parentheses. The dependent variable in column (1) is the average annual growth rate of real GDP. The dependent variable in column (2) is the average annual growth rate of real GDP per capita. The dependent variable in column (3) is the average annual growth rate of real GDP per worker. The additional control variables are the log value of population size, the log value of capital stock, and the degree of trade openness.

[8] We consider the average value within each period.

3 A Schumpeterian model with productive government spending

We introduce government spending modeled as in Barro (1990) in the Schumpeterian growth model with endogenous takeoff developed in Peretto (2015). The model features both the entry of new products (horizontal innovation) and the quality improvement of existing products (vertical innovation). We characterize the entire transition path, from pre-industrial stagnation to endogenous innovation-driven growth. The model also features the property of robust endogenous growth first discussed in Peretto (2018) because the elasticity of output with respect to the quality of products in the aggregate production function does not need to be one to generate endogenous growth.

3.1 Household

There is a representative household with lifetime utility function

$$U = \int_0^\infty e^{-(\rho-\lambda)t} \left[\ln c_t + \eta \ln (1 - l_t)\right] dt, \tag{1}$$

where the parameter $\rho > 0$ is the subjective discount rate and the parameter $\lambda \in (0, \rho)$ is the growth rate of the mass of identical household members (population), $L_t = L_0 e^{\lambda t}$, with $L_0 = 1$. The variable c_t denotes consumption per capita of a final good (our numeraire good). Accordingly, aggregate consumption is $C_t \equiv c_t L_t$. Finally, the variable $l_t \in (0, 1]$ is the fraction of time that each member of the household allocates to work and the parameter $\eta \geq 0$ determines the importance of leisure relative to consumption.

The household maximizes utility subject to the asset-accumulation equation

$$\dot{a}_t = (r_t - \lambda) a_t + (1 - \tau_t) w_t l_t - c_t, \tag{2}$$

where a_t is the real value of assets held by each member of the household and the aggregate value of assets is $A_t \equiv a_t L_t$. r_t is the real interest rate and w_t is the real wage rate. The government levies a tax rate τ_t on wage income. Dynamic optimization yields the consumption growth rate given by

$$\frac{\dot{c}_t}{c_t} = r_t - \rho \tag{3}$$

and the labor supply of each individual given by

$$l_t = 1 - \frac{\eta c_t}{(1 - \tau_t) w_t}. \tag{4}$$

3.2 Final good

The final good is produced by competitive firms with the production function

$$Y_t = \int_0^{N_t} X_t^\theta (i) \left[Z_t^\alpha (i) \left(\frac{G_t}{L_t} \right)^\kappa \frac{L_{y,t}}{N_t^{1-\sigma}} \right]^{1-\theta} di, \tag{5}$$

where $\{\theta, \sigma\} \in (0,1)$, $\alpha > 0$ and $\kappa \geq 0$. We interpret this Cobb-Douglas production structure as featuring constant returns to scale (CRS) with respect to two rival inputs: intermediate goods and augmented labor. It then follows directly that $\theta \in (0,1)$. For the purposes of this study, the core property of this structure is that it features two augmentation terms: the quality of intermediate goods and the provision of government services. Moreover, it features two forms of congestion due to the rival nature of the physical inputs intermediate goods and labor.

Specifically, $X_t(i)$ is the quantity of non-durable intermediate good $i \in [0, N_t]$, where N_t is the variety of intermediate goods available at time t. We allow for partial congestion of labor across intermediate goods (one intermediate good cannot be used by all the workers) and partial congestion of intermediate goods across units of labor (one worker cannot use all of the intermediate goods) and measure this effect with the parameter $\sigma > 0$.

Next, the contribution of intermediate good i to the productivity of labor, the model's first augmentation term, depends on its own quality $Z_t(i)$ with parameter $\alpha > 0$. This representation defines quality as the contribution of an intermediate good to increasing the flow of labor services obtained by the good's user (the final producer) from each unit of labor supplied by the household. The second augmentation term is similar: it is the contribution of government services G_t to increasing the flow of labor services obtained by the final producer from each unit of labor supplied by the household. We capture the magnitude of this channel with the parameter $\kappa \geq 0$. We also make public services subject to congestion on a per capita basis as in, among many others, Peretto (2007b).[9]

This production structure delivers endogenous growth even with $\kappa = 0$, a restriction that reduces the model to a variant of the one in Peretto (2015), further elaborated in Peretto (2018), which does not consider productive government spending. In this study, we introduce G with $\kappa > 0$ to investigate the interaction between productive government spending and endogenous innovation-driven growth. Crucially, we do not impose a priori restrictions designed to produce steady-state constant endogenous growth on the parameters $\{\sigma, \alpha, \kappa\}$, but we will derive such restrictions later on from the equilibrium of the model under the criterion that the model delivers a transition from initial stagnation to the takeoff and then to the steady state with endogenous growth driven by the accumulation of knowledge by firms. The analysis will reveal that the model requires an upper bound on σ, reflecting market share dilution in equilibrium, and allows for $\alpha + \kappa$ being less than, equal to or greater than one because our model generates endogenous growth regardless of whether the aggregate production function in equilibrium is concave, linear or convex in the average level of quality Z_t, which is the model's key accumulated factor.

To close this subsection, we now characterize the behavior of the representative final producer. Profit maximization yields the conditional demand for labor,

$$L_{y,t} = (1-\theta)\,\frac{Y_t}{w_t}, \tag{6}$$

[9] We could specify this form of congestion as partial by raising L_t to some additional parameter ranging from zero to one. The results would not change in any interesting way. Note also that our Cobb-Douglas structure nests congestion of public services across intermediate good in the parameter σ already characterized.

and the conditional demand for each intermediate good,

$$X_t(i) = \left[\frac{\theta}{P_t(i)}\right]^{1/(1-\theta)} Z_t^{\alpha}(i) \left(\frac{G_t}{L_t}\right)^{\kappa} \frac{L_{y,t}}{N_t^{1-\sigma}}, \tag{7}$$

where $P_t(i)$ is the price of good i. Accordingly, the final producer pays $(1-\theta) Y_t = w_t L_{y,t}$ for labor and $\theta Y_t = \int_0^{N_t} P_t(i) X_t(i) \, di$ for intermediate goods.

3.3 Intermediate goods and in-house R&D

A monopolistic firm produces differentiated intermediate good i with a linear technology that uses $X_t(i)$ units of final good to produce $X_t(i)$ units of intermediate good i at quality $Z_t(i)$. This implies that the marginal cost of production is one. The firm also pays $\phi Z_t^{\alpha}(i) Z_t^{1-\alpha}$ units of final good as a fixed operating cost, where $Z_t \equiv \int_0^{N_t} Z_t(j) \, dj / N_t$ is the average quality of all intermediate goods.[10] Finally, to improve the quality of its product, the monopolistic firm devotes $I_t(i)$ units of final good to in-house R&D with the innovation technology

$$\dot{Z}_t(i) = I_t(i). \tag{8}$$

With this structure, the monopolist's before-R&D profit flow is

$$\Pi_t(i) = [P_t(i) - 1] X_t(i) - \phi Z_t^{\alpha}(i) Z_t^{1-\alpha} \tag{9}$$

and the value of the monopolistic firm is

$$V_t(i) = \int_t^{\infty} \exp\left(-\int_t^s r_u du\right) [\Pi_s(i) - I_s(i)] \, ds. \tag{10}$$

The monopolistic firm maximizes (10) subject to (7) and (8).

It is important to note here that this firm-level problem is well defined if and only if it features concavity with respect to the firm-specific state variable $Z_t(i)$. This is the case when $0 < \alpha < 1$. This is the first restriction implied by the model's structure for our core parameters. It is worth stressing that it is a restriction that has nothing to do with the ability of the model to generate endogenous growth but it stems from the model's deeper micro structure, specifically, the requirement that the investment problem of the typical intermediate firm be well-defined.

Dynamic optimization of the monopolistic firm yields the unconstrained profit-maximizing markup ratio $1/\theta$ (see Appendix A). However, we follow Chu *et al.* (2020) to allow for diffusion of knowledge from monopolistic firms to competitive fringe firms, which can produce $X_t(i)$ with the same quality $Z_t(i)$ but at the higher marginal cost $\mu > 1$. To price these fringe firms out of the market, the monopolistic firm sets

$$P_t(i) = \min\{\mu, 1/\theta\} = \mu, \tag{11}$$

[10] Our results are robust to a more general specification for the fixed operating cost: $\phi Z_t^{\chi}(i) Z_t^{1-\chi}$, where $\chi \in (0,1)$.

where we assume $\mu < 1/\theta$. The firm's optimization problem also yields the rate of return to in-house R&D

$$r_t^q(i) = \alpha \left\{ (\mu - 1) \frac{X_t(i)}{Z_t(i)} - \phi \left[\frac{Z_t}{Z_t(i)} \right]^{1-\alpha} \right\},$$

which is increasing in quality-adjusted firm size $X_t(i)/Z_t(i)$.

3.4 Entrants

A new firm pays βX_t units of final good (where $\beta > 0$ is an entry-cost parameter) to develop a new differentiated good with average quality Z_t and start serving the market.[11] Once in the market, the new firm behaves like the incumbent firm characterized above. Therefore, at any point in time the value of all firms — incumbents and entrants — is governed by the asset-pricing equation

$$r_t = \frac{\Pi_t - I_t}{V_t} + \frac{\dot{V}_t}{V_t}. \tag{12}$$

When entry is positive, the free-entry condition

$$V_t = \beta X_t \tag{13}$$

holds. Substituting (7), (8), (9), (11) and (13) into (12) yields the return to entry as

$$r_t^e = \frac{1}{\beta} \left(\mu - 1 - \frac{\phi + z_t}{X_t/Z_t} \right) + \frac{\dot{X}_t}{X_t},$$

where $z_t \equiv \dot{Z}_t/Z_t$ is the growth rate of average quality. The return to entry r_t^e is also increasing in quality-adjusted firm size X_t/Z_t.

3.5 Government

The government balances its fiscal budget at each point in time and finances its spending with the revenues from a flat rate tax on labor income. Therefore, the government's budget constraint is

$$G_t = \tau_t w_t L_{y,t}. \tag{14}$$

Following Barro (1990), we focus on spending as the key policy variable and thus assume that the government sets

$$G_t = \gamma Y_t, \tag{15}$$

where $\gamma \in (0, 1 - \theta)$ is the fiscal policy instrument. Consequently, in our analysis we take γ as exogenous and τ_t as the endogenous tax rate that balances the fiscal budget (14).

[11] This characterization of entry preserves the symmetry of the intermediate goods market equilibrium at all times.

3.6 Equilibrium

The equilibrium of this economy is a time path of allocations $\{A_t, C_t, l_t, Y_t, X_t, I_t, G_t\}$, prices $\{r_t, w_t, P_t(i), V_t\}$ and labor income tax rate τ_t such that:

- the household chooses consumption c_t and labor supply l_t to maximize utility taking $\{r_t, w_t, \tau_t\}$ as given;
- competitive firms produce Y_t to maximize profits taking $\{w_t, P_t(i)\}$ as given;
- monopolistic intermediate-good firms choose $\{P_t(i), I_t\}$ to maximize V_t taking r_t as given;
- entrants make entry decisions taking the maximized value V_t as given;
- the aggregate value of monopolistic firms equals the household's wealth, $a_t L_t = N_t V_t$;
- the government balances the fiscal budget, $G_t = \tau_t w_t L_{y,t}$;
- the labor market clears, $L_{y,t} = l_t L_t$;
- the market for the final good clears, $Y_t = C_t + G_t + N_t \left(X_t + \phi Z_t + I_t\right) + \dot{N}_t \beta X_t$.

