The Solow-Swan Growth Model

Let
- Y = GDP;
- K = capital;
- L = labor;
- I = investment;
- S = savings;
- $y = Y/L$ = GDP per worker;
- $k = K/L$ = capital per worker;
- s = marginal propensity to save (assumed for simplicity to equal the average propensity to save).

For any variable X, let $\dot{X} \equiv \partial X / \partial t =$ the time rate of change of X (t = time).

Then, $\frac{\dot{X}}{X}$ = the rate of growth of X (i.e., the percentage rate of change per unit time).

Note that $I \equiv \dot{K}$.

Now, assume a constant-returns-to-scale production function with capital and labor smoothly substitutable and no depreciation or technical change:

(1) $$Y = F(K, L).$$

Dividing (1) by L yields

(2) $$Y/L = F(K/L, 1)$$

or

(3) $$y = f(k),$$

where $f(k) \equiv F(K/L, 1)$. $f(k)$ is referred to as the production function in *intensive form* – i.e., it is expressed in per worker units rather than natural units.

Solow and Swan consider the case of economies in moving equilibrium – i.e., economies for which there are no unemployed resources and all plans are fulfilled (or, as Harrod would put it, economies that are always on their warranted growth paths). The macroeconomic equilibrium is, therefore defined by

(4) $$S^{planned} = I^{planned}.$$

It is convenient to dropped the "*planned*" superscript, remembering however that we are always in this moving equilibrium. Equation (4) then implies

(5) $$S = sY = I = \dot{K}.$$

We can define a *balanced* or *steady-state* growth path as one in which K and L grow at the same rate. Thus,

(6) $$\frac{\dot{K}}{K} = \frac{\dot{L}}{L}.$$

The right-hand side of (6) is just the growth rate of labor and, since there is no productivity growth, the growth rate of labor is just Harrod's *natural rate of growth* g_N. Using (1) and (5), the numerator on the left-hand side is $sF(K, L)$; and, using the definition of a growth rate, the right-hand side is g_N, so that (6) becomes

(7) $$\frac{sF(K,L)}{K} = g_N.$$

Multiplying both sides by K and dividing both sides by L yields

(8) $$sF(K/L,1) = sf(k) = g_N \frac{K}{L} = g_N k$$

or

(8′) $$sf(k) = g_N k.$$

This is the condition that must be met if the economy is to be on a balanced growth path. The left-hand side is savings per worker,which must equal investment per worker, since the economy is constantly at full employment. The right-hand side gives the amount of output per worker that must be devoted to investment in order to keep the number of workers (growing at rate g_N) with sufficient capital that k remains constant: that is, it meets the needs for *capital widening* – just enough capital to give new workers the same amount of capital that old workers had. Thus, if $sf(k) > g_N k$, more than the amount of capital needed for capital widening is available, and k rises, which is called *capital deepening*; while if $sf(k) < g_N k$, savings and investment fall short of the needs of capital widening and k falls, which is negative capital deepening.

The situation can be pictured in the diagram below. The upper curve is the production function, giving total output per worker. The middle curve is the savings or investment curve (just that proportion of the total output per worker that is actually saved or invested). The ray from the origin is the graph of $g_N k$. Point A, where the middle curve and the ray cross, corresponds to equation (8′) and represents the balanced or steady-state growth path. At point A, capital per worker k^* is used to produce GDP per worker y^*.

Now, imagine that the economy is not at point A, but, say, at at a level of capital per worker given by k_1. The diagram shows that at such a point, $sf(k) > g_N k$, so that k must be rising – as it must for any point below k^*. Similarly, at a point above A, say, at a level of capital per worker given by k_2, $sf(k) < g_N k$, k must be falling – as it must for any point above k^*.

Consequently, if the economy is ever away from point A, k will reconverge to k^*over time. Since in this analysis planned savings always equals planned investment, the economy is always on a warranted growth path. The particular warranted rate of growth changes, depending on k. It is only at A that the warranted rate of growth becomes constant and capital and labor are growing at the same rate g_N, and, therefore, only at A that the economy is on a balanced growth path. The diagram shows that while the economy can be on many warranted growth paths, it reconverges stably to the same balanced growth path at the natural rate of growth (g_N).

Solow took this to mean that Harrod was wrong about the instability of growth. But this was a misreading of Harrod. Harrod compares the actual rate of growth to the warranted rate of growth and shows that, if an economy is on its warranted growth path, then anything that raises or lowers the warranted rate (e.g., a change in the savings rate) pushes the actual rate of growth in the opposite direction. Solow shows that, in a situation in which the actual rate of growth is never allowed to deviate from the warranted rate of growth, anything that pushes the warranted rate of growth away from the natural rate of growth (again, e.g., a change in the rate of savings), will result in reconvergence to the natural rate. Harrod compares g_A to g_W; while Solow compares g_W to g_N. It is an apples-and-oranges comparison for Solow to claim that his model refutes Harrod's analysis, because Harrod never assumes constant full employment or, equivalently, that planned savings always equals planned investment.

One important property of Solow's model is that the balanced growth path is unaffected by the rate of saving or investment, which some people found to be counter-intutitive. This is because the natural rate of growth (g_N) is simply the *exogenous* rate of labor-force (or population) growth, which is assumed to be independent of the savings rate. Changing the savings rate (s) can neither raise not lower (g_N). What it can do, however, is change the location of point A, which changes the level of k^* and y^*. Thus, an economy that wants to raise its per capita income will be advised to save at a higher rate: in the long run, its people will be richer per capita, but the economy will not grow any faster than it did at the lower savings rate.

The Solow-Swan (or Neoclassical) Growth Model