3.7 Aggregation

Under the conditions discussed in Peretto (2015), the equilibrium of this model is symmetric: intermediate firms charge the same price, produce the same quantity and grow at the same rate. In such equilibrium, (7), (11) and (15) yield the reduced-form aggregate production function

$$Y_t = \left[\left(\frac{\theta}{\mu}\right)^{\theta/(1-\theta)} \gamma^{\kappa} l_t N_t^{\sigma} Z_t^{\alpha}\right]^{1/(1-\kappa)} L_t, \tag{16}$$

where the elasticity of output with respect to product variety, N_t, is $\sigma/(1-\kappa)$ and the elasticity of output with respect to (average) product quality, Z_t, is $\alpha/(1-\kappa)$. For this to be a sensible representation of production, we must impose $\kappa < 1$ so that these two elasticities are finite and positive.

Now, we stated in the introduction that our model generates endogenous growth even when $\kappa = 0$, in which case the two elasticities are σ and $\alpha < 1$. An important property of this class of models is that product variety expansion is not an engine of endogenous growth because of the fixed operating cost borne by firms. Therefore, whether the model produces endogenous growth or not depends only on the elasticity α. Since $\alpha < 1$, the conventional wisdom in the literature says that this model cannot generate endogenous growth. Peretto (2018) has challenged such conventional wisdom in a simpler variant of this model. In this study, we generalize the challenge by nesting the basic model in a structure that (a) allows for the additional labor augmentation channel via government spending in the spirit of

Barro (1990) and (b) removes the knowledge spillover term that early versions of the theory posited.[12]

For future use, we close this discussion with the derivation the growth rate of final output per capita, $y_t = Y_t/L_t$. According to the production function (16), the growth rate is

$$g_t \equiv \frac{\dot{y}_t}{y_t} = \frac{1}{1-\kappa}\left(\sigma n_t + \alpha z_t + \frac{\dot{l}_t}{l_t}\right) \tag{17}$$

and consists of three components: the variety growth rate $n_t \equiv \dot{N}_t/N_t$; the quality growth rate z_t; and the growth rate of individual labor supply $\dot{l}_t/l_t$.

4 Productive government spending and takeoff

In this section, we first solve for the entire path of the economy from stagnation to steady-state growth. The economy experiences four stages of economic growth governed by the evolution of firm size. It begins in a pre-industrial era in which the growth rate of final output per capita is zero. It then enters the industrial era, which consists of two phases. In the first phase, the entry of new firms that bring to market new products drives the growth rate of output per capita. In the second phase, the improvement of the quality of existing products by existing firms adds its contribution to economic growth and produces an acceleration of the growth rate.[13] The economy finally converges to a balanced growth path that features constant growth in output per capita fueled by both vertical and horizontal innovation.

Next, we show that productive government spending shapes this process of phase transitions and convergence, by determining the timing of the first phase transition — the endogenous takeoff of the economy — and the timing of the second phase transition — the activation of vertical innovation, which further accelerates economic growth. Importantly, we find that due to the model's scale invariance, productive government spending does not affect the steady-state growth rate.

4.1 Dynamics

Our characterization of government spending in (15) and the reduced-form production function (16) yield that per capita public spending is

$$\frac{G_t}{L_t} = \left[\left(\frac{\theta}{\mu}\right)^{\theta/(1-\theta)} \gamma l_t N_t^{\sigma} Z_t^{\alpha}\right]^{1/(1-\kappa)}. \tag{18}$$

[12] Specifically, those models specify the augmentation term for quality as $Z_t^{\alpha}(i)\, Z_t^{1-\alpha}$, which in symmetric equilibrium becomes Z_t. This specification accepts the conventional wisdom that endogenous growth requires production Y_t to be linear in Z_t. Here, we reject such conventional wisdom and remove the second term from this expression, obtaining nevertheless endogenous growth even under $\kappa = 0$.

[13] We consider the realistic case in which the activation of variety innovation happens before the activation of quality innovation. See Peretto (2015) for a comprehensive discussion of this property of the baseline growth model.

Using (7), (11) and (18), we express quality-adjusted firm size as

$$\frac{X_t(i)}{Z_t(i)} = \frac{X_t}{Z_t} = \left[\left(\frac{\theta}{\mu}\right)^{1/(1-\theta)-\kappa} \gamma^\kappa l_t\right]^{\frac{1}{1-\kappa}} \frac{Z_t^{\frac{\alpha}{1-\kappa}-1} L_t}{N_t^{1-\frac{\sigma}{1-\kappa}}}.$$

We then define the composite variable

$$x_t \equiv \frac{1}{(\gamma^\kappa l_t)^{\frac{1}{1-\kappa}}} \frac{X_t}{Z_t} = \left(\frac{\theta}{\mu}\right)^{\frac{1-\kappa(1-\theta)}{(1-\kappa)(1-\theta)}} \frac{Z_t^{\frac{\alpha}{1-\kappa}-1} L_t}{N_t^{1-\frac{\sigma}{1-\kappa}}}. \tag{19}$$

This variable compresses the three state variables L_t (population), Z_t (average quality) and N_t (mass of products/firms) into the ratio $Z_t^{\alpha/(1-\kappa)-1} L_t / N_t^{1-\sigma/(1-\kappa)}$ and, therefore, makes the analysis of the model's dynamics simple. Moreover, this expression shows that for the model to exhibit the sensible property that in equilibrium firm size is decreasing in the mass of firms, we must assume $1 > \sigma/(1-\kappa)$. This is the second restrictions on the parameters that we impose not to obtain endogenous growth but to ensure that the model's microstructure produces realistic properties in equilibrium.

To see in more detail the previous point, we use equation (19) to write the rate of return to quality improvement as

$$r_t^q = \alpha \left[(\mu - 1)(\gamma^\kappa l_t)^{\frac{1}{1-\kappa}} x_t - \phi\right] \tag{20}$$

and the rate of return to entry as

$$r_t^e = \frac{1}{\beta}\left[\mu - 1 - \frac{\phi + z_t}{(\gamma^\kappa l_t)^{\frac{1}{1-\kappa}} x_t}\right] + \frac{\dot{x}_t}{x_t} + z_t + \frac{1}{1-\kappa}\frac{\dot{l}_t}{l_t}. \tag{21}$$

Both rates of return are increasing in quality-adjusted firm size $(\gamma^\kappa l_t)^{1/(1-\kappa)} x_t$ and are thus decreasing in the mass of firms for $1 > \sigma/(1-\kappa)$. This property captures the main force driving this class of models: as the mass of firms rises, each firm captures a smaller share of the market and experiences falling profitability and thereby a weaker incentive to innovate.

The government budget constraint, $G_t = \tau_t w_t L_{y,t}$, yields the labor income tax rate

$$\tau_t = \tau = \frac{\gamma}{1-\theta}, \tag{22}$$

which is increasing in γ. The combination of labor supply (4) and labor demand (6) yields the equilibrium fraction of time allocated to work

$$l_t = \left(1 + \frac{\eta}{1-\theta-\gamma}\frac{c_t}{y_t}\right)^{-1}. \tag{23}$$

This equation says that for a given consumption-output ratio c_t/y_t, the fraction of time allocated to work l_t is decreasing in the government spending ratio γ via the higher tax rate τ.

Finally, equation (19) yields the equilibrium law of motion of the state variable x_t,

$$\frac{\dot{x}_t}{x_t} = \lambda + \left(\frac{\alpha}{1-\kappa} - 1\right) z_t - \left(1 - \frac{\sigma}{1-\kappa}\right) n_t. \tag{24}$$

In this expression, the entry rate n_t and the quality growth rate z_t are either zero or increasing functions of quality-adjusted firm size $(\gamma^\kappa l_t)^{1/(1-\kappa)} x_t$ (as we show below). If these two functions have the required properties, the composite variable x_t converges to its constant steady-state value. Thus, the core of the analysis in this section is the characterization of these two functions as equilibrium objects.

4.2 The pre-industrial era

We follow the configuration of the pre-industrial intermediate goods sector in Chu *et al.* (2022). In the pre-industrial era, initial demand for each intermediate good is insufficient for a would-be monopolist operating the increasing-returns technology characterized in Section 3 to earn positive profit (see Appendix A for details). As a result, competitive firms produce the existing N_0 intermediate goods. They make zero profit at the limit price $P_t(i) = \mu$ and consequently have zero stock-market value. Anticipating such zero value, entrepreneurs do not pay the sunk entry cost, which implies that there is no variety innovation (no entry of products). Therefore, all technologies in the pre-industrial era exhibit constant returns to scale, and x_t grows solely due to exogenous population growth (i.e., $\dot{x}_t/x_t = \lambda$). In this pre-industrial era, the initial mass of intermediate goods N_0 is exogenous and predetermined, whereas the market structure in each product line (i.e., the number of firms and the size of each firm) is indeterminate.

The demand for intermediate goods eventually becomes sufficiently high for a would-be monopolist operating the increasing-returns technology to earn positive profit. However, although the increasing returns technology becomes viable, agents do not deploy it yet because this technology requires the sunk entry cost. In other words, only innovation allows a new firm to monopolize an existing market. Therefore, the pre-industrial era ends only when the present value of monopolistic firms becomes sufficiently high for the free-entry condition (13) to hold.

As a result of the pre-industrial market structure outlined above, in the pre-industrial era the household's financial wealth is zero and the household's consumption is $c_t = (1-\tau_t)w_t l_t = (1-\theta-\gamma)y_t$, which yields

$$\frac{c}{y} = 1 - \theta - \gamma. \tag{25}$$

Substituting this result into (23) yields

$$l = \frac{1}{1+\eta}. \tag{26}$$

This says that the equilibrium fraction of time allocated to work, l, in the pre-industrial era is stationary and independent of γ. The associated growth rate of output per capita is

$$g_t = \frac{1}{1-\kappa}\left(\sigma n_t + \alpha z_t + \frac{\dot{l}_t}{l_t}\right) = 0 \tag{27}$$

because $n_t = z_t = \dot{l}_t/l_t = 0$.

4.3 The industrial era: phase 1

Horizontal innovation (but not yet vertical innovation) starts when firm size grows sufficiently large. This event marks the beginning of the industrial era. In the first phase of this era, we have a positive variety growth rate $n_t > 0$ and a zero quality growth rate $z_t = 0$. To see this, note first that when the free-entry condition holds, the consumption-output ratio c_t/y_t and the fraction of time allocated to work l_t jump to the following steady-state values (derived in Appendix A):

$$\left(\frac{c}{y}\right)^* = 1 - \theta - \gamma + \frac{\beta\theta}{\mu}(\rho - \lambda); \tag{28}$$

$$l^* = \frac{1}{1 + \eta\left[1 + \frac{\beta\theta(\rho-\lambda)}{\mu(1-\theta-\gamma)}\right]}. \tag{29}$$

The second equation shows that the equilibrium fraction of time allocated to work in the industrial era, l^*, is decreasing in the government spending ratio γ.

In the first phase of the industrial era, the growth rate of output per capita becomes $g_t = [\sigma/(1-\kappa)]n_t$ because $z_t = 0$. Using the fact that in equilibrium $r_t^e = r_t = \rho + g_t = \rho + [\sigma/(1-\kappa)]n_t$, we derive the growth rate of product variety n_t as

$$n_t = \frac{1}{\beta}\left[\mu - 1 - \frac{\phi}{(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}} x_t}\right] + \lambda - \rho, \tag{30}$$

which is increasing in firm size $(\gamma^\kappa l^*)^{1/(1-\kappa)} x_t$ and is positive if and only if

$$x_t > \frac{\phi}{(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}}\left[\mu - 1 - \beta(\rho - \lambda)\right]} \equiv x_N, \tag{31}$$

where x_N is decreasing in $\gamma^\kappa l^*$. Given this activation threshold, we use the fact that in the pre-industrial era $\dot{x}_t/x_t = \lambda$ to compute the time of the industrial takeoff,

$$T_N = \frac{1}{\lambda}\log\left(\frac{x_N}{x_0}\right), \tag{32}$$

where x_0 is the initial value of the composite state variable x_t. This result says that given $x_t = x_0$ at time 0, it takes T_N years for the economy to reach the threshold x_N and thus experience the industrial takeoff.

With these expressions in hand, we can investigate the effects of government spending. If labor supply is perfectly inelastic (i.e., $\eta = 0$), the term $\gamma^\kappa l^*$ is monotonically increasing in the government spending ratio γ because $l^* = 1$. In this case, raising productive government spending leads to an earlier takeoff by decreasing x_N and a higher growth rate by increasing n_t. If labor supply is elastic (i.e., $\eta > 0$), the term $\gamma^\kappa l^*$ becomes an inverted-U function of the government spending ratio γ because l^* is decreasing in γ. In this case, raising productive government spending γ has a U-shaped effect on the date of the industrial takeoff, T_N, and an inverted-U effect on the transitional growth rate g_t.

To fully characterize this phase, we note that equations (24) and (30) yield that the dynamics of the economy are governed by the linear differential equation

$$\dot{x}_t = \frac{\left(1-\frac{\sigma}{1-\kappa}\right)\frac{\phi}{\beta}}{\left(\gamma^\kappa l^*\right)^{\frac{1}{1-\kappa}}} - \left[\left(1-\frac{\sigma}{1-\kappa}\right)\left(\frac{\mu-1}{\beta}-\rho\right)-\frac{\sigma\lambda}{1-\kappa}\right]x_t, \tag{33}$$

where we argued above that $1 > \sigma/\left(1-\kappa\right)$.

4.4 The industrial era: phase 2

When firm size is sufficiently large, horizontal and vertical innovation occur simultaneously. This is the second phase of the industrial era. Given active horizontal innovation, the consumption-output ratio and the fraction of time allocated to work l_t remain at the steady-state values (28)-(29). Therefore, we can use the relation $r_t^q = r_t = \rho + g_t$ to write the growth rate g_t as

$$g_t = \alpha\left[(\mu-1)(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}}x_t - \phi\right] - \rho, \tag{34}$$

which is linearly increasing in firm size $(\gamma^\kappa l^*)^{1/(1-\kappa)}x_t$. Therefore, raising productive government spending γ has an inverted-U (a positive) effect on the transitional growth rate g_t if labor supply is elastic (perfectly inelastic).

Next, we use the fact that $r_t^e = r_t = \rho + g_t = \rho + [\sigma/(1-\kappa)]n_t + [\alpha/(1-\kappa)]z_t$ to write the entry process driving the dynamics of x_t as

$$n_t = \frac{1}{\beta}\left[\mu - 1 - \frac{\phi + z_t}{(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}}x_t}\right] + \lambda - \rho. \tag{35}$$

Using (34), (35) and $g_t = [\sigma/(1-\kappa)]n_t + [\alpha/(1-\kappa)]z_t$ a little bit of algebra yields the growth rate of quality z_t as a function of the state variable x_t, namely, $z_t = z\left(x_t\right)$, where

$$z\left(x_t\right) \equiv \frac{\left[(\mu-1)(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}}x_t - \phi\right]\left[\alpha(1-\kappa) - \frac{\sigma}{\beta(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}}x_t}\right] - \rho(1-\kappa) + \sigma\left(\rho-\lambda\right)}{\alpha - \frac{\sigma}{\beta(\gamma^\kappa l^*)^{1/(1-\kappa)}x_t}}. \tag{36}$$

This expression says that quality growth is positive if and only if $x_t > x_Z$, where

$$x_Z \equiv \arg\underset{x}{\text{solve}}\left\{\frac{(\mu-1)(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}}x - \phi}{\rho(1-\kappa)-\sigma\left(\rho-\lambda\right)}\left[\alpha(1-\kappa) - \frac{\sigma}{\beta(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}}x}\right] = 1\right\} \tag{37}$$

and, as argued earlier, we work with a configuration of parameters that yields $x_Z > x_N$. Substituting (36) in (35) and rearranging terms, we write $n_t = n\left(x_t\right)$, where

$$n\left(x_t\right) \equiv \frac{[\kappa(\mu-1) - \beta(\rho-\lambda)](\gamma^\kappa l^*)^{\frac{1}{1-\kappa}}x_t + (1-\kappa)\rho/\alpha - \kappa\phi}{\beta(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}}x_t - \sigma/\alpha}, \tag{38}$$

which expresses the rate of entry as a function of the state variable x_t. Finally, using (36) and (38), we write the equilibrium law of motion

$$\frac{\dot{x}_t}{x_t} = \lambda + \left(\frac{\alpha}{1-\kappa} - 1\right)z\left(x_t\right) - \left(1 - \frac{\sigma}{1-\kappa}\right)n\left(x_t\right). \tag{39}$$

This equation is non-linear but relatively straightforward to study.

Summing up the results of the analysis of each phase, we have reduced a seemingly complex model to a representation of the equilibrium dynamics that consists of a piece-wise differential equation in the composite state variable x_t. The equation has the properties that the first two pieces are linear while the last piece is non-linear but not particularly challenging. Armed with this representation, we next discuss the conditions under which the model converges to a steady state with endogenous growth and how the process depends on government spending.

4.5 Convergence to the balanced growth path

We showed above that the state variable x_t grows exponentially in the pre-industrial era due to the exogenous growth of the population. Therefore, the economy experiences the takeoff in finite time as long as the threshold x_N is finite. This property gives us the third restriction that we impose to characterize the global dynamics of the model, namely, $\mu - 1 > \beta(\rho - \lambda)$. This restriction simply says that the fundamentals are such that the gross profit rate earned by monopolistic firms can cover the flow cost due to the decision to undertake entry. Specifically, the right hand side of the inequality is the initial sunk cost of entry, β, multiplied by the interest rate that the firm must pay at each point in time to finance that initial expenditure (think of the entrant firm taking a loan to finance β or, equivalently, issuing equity that must promise the market rate of return).

In phase 1 of the industrial era, the economy obeys the linear differential equation (33), which says that the economy crosses the threshold x_Z in finite time if (33) has the property

$$\dot{x}_t\left(x_Z\right) = \frac{\left(1 - \frac{\sigma}{1-\kappa}\right)\frac{\phi}{\beta}}{\left(\gamma^{\kappa} l^{*}\right)^{\frac{1}{1-\kappa}}} - \left[\left(1 - \frac{\sigma}{1-\kappa}\right)\left(\frac{\mu - 1}{\beta} - \rho\right) - \frac{\sigma\lambda}{1-\kappa}\right] x_Z > 0.$$

This is simply a restriction on the parameters that ensures that the process of entry does not saturate the market so much that incumbent firms cannot cross the threshold of profitability that activates in-house quality innovation. It is the analog of the condition discussed above that guarantees that the first phase transition occurs. In other words, this condition guarantees that the second phase transition occurs. On reflection, we can express these conditions in more compact terms. We rewrite the inequality above as

$$\rho\beta + \left[\frac{\left(1 - \frac{\sigma}{1-\kappa}\right)\frac{\phi}{\beta}}{\left(\gamma^{\kappa} l^{*}\right)^{\frac{1}{1-\kappa}} x_Z} + \frac{\sigma\lambda}{1-\kappa}\right]\frac{\beta}{1 - \frac{\sigma}{1-\kappa}} > \mu - 1.$$

Combining this inequality with the inequality derived for the first phase transition, we obtain

$$\rho\beta + \left[\frac{\left(1 - \frac{\sigma}{1-\kappa}\right)\frac{\phi}{\beta}}{\left(\gamma^{\kappa} l^{*}\right)^{\frac{1}{1-\kappa}} x_Z} + \frac{\sigma\lambda}{1-\kappa}\right]\frac{\beta}{1 - \frac{\sigma}{1-\kappa}} > \mu - 1 > \beta(\rho - \lambda) \tag{40}$$

as the *sufficient* condition for the dynamics of the model to yield the full sequence: pre-industrial era $\rightarrow$ industrial era phase 1 $\rightarrow$ industrial era phase 2.

Perhaps intuitively, the conditions just discussed are subsumed in the conditions that we obtain by looking directly at the model's global dynamics, paying special attention to what happens in phase 2 of the industrial era. Figure 1 plots the phase diagrams for the three cases ($\alpha+\kappa=1$, $\alpha+\kappa>1$ and $\alpha+\kappa<1$) and shows that in each case x_t can converge to the steady-state value x^* that features both quality improvement ($z^*>0$) and variety expansion ($n^*>0$) in the long run. Note that under the sufficient condition (40) in all three panels the piece of the differential equation in the interval $[x_N, x_Z]$ is above the horizontal axis.[14] The conditions for the model to produce endogenous growth conventionally defined, therefore, are the conditions under which in each case the steady state x^* exists and is the global attractor of the economy's equilibrium dynamics. We now characterize each case.

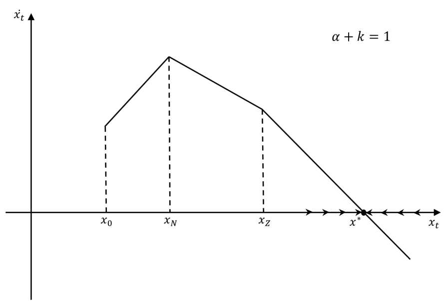

Figure 1a: $\alpha+\kappa=1$

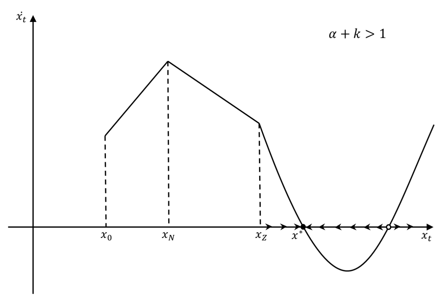

Figure 1b: $\alpha+\kappa>1$

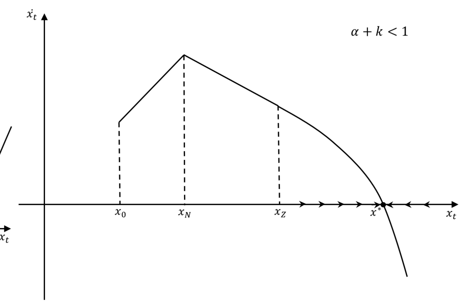

Figure 1c: $\alpha+\kappa<1$

Case 1. Consider $\alpha+\kappa=1$. In Appendix A, we show that the steady-state value of the state variable x_t is

$$x^*=\frac{1}{(\gamma^\kappa l^*)^{\frac{1}{1-\kappa}}}\frac{(\kappa\phi-\rho)\left(1-\frac{\sigma}{1-\kappa}\right)-\frac{\sigma\lambda}{\alpha}}{[\kappa(\mu-1)-\beta(\rho-\lambda)]\left(1-\frac{\sigma}{1-\kappa}\right)-\beta\lambda}. \tag{41}$$

The associated growth rate of income per capita is

$$g^*=\alpha\left[(\mu-1)\frac{(\kappa\phi-\rho)\left(1-\frac{\sigma}{1-\kappa}\right)-\frac{\sigma\lambda}{\alpha}}{[\kappa(\mu-1)-\beta(\rho-\lambda)]\left(1-\frac{\sigma}{1-\kappa}\right)-\beta\lambda}-\phi\right]-\rho, \tag{42}$$

which is fueled by the variety growth rate $n^*=(1-\kappa)\lambda/(1-\kappa-\sigma)>0$ and the quality growth rate $z^*=g^*-(\sigma/\alpha)n^*$. Note that the steady-state firm size, $(\gamma^\kappa l^*)^{1/(1-\kappa)}x^*$, the steady-state growth rate, g^*, and the steady-state rate of quality innovation, z^*, are all independent of the government spending ratio γ due to the model's scale invariance.

The conditions for endogenous growth are (i) that the values x^*, g^* and z^* exist and are positive and (ii) that the model's dynamics allow the state variable x_t to converge to the steady state x^*. Inspecting the phase diagrams shows that under condition (40) if the steady state x^* exists, then it is locally stable and, therefore, the global attractor of the full dynamical system. We provide a more formal characterization of this property in the proof of Proposition 1 below. The condition for $x^*>0$ is

$$\left(1-\frac{\sigma}{1-\kappa}\right)>\max\left\{\frac{\sigma\lambda}{\alpha\left(\kappa\phi-\rho\right)},\frac{\beta\lambda}{\kappa(\mu-1)-\beta(\rho-\lambda)}\right\}, \tag{43}$$

[14] In an online appendix (see Appendix C), we show the phase diagrams for other possibilities.

which says that both the numerator and the denominator of (41) are positive. The conditions for $g^* > 0$ and $z^* > 0$ add, respectively, the inequalities:

$$\alpha\left[(\mu-1)\frac{(\kappa\phi-\rho)\left(1-\frac{\sigma}{1-\kappa}\right)-\frac{\sigma\lambda}{\alpha}}{[\kappa(\mu-1)-\beta(\rho-\lambda)]\left(1-\frac{\sigma}{1-\kappa}\right)-\beta\lambda}-\phi\right]>\rho;$$

$$\alpha\left[(\mu-1)\frac{(\kappa\phi-\rho)\left(1-\frac{\sigma}{1-\kappa}\right)-\frac{\sigma\lambda}{\alpha}}{[\kappa(\mu-1)-\beta(\rho-\lambda)]\left(1-\frac{\sigma}{1-\kappa}\right)-\beta\lambda}-\phi\right]>\rho+\frac{\sigma\lambda}{1-\kappa-\sigma}. \tag{44}$$

Note that the latter implies the former, which we can thus ignore. Summarizing, the conditions that ensure that the economy converges to x^* under the model's equilibrium dynamics, where x^* exhibits endogenous growth, are (40), (43) and (44). These are *inequality* restrictions. Nowhere the model requires an equality restriction representing a knife-edge condition on the parameters.

To characterize the next two cases, we derive in Appendix A the steady-state firm size and the steady-state growth rate given, respectively, by:

$$(\gamma^\kappa l^*)^{1/(1-\kappa)}x^*=\frac{a_2-\sqrt{a_2^2-4a_1a_3}}{2a_1}; \tag{45}$$

$$g^*=\alpha\left[(\mu-1)\frac{a_2-\sqrt{a_2^2-4a_1a_3}}{2a_1}-\phi\right]-\rho. \tag{46}$$

To work with compact notation, we defined the coefficients:

$$a_1\equiv\alpha\beta(\mu-1)(1-\kappa)(\alpha+\kappa-1);$$

$$\begin{aligned}a_2\equiv\ &(\alpha+\kappa-1)\{[\mu-1-\beta(\rho-\lambda)]\sigma+\beta(1-\kappa)(\rho+\phi\alpha)\}\\&+\alpha(1-\kappa-\sigma)[(\mu-1)\kappa-\beta(\rho-\lambda)]-\alpha\beta\lambda(1-\kappa);\end{aligned}$$

$$a_3\equiv(\alpha+\kappa-1)\phi\sigma+(1-\kappa-\sigma)[\alpha\phi\kappa-(1-\kappa)\rho]-\sigma\lambda(1-\kappa).$$

Note that once again both the steady-state firm size, $(\gamma^\kappa l^*)^{1/(1-\kappa)}x^*$, and the steady-state growth rate, g^*, are independent of the government spending ratio γ due to the model's scale invariance. We now examine each case using these expressions. Before doing so, we note the new property that arises in these cases, namely, the equation

$$\lambda+\left(\frac{\alpha}{1-\kappa}-1\right)z^*=\left(1-\frac{\sigma}{1-\kappa}\right)n^*,$$

which says that there exists a relation between variety growth and quality growth dictated by the sign of the coefficient $\alpha+\kappa-1$. Recall that $1>\sigma/(1-\kappa)$. This new property yields that since $g^*=(\sigma n^*+\alpha z^*)/(1-\kappa)$, the breakdown of economic growth in its two components — variety and quality growth — is:

$$z^*=\frac{(1-\kappa)\,g^*-\frac{\sigma\lambda(1-\kappa)}{1-\kappa-\sigma}}{\frac{\sigma(\alpha+\kappa-1)}{1-\kappa-\sigma}+\alpha};$$

$$n^* = \frac{\lambda (1-\kappa) + (\alpha+\kappa-1)\, z^*}{1-\kappa-\sigma}.$$

The new result here is that for $\alpha + \kappa \neq 1$ the rate of entry is not pinned down by the rate of growth of the population but is jointly determined with the rate of quality growth.

Case 2. Consider $\alpha + \kappa > 1$, which yields $a_1 > 0$. Then, the condition for $x^* > 0$ is simply that x^* exists and is real, i.e., $a_2^2 > 4a_1a_3$. This is an inequality restriction on the parameters. We then add to this inequality the restrictions for $g^* > 0$ and $z^* > 0$ and the sufficient condition (40). This case exhibits the property that the steady-state entry rate, n^*, is increasing in the growth rate of quality, z^*. In other words, quality innovation is so effective that it creates room for variety growth faster than in the canonical case $\alpha+\kappa = 1$. To understand why the model generates constant growth under seemingly explosive conditions, we revisit equation (19) that defines the composite state variable x_t. That equation shows that the model's mechanics identifies the ratio

$$\ell_t \equiv L_t / N_t^{1-\frac{\sigma}{1-\kappa}}$$

as the key measure of labor input per intermediate good. The interpretation is that this ratio measures the flow of raw labor services effectively allocated to the typical intermediate good after we account for the two forms of congestion at work in the model. In the steady state, this measure of labor allocation has the following growth rate:

$$\frac{\dot{\ell}_t}{\ell_t} = \lambda - \left(1 - \frac{\sigma}{1-\kappa}\right) n^* = -\left(\frac{\alpha}{1-\kappa} - 1\right) z^* < 0.$$

The interpretation of this steady state, therefore, is that the economy exhibits constant endogenous growth because the mass of firms grows sufficiently faster than the population so that there is continuous dilution of labor services across firms. This dilution offsets the explosive pressure due to the property that production is convex in average knowledge, Z.

Case 3. Consider the case $\alpha + \kappa < 1$, which yields $a_1 < 0$. The condition for $x^* > 0$ then is $a_2 < \sqrt{a_2^2 - 4a_1a_3}$, where $a_2^2 > 4a_1a_3$ must hold for x^* to exist and be real. When $a_2 < 0$, the condition that we seek is simply $a_2^2 > 4a_1a_3$. When $a_2 \geq 0$, we obtain that x^* is positive when $a_2^2 < a_2^2 - 4a_1a_3$, which reduces to $4a_1a_3 < 0$. Since $a_1 < 0$, this holds for $a_3 > 0$. But $a_3 > 0$ and $a_1 < 0$ imply that the inequality $a_2^2 > 4a_1a_3$ always holds so that x^* surely exists and is real. Therefore, the condition that we seek is simply $a_3 > 0$ when $a_2 \geq 0$. These are again inequality restrictions on the parameters. As in the previous case, we add to these inequalities the restrictions for $g^* > 0$ and $z^* > 0$ and the sufficient condition (40). This case exhibits the property that n^* is decreasing in z^* because quality innovation is not sufficiently effective and the economy can sustain endogenous growth if and only if variety growth is slower than in the canonical case $\alpha + \kappa = 1$. As we saw in the previous case, this property has implications for the dynamics of the ratio ℓ_t. In this case, in particular, we have

$$\frac{\dot{\ell}_t}{\ell_t} = \lambda - \left(1 - \frac{\sigma}{1-\kappa}\right) n^* = -\left(\frac{\alpha}{1-\kappa} - 1\right) z^* > 0.$$

The interpretation of this steady state, therefore, is that the economy exhibits constant endogenous growth because the mass of firms grows sufficiently slower than the population so that there is continuous concentration of labor services across firms. This concentration offsets the implosive pressure due to the property that production is concave in average knowledge, Z.

4.6 Summary of results

We can summarize our result on the main global dynamics in Proposition 1.

Proposition 1 *Assume that $x_0 < x_N < x_Z$. Then, the economy begins in the pre-industrial era with no innovation of any kind. It then experiences an industrial takeoff and enters the first phase of the industrial era where horizontal innovation alone fuels industrial growth. After that, the economy enters the second phase of the industrial era with both vertical and horizontal innovation and converges to the balanced growth path.*

Proof. See Appendix A. ■

In Proposition 2, we summarize the effects of productive government spending, which depend on whether labor supply is elastic or not. If labor supply is perfectly inelastic, then the labor income tax has no distortionary effect. In this case, raising productive government spending causes an earlier industrial takeoff and increases the transitional growth rate by enlarging firm size in the short run; however, it has no effect on long-run economic growth due to the absence of the scale effect. If labor supply is elastic, then the labor income tax has a distortionary effect on employment. In this case, raising productive government spending has a U-shaped effect on the timing of industrial takeoff and an inverted-U effect on the transitional growth rate.

Proposition 2 *The effects of productive government spending are as follows. If labor supply is perfectly inelastic (i.e., $\eta = 0$), then raising productive government spending γ leads to an earlier industrial takeoff and a higher transitional growth rate g_t during the industrial era. If labor supply is elastic (i.e., $\eta > 0$), then there exists a threshold value $\tilde{\gamma} \in (0, 1-\theta)$ such that raising productive government spending γ leads to an earlier (a delayed) industrial takeoff when $\gamma < \tilde{\gamma}$ ($\gamma > \tilde{\gamma}$). During the industrial era, i.e., $x_t \in (x_N, x^*)$, an increase in γ increases (decreases) the transitional growth rate g_t when $\gamma < \tilde{\gamma}$ ($\gamma > \tilde{\gamma}$). In the long run, the government spending share γ does not affect the steady-state growth rate (regardless of whether labor supply is elastic or perfectly inelastic).*

Proof. See Appendix A. ■

It is useful to note that the threshold level $\tilde{\gamma}$ of government spending that gives rise to the earliest takeoff and the highest transition growth rate is the same. Once again, due to the endogenous market structure removing the scale effect, any changes in government spending have no effect on long-run economic growth.

4.7 Quantitative analysis

In this section, we calibrate the model to US data and perform a quantitative analysis. The model features the following 11 parameters: $\{\kappa, \lambda, \alpha, \mu, \gamma, \theta, \rho, \beta, \phi, \eta, \sigma\}$. In (5), the exponent on productive government spending G_t is κ; therefore, κ is the key parameter that determines the strength of the effects of productive government spending on the timing of industrial takeoff and the transitional growth rate. In light of its importance, we consider a wide range of values for $\kappa \in [0.10, 0.55]$.[15] Then, we set $\lambda = 1.6\%$, equal to the average growth rate of employment in 1978-2019 from the Business Dynamics Statistics (BDS). We follow Iacopetta and Peretto (2021) to set the elasticity of profit with respect to own knowledge α to 0.333. We set markup ratio μ to 1.3, which is within the range of aggregate markup ratios estimated in De Loecker *et al.* (2020). Government spending as a share γ of GDP in the US is about 0.2 on average in recent decades. The labor income share $1-\theta$ is around 60%, such that $\theta = 0.4$. We set the discount rate ρ to a conventional value of 0.03. Then, we calibrate the remaining parameters $\{\beta, \phi, \eta, \sigma\}$ by matching the following moments. The parameters $\{\beta, \phi\}$ mainly target a long-run growth rate of output per capita of 2% and R&D as a share of output of 2.7%. The relative importance of leisure η matches labor supply as a share of labor endowment of 0.3. The social return to variety σ matches a net firm entry rate of 1% (also from the BDS). We summarize the calibrated parameter values in Table 3, which shows that we have $\alpha + \kappa < 1$ for our entire range of values for κ.

Table 3. Calibrated parameter values

κ	λ	α	μ	γ	θ	ρ	β	ϕ	η	σ	$\tilde{\gamma}$
0.100	0.016	0.333	1.300	0.200	0.400	0.030	5.837	0.116	2.195	0.933	0.359
0.250	0.016	0.333	1.300	0.200	0.400	0.030	5.837	0.116	2.195	0.633	0.435
0.400	0.016	0.333	1.300	0.200	0.400	0.030	5.837	0.116	2.195	0.333	0.466
0.550	0.016	0.333	1.300	0.200	0.400	0.030	5.837	0.116	2.195	0.033	0.485

Table 3 also computes the value of productive government spending share $\tilde{\gamma}$ that maximizes $\gamma^{\kappa} l^*$ under different values of κ. It is useful to note from (31), (30) and (34) that a larger $\gamma^{\kappa} l^*$ implies a lower industrial-takeoff threshold x_N and a higher transitional growth rate g_t in the two phases of the industrial era. To decide on the value of κ, we choose a conservative value of 0.250. In this case, the growth-maximizing government spending share is $\tilde{\gamma} = 0.435$. Furthermore, under these parameter values, the growth rate of average employment L_t/N_t is $\lambda - n^* = 0.6\%$, which is in line with the annual growth rate of average employment per firm in the BDS of 0.62% from 1978 to 2019. Furthermore, given that in-house R&D share of output $N_t I_t / Y_t$ is a constant in the steady-state, the model-produced growth rate of firms' R&D spending is $g_I^* \equiv \dot{I}_t / I_t = g^* + \lambda - n^* = 2.6\%$, which is roughly in line with the empirical estimate of 2.9%.[16]

[15] As κ rises above 0.566, σ would become negative, which is empirically implausible because a negative σ implies that variety innovation contributes negatively to economic growth.

[16] From OECD statistics, the average annual growth rate of nominal R&D expenditures is 6.1% in the US, which translates to a real growth rate of 3.9% and a real growth rate of R&D per firm of 2.9% (recall that the net entry rate of firms is 1%).

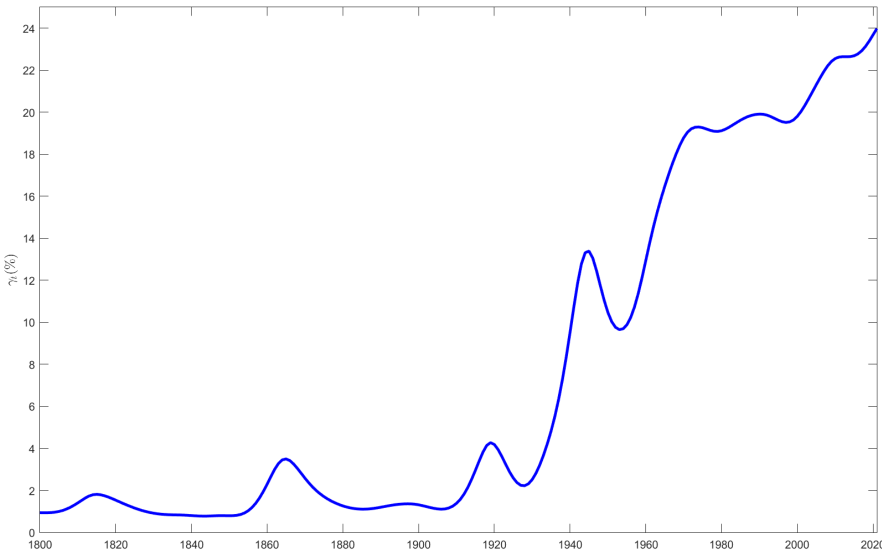

Figure 2: Trend of the US government spending share of GDP

Figure 2 presents the HP-filter trend of the historical data on the US government spending share of GDP, which rises from 1% in 1800 to 24% in 2020.[17] Then, we input this data series into our model to simulate the growth rate in Figure 3,[18] which presents the equilibrium growth rate g_t from pre-industrial stagnation to the takeoff since the early 19th century and also the trend of the empirical growth rate of real GDP per capita in the US. The correlation between the data series and the simulated series is 0.411, which is reasonably high. Furthermore, the simulated path replicates the data especially well in the 20th century. As a counterfactual exercise, Figure 3 also simulates how the equilibrium growth rate g_t would change when the government raises its spending share γ in the 18th century from the historical value of 0.01 to $\tilde{\gamma} = 0.435$ for the benchmark case of $\kappa = 0.25$. It shows that the industrial takeoff would occur earlier by over six decades and that within about half a century since the takeoff, the annual growth rate would reach 2% by the early 19th century as compared to the economy remaining at stagnation for the same period without the increase in government spending. In the long run, although the growth rate would converge to the same steady-state value of 2%, the balanced-growth level of output would increase by 129%.

[17] Data source: IMF Government Finance Statistics. We have chosen an HP-filter parameter of 100 for the annual data.

[18] Here, we model the changes in γ as unanticipated shocks (i.e., MIT shocks).

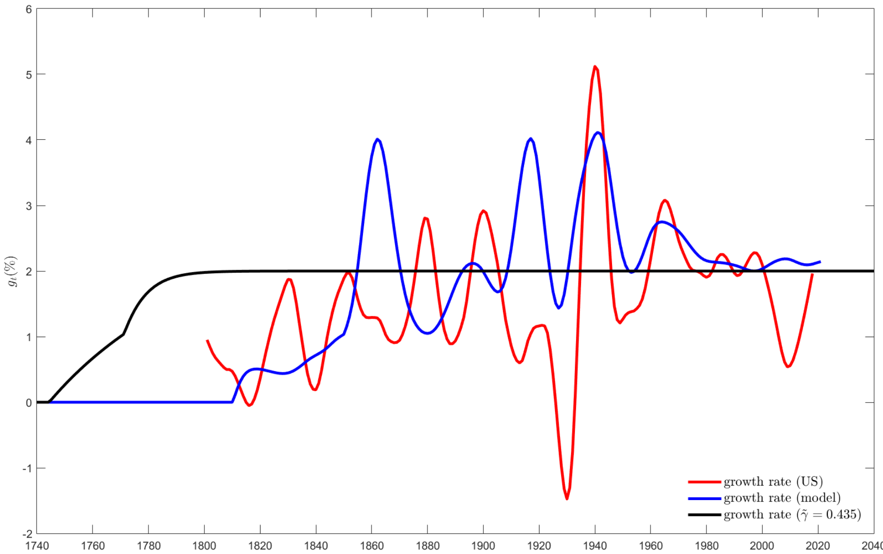

Figure 3: Effects of productive government spending ($\kappa = 0.25$)

5 Extensions

In this section, we consider two extensions of the model with different tax instruments. Section 5.1 considers a consumption tax. Section 5.2 considers a corporate income tax.

5.1 Consumption tax

This section replaces the labor income tax with a consumption tax and explores the effects of productive government spending on the dynamics of the economy from pre-industrial stagnation to innovation-driven growth. To conserve space, we do not repeat all the derivations but focus on the equations that are different from the baseline model.

The household's asset-accumulation equation becomes

$$\dot{a}_t = (r_t - \lambda)a_t + w_t l_t - (1 + \tau_{c,t})c_t,$$

where $\tau_{c,t}$ is the consumption tax rate. Individual labor supply becomes

$$l_t = 1 - \eta(1 + \tau_{c,t})\frac{c_t}{w_t}.$$

As for the fiscal budget constraint, it becomes

$$G_t = \tau_{c,t} C_t.$$

The rest of the model remains the same as before. Following the same derivations in the baseline model, we derive the equilibrium level of labor as

$$l_t = \left[1 + \frac{\eta(1+\tau_{c,t})}{1-\theta}\frac{c_t}{y_t}\right]^{-1}.$$

In the pre-industrial era, the consumption-output ratio jumps to

$$\frac{c}{y} = \frac{1-\theta}{1+\tau_c},$$

where the consumption tax rate is stationary and given by

$$\tau_c = \frac{\gamma}{1-\theta-\gamma},$$

which is increasing in the government spending ratio γ. Therefore, the equilibrium level of labor in the pre-industrial era jumps to

$$l = \frac{1}{1+\eta}.$$

In the industrial era, the consumption-output ratio jumps to

$$\left(\frac{c}{y}\right)^* = \frac{1}{1+\tau_c^*}\left[1-\theta+\frac{\beta\theta}{\mu}(\rho-\lambda)\right],$$

where the consumption tax rate is stationary and given by

$$\tau_c^* = \frac{\gamma}{1-\theta-\gamma+\beta\theta(\rho-\lambda)/\mu},$$

which is increasing in the government spending ratio γ. Therefore, the equilibrium level of labor in the industrial era jumps to

$$l^* = \frac{1}{1+\eta\left[1+\frac{\beta\theta(\rho-\lambda)}{\mu(1-\theta)}\right]},$$

which is independent of the government spending ratio γ because $(1+\tau_c)c/y$ is independent of τ_c. Therefore, raising productive government spending γ continues to have a positive effect on firm size $(\gamma^\kappa l^*)^{1/(1-\kappa)}x_t$ via the term γ^κ, but the negative effect of γ via equilibrium labor l^* in the baseline model disappears because consumption tax does not affect employment l^*.

The dynamics of the economy are captured by (30) to (46) and (A9) to (A11) in Appendix A as in the baseline model. We can now summarize the global dynamics under the consumption tax as follows.

Proposition 3 *When government spending is financed by a consumption tax $\tau_{c,t}$, the effects of productive government spending γ are as follows. In the pre-industrial era, i.e., $x_t \in (x_0, x_N)$, an increase in γ leads to an earlier industrial takeoff. In the industrial era, i.e., $x_t \in (x_N, x^*)$, an increase in γ raises the transitional growth rate g_t but does not affect the steady-state growth rate g^*.*

Proof. From (31), one can show that a larger γ reduces x_N. From (30) and (34), one can show that a larger γ increases g_t in both the first phase and second phase of the industrial era, i.e., $x_t \in (x_N, x^*)$. Then, (46) shows that g^* is independent of γ. ■

5.2 Corporate income tax

This section replaces the consumption tax with a corporate income tax. For simplicity and tractability, we consider the special case with $\kappa = 1 - \alpha$. Before the industrial takeoff, firms make zero corporate income, i.e., $\Pi_t = 0$.[19] Therefore, we keep the labor income tax as in the baseline model to ensure that government spending G_t is positive before the industrial takeoff. Again, we do not repeat all the derivations but focus on the equations that are different from the baseline economy.

Given the corporate income tax, the value of monopolistic firm i becomes

$$V_t(i) = \int_t^\infty \exp\left(-\int_t^s r_u du\right)(1 - \tau_{\Pi,s})\left[\Pi_s(i) - I_s(i)\right] ds.$$

The government levies the corporate income tax $\tau_{\Pi,t}$ on the firm's cash flow net of R&D expenditure, which implies that R&D is fully expensible. Then, the asset-pricing equation becomes

$$r_t = \frac{(1 - \tau_{\Pi,t})(\Pi_t - I_t)}{V_t} + \frac{\dot{V}_t}{V_t}.$$

The fiscal budget constraint becomes

$$G_t = \tau w_t L_{y,t} + \tau_{\Pi,t} N_t (\Pi_t - I_t).$$

In the pre-industrial era, it must be the case that $G_t = \tau w_t L_{y,t}$ because $\tau_{\Pi,t} N_t(\Pi_t - I_t) = 0$. In the industrial era, the endogenous corporate income tax is given by

$$\tau_{\Pi,t} = \frac{\gamma - \tau(1-\theta)}{N_t(\Pi_t - I_t)/Y_t},$$

where we take the government spending ratio γ and the labor-income tax rate τ as exogenous. The rate of return to entry becomes

$$r_t^e = \frac{1 - \tau_{\Pi,t}}{\beta}\left[\mu - 1 - \frac{\phi + z_t}{(\gamma^{1-\alpha} l^*)^{1/\alpha} x_t}\right] + \frac{\dot{x}_t}{x_t} + z_t + \frac{1}{\alpha}\frac{\dot{l}_t}{l_t}.$$

In both the pre-industrial era and the industrial era, the equilibrium level of labor is given by

$$l_t = \left[1 + \frac{\eta}{(1-\theta)(1-\tau)}\frac{c_t}{y_t}\right]^{-1}.$$

In the pre-industrial era, the consumption-output ratio is given by

$$\frac{c}{y} = (1-\tau)(1-\theta),$$

and the equilibrium level of labor is given by

$$l = \frac{1}{1+\eta}.$$

[19] Extending the baseline model to allow for positive monopolistic profits in the pre-industrial era complicates the pre-industrial dynamics but does not change the main results of the paper. Derivations are available upon request.

When the economy enters the industrial era, both the consumption-output ratio c_t/y_t and the equilibrium level of labor l_t jump to their steady-state values:

$$\left(\frac{c}{y}\right)^* = (1-\theta)(1-\tau) + \frac{\beta\theta}{\mu}(\rho - \lambda),$$

$$l^* = \frac{1}{1+\eta\left[1+\frac{\beta\theta(\rho-\lambda)}{\mu(1-\theta)(1-\tau)}\right]},$$

which are independent of $\tau_{\Pi,t}$. In the industrial era, the endogenous corporate income tax can be expressed as (see the derivation in Appendix A)

$$\tau_{\Pi,t} = \frac{\mu[\gamma-(1-\theta)\tau]}{\beta\theta(n_t+\rho-\lambda)+\mu[\gamma-(1-\theta)\tau]},$$

which is decreasing in n_t and increasing in the government spending ratio γ for a given n_t.

In the first phase of the industrial era, the growth rate of output is $g_t = (\sigma/\alpha)n_t$ and the variety growth rate n_t can be expressed as

$$n_t = \frac{1}{\beta}\left\{\mu - 1 - \frac{\phi}{(\gamma^{1-\alpha}l^*)^{1/\alpha}x_t} - \frac{\mu}{\theta}[\gamma-(1-\theta)\tau]\right\} + \lambda - \rho,$$

which is increasing in firm size $(\gamma^{1-\alpha}l^*)^{1/\alpha}x_t$ as before and is positive if only if

$$x_t > \frac{\phi}{(\gamma^{1-\alpha}l^*)^{1/\alpha}\{\mu-1-\beta(\rho-\lambda)-[\gamma-(1-\theta)\tau]\mu/\theta\}} \equiv x_N.$$

In Appendix A, we show that if the labor income tax rate τ is sufficiently small, then the industrial threshold x_N is a U-shaped function in productive government spending γ, which lowers the industrial threshold x_N (when γ is small) by raising firm size via $\gamma^{1-\alpha}$ but increases the industrial threshold x_N (when γ is large) by discouraging firm entry via corporate income tax $\tau_{\Pi,t}$. Therefore, a small increase in productive government spending γ can reduce the industrial threshold x_N and trigger an immediate industrialization of the economy when $x_N \leq x_0$.

In the second phase of the industrial era, the growth rate of output becomes $g_t = (\sigma/\alpha)n_t + z_t$ and converges to the steady state. On the balanced growth path, the steady-state variety growth rate is given by $n^* = \alpha\lambda/(\alpha-\sigma)$, and the steady-state corporate income tax rate is given by

$$\tau_\Pi^* = \frac{\mu[\gamma-(1-\theta)\tau]}{\beta\theta[\rho+\sigma\lambda/(\alpha-\sigma)]+\mu[\gamma-(1-\theta)\tau]},$$

which is increasing in the government spending ratio γ. Finally, the steady-state per capita output growth rate is given by

$$\begin{aligned} g^* &= \frac{\alpha[\beta\phi-(1-\tau_\Pi^*)(\mu-1)]\,[\rho+\sigma\lambda/(\alpha-\sigma)]}{(1-\tau_\Pi^*)(1-\alpha)(\mu-1)-\beta\,[\rho+\sigma\lambda/(\alpha-\sigma)]} - \rho \\ &= \frac{\alpha\{[\beta\phi-(\mu-1)][\rho+\sigma\lambda/(\alpha-\sigma)]+[\gamma-(1-\theta)\tau]\mu\phi/\theta\}}{(1-\alpha)(\mu-1)-\beta[\rho+\sigma\lambda/(\alpha-\sigma)]-[\gamma-(1-\theta)\tau]\mu/\theta} - \rho \end{aligned}$$

which is increasing in τ_{Π}^{*} and, hence, increasing in γ. Intuitively, corporate income tax raises steady-state growth by discouraging firm entry and enlarging firm size in the long run.

We can now summarize the global dynamics under the corporate-income tax in Proposition 4, which focuses on industrial takeoff and long-run growth for simplicity.

Proposition 4 *When government spending is financed by a corporate income tax $\tau_{\Pi,t}$, the effects of productive government spending γ are as follows. In the pre-industrial era, an increase in γ leads to an earlier (a delayed) industrial takeoff if γ is below (above) a threshold. In the industrial era, an increase in γ raises the steady-state growth rate g^{*}.*

Proof. See Appendix A. ■

6 Conclusion

In this study, we developed a Schumpeterian growth model with productive government spending and endogenous takeoff. We took the seminal study by Barro (1990) as our point of departure and focused on the government's provision of productive public services to the private sector. We then took the analysis in three new directions. First, we investigated the role of government spending in a scale-invariant Schumpeterian model of endogenous innovation. Second, we investigated the role of public spending as the catalyst of the takeoff of the economy. Third, we postulated a production structure that violates the condition for endogenous growth stressed in Barro (1990), namely, that the economy's reduced-form production function must be linear in the accumulated factor. Consequently, our results expand significantly our understanding of the conditions under which general equilibrium models produce constant exponential growth in steady state, in particular innovation-driven growth that starts at a specific date, accelerates throughout the secular transition, and in steady state is scale invariant and subject to policy action.

Our main results can be summarized as follows. When public spending is financed with a distortionary labor-income tax, there is a value of productive government spending that leads to the earliest industrial takeoff and also maximizes the transitional growth rate. This theoretical prediction of an inverted-U effect of productive government spending on economic growth is consistent with the stylized facts reported in the literature that we revisited here using cross-country panel data. Calibrating the model to US data, we found that raising productive government spending to its growth-maximizing value causes the industrial takeoff to occur by over six decades earlier and the long-run level of output to increase by 129%, confirming the importance of productive public spending for industrialization and innovation. In policy perspective, building on this example, our growth-theoretic framework that can be fruitfully applied to explore the United Nations Sustainable Development Goals, which specifically mention building resilient infrastructure, promoting sustainable industrialization and fostering innovation as necessary ingredients to success.

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Appendix A: Proofs

Dynamic optimization of the monopolistic firm. The current-value Hamiltonian for monopolistic firm i is

$$H_t(i) = \Pi_t(i) - I_t(i) + \zeta_t(i)\,\dot{Z}_t(i) + \xi_t(i)\left[\mu - P_t(i)\right], \tag{A1}$$

where $\zeta_t(i)$ is the co-state variable on $\dot{Z}_t(i)$ and $\xi_t(i)$ is the multiplier on $P_t(i) \leq \mu$. We substitute (7), (8) and (9) into (A1) and derive

$$\frac{\partial H_t(i)}{\partial P_t(i)} = 0 \Rightarrow \frac{\partial \Pi_t(i)}{\partial P_t(i)} = \xi_t(i), \tag{A2}$$

$$\frac{\partial H_t(i)}{\partial I_t(i)} = 0 \Rightarrow \zeta_t(i) = 1, \tag{A3}$$

$$\begin{aligned}\frac{\partial H_t(i)}{\partial Z_t(i)} &= \alpha\left\{\left[P_t(i) - 1\right]\left[\frac{\theta}{P_t(i)}\right]^{1/(1-\theta)} Z_t^{\alpha-1}(i)\left(\frac{G_t}{L_t}\right)^{\kappa}\frac{L_{y,t}}{N_t^{1-\sigma}} - \phi\left(\frac{Z_t}{Z_t(i)}\right)^{1-\alpha}\right\} \\ &= r_t\zeta_t(i) - \dot{\zeta}_t(i).\end{aligned} \tag{A4}$$

When $P_t(i) < \mu$, $\xi_t(i) = 0$, which implies $\partial\Pi_t(i)/\partial P_t(i) = 0$ such that $P_t(i) = 1/\theta$. When the constraint is binding, i.e., $P_t(i) = \mu$, $\xi_t(i) > 0$. Thus, we have proven (11). The assumption $\mu < 1/\theta$ yields $P_t(i) = \mu$. Using (A3), (19), $P_t(i) = \mu$ and symmetry in (A4) yields (20). ■

Monopolistic profit in the pre-industrial era. In the pre-industrial era, firm size is not large enough for monopolistic firms with increasing-returns technology to earn positive profit, i.e.,

$$(\gamma^{\kappa} l)^{1/(1-\kappa)} x_t < \phi/(\mu - 1) \Leftrightarrow \Pi_t < 0,$$

where l is given in (26). As a result, competitive firms produce existing N_0 intermediate goods and make zero profit. When $(\gamma^{\kappa} l)^{1/(1-\kappa)} x_t \geq \phi/(\mu - 1)$, we assume that agents do not deploy increasing-returns technology until $x_t \geq x_N$. ■

Dynamics of the consumption-output ratio in the industrial era. The value of assets owned by each member of the household is

$$a_t = V_t N_t / L_t. \tag{A5}$$

If $n_t > 0$, then $V_t = \beta X_t$ in (13) holds. Substituting (13) and $\mu X_t N_t = \theta Y_t$ into (A5) yields

$$a_t = \beta X_t N_t / L_t = (\theta/\mu)\,\beta Y_t / L_t = (\theta/\mu)\,\beta y_t, \tag{A6}$$

which implies that a_t/y_t is constant. Substituting (A6), (3), (6) and (22) into (2) yields

$$\begin{aligned}\frac{\dot{y}_t}{y_t} = \frac{\dot{a}_t}{a_t} &= r_t - \lambda + \frac{(1-\tau_t)\,w_t l_t - c_t}{a_t} \\ &= \frac{\dot{c}_t}{c_t} + \rho - \lambda + \frac{(1-\theta-\gamma)\,\mu}{\beta\theta} - \frac{\mu}{\beta\theta}\frac{c_t}{y_t},\end{aligned} \tag{A7}$$

Equation (A7) can be rearranged as

$$\frac{\dot{c}_t}{c_t} - \frac{\dot{y}_t}{y_t} = \frac{\mu}{\beta\theta}\frac{c_t}{y_t} - \frac{(1-\theta-\gamma)\,\mu}{\beta\theta} - \rho + \lambda, \tag{A8}$$

which implies that c_t/y_t jumps to its steady-state value in (28) whenever $n_t > 0$. Substituting (28) into (23) yields (29). ■

Proof of Proposition 1. In the pre-industrial era, firm size is not large enough for horizontal innovation and vertical innovation to be viable such that $n_t = z_t = 0$. As a result, labor supply l is given by (26), government spending share is given by (22), and the state variable $x_t = (\theta/\mu)^{\frac{1-\kappa(1-\theta)}{(1-\kappa)(1-\theta)}} Z_0^{\frac{\alpha}{1-\kappa}-1} L_t/N_0^{1-\sigma/(1-\kappa)}$ increases at the exogenous population growth rate λ. Therefore, the dynamics of x_t in the pre-industrial era is given by

$$\dot{x}_t = \lambda x_t > 0. \tag{A9}$$

In the first phase of the industrial era, firm size becomes large enough for horizontal innovation (but not for vertical innovation) to be viable such that $n_t > 0$ and $z_t = 0$. The variety growth rate n_t is positive if and only if (31) holds. The dynamics of $x_t = (\theta/\mu)^{\frac{1-\kappa(1-\theta)}{(1-\kappa)(1-\theta)}} Z_0^{\frac{\alpha}{1-\kappa}-1} L_t/N_t^{1-\sigma/(1-\kappa)}$ is

$$\dot{x}_t = \left[\lambda - \left(1 - \frac{\sigma}{1-\kappa}\right) n_t\right] x_t = \frac{1-\sigma-\kappa}{\beta(1-\kappa)} \left\{ \frac{\phi}{(\gamma^\kappa l^*)^{1/(1-\kappa)}} - \left[\mu - 1 - \beta\left(\rho + \frac{\sigma\lambda}{1-\sigma-\kappa}\right)\right] x_t \right\}, \tag{A10}$$

which uses (30) for n_t.

In the second phase of the industrial era, firm size becomes large enough for both horizontal and vertical innovation to be viable such that $n_t > 0$ and $z_t > 0$. The quality growth rate z_t is positive if and only if (37) holds. We use (34), (35) and $z_t = [(1-\kappa)/\alpha]g_t - (\sigma/\alpha)n_t$ to derive n_t and the dynamics of $x_t = (\theta/\mu)^{\frac{1-\kappa(1-\theta)}{(1-\kappa)(1-\theta)}} Z_t^{\frac{\alpha}{1-\kappa}-1} L_t/N_t^{1-\sigma/(1-\kappa)}$ as

$$\dot{x}_t = \frac{a_1\left[(\gamma^\kappa l^*)^{1/(1-\kappa)} x_t\right]^2 - a_2(\gamma^\kappa l^*)^{1/(1-\kappa)} x_t + a_3}{(1-\kappa)\left[\alpha\beta(\gamma^\kappa l^*)^{1/(1-\kappa)} - \sigma/x_t\right]}, \tag{A11}$$

where

$$a_1 = \alpha\beta(\mu-1)(1-\kappa)(\alpha+\kappa-1),$$
$$a_2 = (\alpha+\kappa-1)\{[\mu-1-\beta(\rho-\lambda)]\sigma+\beta(1-\kappa)(\rho+\phi\alpha)\}+\alpha(1-\kappa-\sigma)[(\mu-1)\kappa-\beta(\rho-\lambda)]-\alpha\beta\lambda(1-\kappa),$$
$$a_3 = (\alpha+\kappa-1)\phi\sigma + (1-\kappa-\sigma)[\alpha\phi\kappa - (1-\kappa)\rho] - \sigma\lambda(1-\kappa).$$

Using $\dot{x}_t = 0$ we can derive the steady-state firm size

$$(\gamma^\kappa l^*)^{1/(1-\kappa)} x^* = \frac{a_2 \pm \sqrt{a_2^2 - 4a_1 a_3}}{2a_1},$$

which is independent of γ because a_1, a_2 and a_3 are independent of γ.

Case 1. When $\alpha + \kappa = 1$, $a_1 = 0$, $a_2 = \alpha(1-\kappa-\sigma)\{\kappa(\mu-1) - \beta[\rho + \sigma\lambda/(1-\kappa-\sigma)]\}$, and $a_3 = \alpha(1-\kappa-\sigma)\{\kappa\phi - [\rho + \sigma\lambda/(1-\kappa-\sigma)]\}$. In this case, the dynamics of x_t in the pre-industrial and first phase of industrial eras are the same as in (A9) and (A10), and in the second phase of industrial era it becomes

$$\dot{x}_t = \frac{1-\kappa-\sigma}{(1-\kappa)\beta - \sigma/[(\gamma^\kappa l^*)^{1/(1-\kappa)}x_t]} \left\{ \left[\kappa\phi - \left(\rho + \frac{\sigma\lambda}{1-\kappa-\sigma}\right)\right] \frac{1}{(\gamma^\kappa l^*)^{1/(1-\kappa)}} - \left[\kappa(\mu-1) - \beta\left(\rho + \frac{\sigma\lambda}{1-\kappa-\sigma}\right)\right] x_t \right\}.$$

The differential equation $\dot{x}_t = 0$ only has one stable real root $x^* > 0$ under the parameter conditions in (43) and (44), which ensure $z^* > 0$.

Case 2. When $\alpha + \kappa > 1$, $a_1 > 0$. If $\Delta = a_2^2 - 4a_1a_3 > 0$, then we can obtain the following results for $\Phi[(\gamma^\kappa l^*)^{1/(1-\kappa)}x^*] = a_1\left[(\gamma^\kappa l^*)^{1/(1-\kappa)}x^*\right]^2 - a_2(\gamma^\kappa l^*)^{1/(1-\kappa)}x^* + a_3 = 0$:

$$\begin{cases} \text{If } a_2 > 0, a_3 > 0, & \text{2 positive real roots: } \frac{a_2+\sqrt{\Delta}}{2a_1}\text{(unstable)}, \frac{a_2-\sqrt{\Delta}}{2a_1}\text{(stable)}; \\ \text{If } a_2 > 0, a_3 \le 0, & \text{1 positive real root: } \frac{a_2+\sqrt{\Delta}}{2a_1}\text{(unstable)}; \\ \text{If } a_2 \le 0, a_3 \ge 0, & \text{no positive real root}; \\ \text{If } a_2 \le 0, a_3 < 0, & \text{1 positive real roots: } \frac{a_2+\sqrt{\Delta}}{2a_1}\text{(unstable)}. \end{cases}$$

If $\Delta \le 0$, then $\Phi[(\gamma^\kappa l^*)^{1/(1-\kappa)}x^*] = 0$ does not have a positive stable real root. In summary, when $\alpha + \kappa > 1$, there exists a stable steady-state value $x^* > 0$ only if $\Delta > 0$, $a_2 > 0$ and $a_3 > 0$.

Case 3. When $\alpha + \kappa < 1$, $a_1 < 0$. If $\Delta > 0$, then we can obtain the following results for $\Phi(x^*) = 0$:

$$\begin{cases} \text{If } a_2 \ge 0, a_3 > 0, & \text{1 positive real root: } \frac{a_2-\sqrt{\Delta}}{2a_1}\text{(stable)}; \\ \text{If } a_2 \ge 0, a_3 \le 0, & \text{no positive real root}; \\ \text{If } a_2 < 0, a_3 \ge 0, & \text{1 positive real root: } \frac{a_2-\sqrt{\Delta}}{2a_1}\text{(stable)}; \\ \text{If } a_2 < 0, a_3 < 0, & \text{2 positive real roots: } \frac{a_2+\sqrt{\Delta}}{2a_1}\text{(unstable)}, \frac{a_2-\sqrt{\Delta}}{2a_1}\text{(stable)}. \end{cases}$$

If $\Delta \le 0$, then $\Phi[(\gamma^\kappa l^*)^{1/(1-\kappa)}x^*] = 0$ does not have a positive stable real root. In summary, when $\alpha + \kappa < 1$, there exists a stable steady-state value $x^* > 0$ only if $\Delta > 0$, $a_2 \ge 0$ and $a_3 > 0$, or $\Delta > 0$ and $a_2 < 0$.

In summary, given $x_0 < x_N < x_Z$ and the parameter conditions discussed above, the autonomous dynamics of x_t is stable and governed by (A9), (A10) and (A11). Given an initial value x_0, the state variable x_t increases according to (A9) until x_t reaches the first threshold x_N. Then, x_t increases according to (A10) until x_t reaches the second threshold x_Z. Finally, x_t increases according to (A11) until x_t converges to its steady-state value x^* in (45). ■

Proof of Proposition 2. Taking derivative for $\ln x_N$ with respect to γ yields

$$\frac{\partial \ln x_N}{\partial \gamma} = -\frac{\kappa}{1-\kappa}\frac{1}{\gamma} - \frac{1}{1-\kappa}\frac{\partial l^*/\partial\gamma}{l^*},$$

where

$$\frac{\partial l^*}{\partial \gamma} = -(l^*)^2 \frac{\eta\beta\theta(\rho-\lambda)}{\mu(1-\theta-\gamma)^2}.$$

Substituting $\partial l^*/\partial\gamma$ into $\partial \ln x_N/\partial\gamma$ yields

$$\frac{\partial \ln x_N}{\partial \gamma} = \frac{\eta\beta\theta(\rho-\lambda)\gamma - \kappa(1-\theta-\gamma)[(1+\eta)(1-\theta-\gamma)\mu + \eta\beta\theta(\rho-\lambda)]}{\gamma(1-\kappa)(1-\theta-\gamma)[(1+\eta)(1-\theta-\gamma)\mu + \eta\beta\theta(\rho-\lambda)]},$$

The sign of $\partial \ln x_N/\partial\gamma$ is determined by the numerator. It can be shown that the numerator is increasing in γ, and we denote the numerator as Λ.

$$\lim_{\gamma\to 0} \Lambda = -\kappa(1-\theta)[(1+\eta)(1-\theta)\mu + \eta\beta\theta(\rho-\lambda)] < 0;$$

$$\lim_{\gamma\to 1-\theta} \Lambda = \eta\beta\theta(1-\theta)(\rho-\lambda) > 0.$$

Therefore, there exists a threshold value $\tilde{\gamma} \in (0, 1-\theta)$ such that $\partial \ln x_N/\partial\gamma < 0$ for $\gamma \in (0, \tilde{\gamma})$ and $\partial \ln x_N/\partial\gamma > 0$ for $\gamma \in (\tilde{\gamma}, 1)$. Therefore, for a relatively small (large) $\gamma < \tilde{\gamma}$ ($\gamma > \tilde{\gamma}$), an increase in γ leads to a smaller (larger) x_N, which causes an earlier (a delayed) takeoff.

From (30) and (34), for a given $x_t \in (x_N, x^*)$, an increase in γ increases (decreases) the equilibrium growth rate when $\gamma < \tilde{\gamma}$ ($\gamma > \tilde{\gamma}$). From (46), γ does not affect the steady-state growth rate due to the scale-invariant property of the model. ■

Dynamics of corporate income tax rate in the industrial era. The profit as a share of output is given by

$$\frac{N_t\left(\Pi_t - I_t\right)}{Y_t} = \frac{\theta}{\mu}\left[\mu - 1 - \frac{\phi + z_t}{(\gamma^{1-\alpha} l^*)^{1/\alpha} x_t}\right] = \frac{\beta\theta(n_t + \rho - \lambda)}{\mu(1-\tau_{\Pi,t})},$$

which uses the growth rate of variety given by

$$n_t = \frac{1-\tau_{\Pi,t}}{\beta}\left[\mu - 1 - \frac{\phi + z_t}{(\gamma^{1-\alpha} l^*)^{1/\alpha} x_t}\right] + \lambda - \rho.$$

Substituting the profit share into the government budget constraint yields

$$\tau_{\Pi,t} = \frac{\mu[\gamma - \tau\left(1-\theta\right)](1-\tau_{\Pi,t})}{\beta\theta(n_t + \rho - \lambda)}.$$

Solving for $\tau_{\Pi,t}$ yields the corporate income tax rate in Section 4.2. ■

Dynamics of x_t under corporate income tax. In the pre-industrial era, the dynamics of x_t is the same as before:

$$\dot{x}_t = \lambda x_t > 0.$$

In the first phase of industrial era, the dynamics of x_t becomes

$$\dot{x}_t = \frac{\alpha-\sigma}{\alpha\beta}\left\{\frac{\phi(1-\tau_{\Pi,t})}{(\gamma^{1-\alpha} l^*)^{1/\alpha}} - \left[(1-\tau_{\Pi,t})(\mu-1) - \beta\left(\rho + \frac{\sigma\lambda}{\alpha-\sigma}\right)\right] x_t\right\}.$$

In the second phase of industrial era, the dynamics of x_t becomes

$$\dot{x}_t = \frac{\alpha-\sigma}{\alpha\beta}\left\{\left[(1-\alpha)\phi - \left(\rho + \frac{\sigma\lambda}{\alpha-\sigma}\right) - \frac{\dot{\tau}_{\Pi,t}}{1-\tau_{\Pi,t}}\right]\frac{1-\tau_{\Pi,t}}{(\gamma^{1-\alpha} l^*)^{1/\alpha}} \right.$$
$$\left. - \left[\left(1-\tau_{\Pi,t}\right)\left(\mu-1\right)\left(1-\alpha\right) - \beta\left(\rho + \frac{\sigma\lambda}{\alpha-\sigma}\right)\right] x_t\right\},$$

where we have used $(1-\tau_{\Pi,t})\sigma/[(\gamma^{1-\alpha}l^*)^{1/\alpha}x_t] \cong 0$. ■

Proof of Proposition 3. Taking derivative for $\ln x_N$ with respect to γ yields

$$\begin{aligned}\frac{\partial \ln x_N}{\partial \gamma} &= -\frac{1-\alpha}{\alpha}\frac{1}{\gamma} + \frac{\mu/\theta}{\mu-1-\beta(\rho-\lambda)-[\gamma-(1-\theta)\tau]\mu/\theta} \\ &= \frac{(\mu/\theta)\gamma-(1-\alpha)[\mu-1-\beta(\rho-\lambda)+(1-\theta)\tau\mu/\theta]}{\alpha\gamma\{\mu-1-\beta(\rho-\lambda)-[\gamma-(1-\theta)\tau]\mu/\theta\}},\end{aligned}$$

The sign of $\partial \ln x_N/\partial\gamma$ is determined by the numerator, because the denominator must be positive to ensure $x_N > 0$. It is useful to note that the numerator is increasing in γ, and we denote the numerator as Γ.

$$\lim_{\gamma\to(1-\theta)\tau} \Gamma = \alpha\tau\mu(1-\theta)/\theta - (1-\alpha)[\mu-1-\beta(\rho-\lambda)].$$

Then, $\partial \ln x_N/\partial\gamma = 0$ yields

$$\hat{\gamma} = (1-\alpha)\{[\mu-1-\beta(\rho-\lambda)]\theta/\mu + (1-\theta)\tau\}.$$

When $\hat{\gamma} > (1-\theta)\tau$, $\lim_{\gamma\to(1-\theta)\tau}\Gamma < 0$ implies that

$$\tau < \min\left\{\frac{(1-\alpha)\theta}{\alpha\mu(1-\theta)}[\mu-1-\beta(\rho-\lambda)], 1\right\}.$$

Therefore, if the labor income tax rate τ is small enough such that $\lim_{\gamma\to(1-\theta)\tau}\Gamma < 0$, then there exists a threshold value $\hat{\gamma}$ such that $\partial \ln x_N/\partial\gamma < 0$ for $\gamma < \hat{\gamma}$ and $\partial \ln x_N/\partial\gamma > 0$ for $\gamma > \hat{\gamma}$. Therefore, for a relatively small (large) $\gamma < \hat{\gamma}$ ($\gamma > \hat{\gamma}$), an increase in γ leads to an earlier (a delayed) takeoff. ■

Appendix B: Data

Table B1: Summary statistics

	obs	mean	std. dev.	min	max
Growth of real GDP	189	0.039	0.030	-0.037	0.169
Growth of real GDP per capita	189	0.034	0.029	-0.030	0.144
Growth of real GDP per worker	189	0.030	0.025	-0.021	0.106
Productive government spending	189	0.165	0.039	0.053	0.252
Log real GDP per capita	189	10.075	0.704	7.860	11.459
Log population	189	6.815	1.806	2.069	11.844
Log capital stock	189	13.934	1.933	8.361	17.963
Trade openness	189	0.908	0.746	0.135	5.080

Appendix C: Phase diagrams

Figure C1 presents the equilibrium with only variety growth for $\alpha + \kappa = 1$.

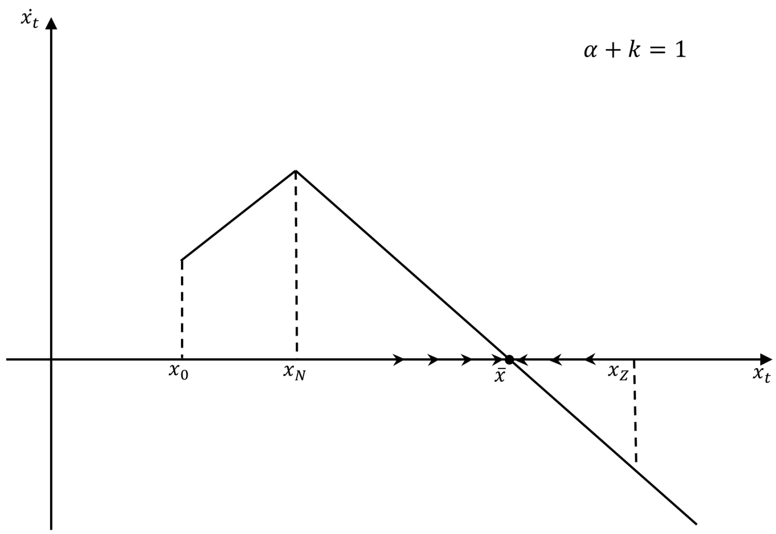

Figure C1: Variety growth only ($\alpha + \kappa = 1$)

Figure C2 presents the equilibrium with only variety growth for $\alpha + \kappa > 1$.

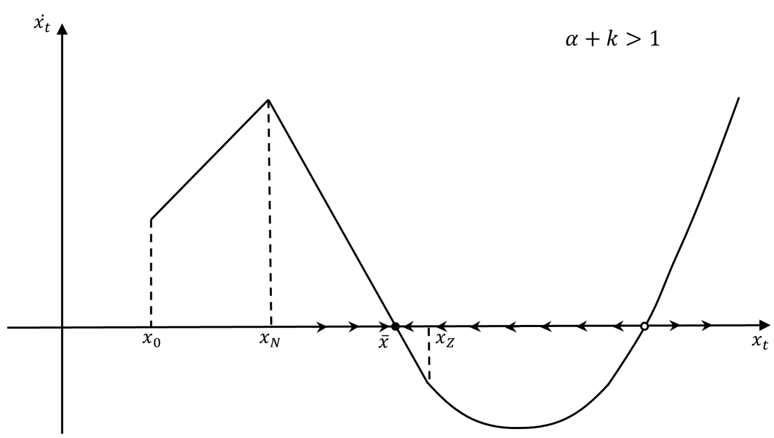

Figure C2: Variety growth only ($\alpha + \kappa > 1$)

Figure C3 presents the equilibrium with explosive growth for $\alpha + \kappa > 1$.

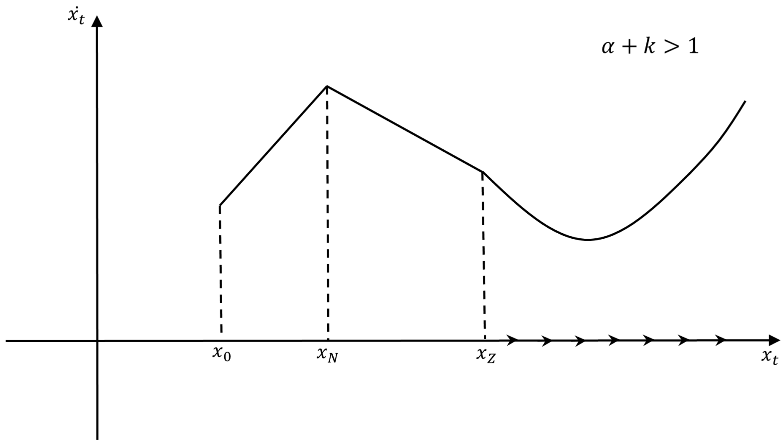

Figure C3: Explosive growth ($\alpha + \kappa > 1$)

Figure C4 presents the equilibrium with only variety growth for $\alpha + \kappa < 1$.

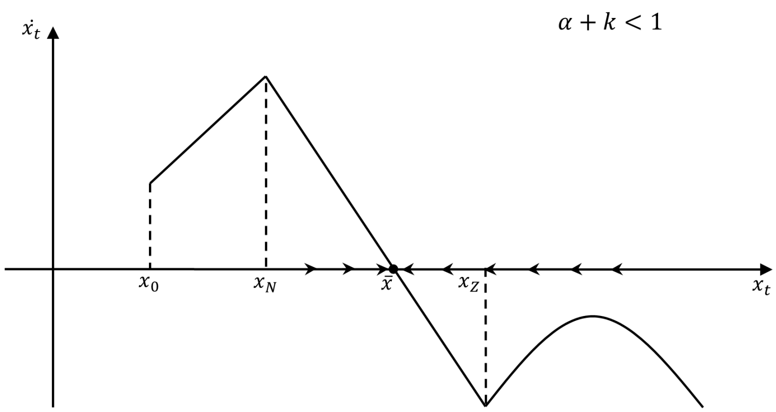

Figure C4: Variety growth only ($\alpha + \kappa < 1$)

Figure C5 presents the equilibrium with multiple equilibria for $\alpha + \kappa < 1$.

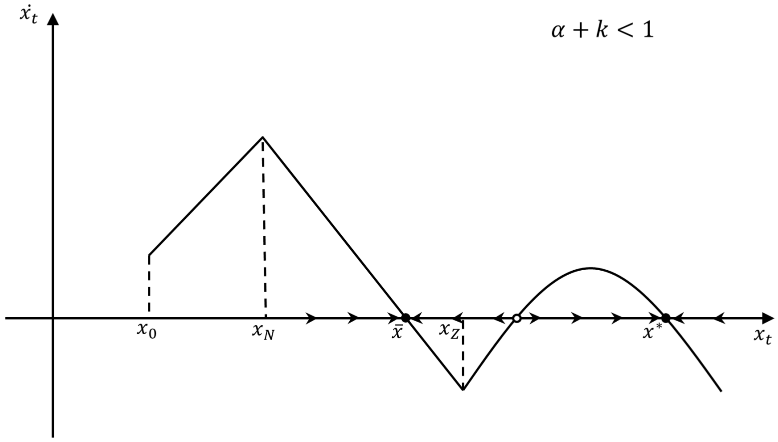

Figure C5: Multiple equilibria ($\alpha + \kappa < 1$)