Economics 883 Spring 2015
Tauchen

Pure Jump Lévy Processes

1 Pure Jump

For these processes the Brownian component is absent $\sigma^2 = 0$ and (for now) the drift $b = 0$ is also absent. Thus the characteristic triplet is $(0, 0, m)$, and the process is purely discontinuous.

Here is one key to understanding this material: Consider a compound Poisson process. Let $m(x) = \lambda f(x)$ be the intensity density, where $f(x)$ is the jump density, nonnegative and $\int f(x)\, dx = 1$, and $\lambda > 0$ is the intensity; of course by construction $\lambda = \int m(x)\, dx$. Now consider the process's realization at time $t = 1$:

$$Y = \sum_{j=1}^{N} X_j, \quad X_i \sim f(x) \tag{1}$$

where $N \sim \text{Poisson}(\lambda)$, i.e.

$$\Pr(N = k) = \frac{\lambda^k}{k!} e^{-\lambda} \tag{2}$$

Now remember that for any number b

$$e^b = \sum_{j=0}^{\infty} \frac{b^k}{j!} = 1 + b + \frac{b^2}{2} + \frac{b^3}{6} \ldots \tag{3}$$

Indeed, it is (3) that makes

$$\sum_{k=0}^{\infty} \Pr(N = k) = 1. \tag{4}$$

Now for the characteristic functions: Let $\phi(u) = \mathrm{E}(e^{iuX})$, and we seek $\mathrm{E}(e^{iuY})$. Conditional on $N = k$ Y is the sum of k independent and identically distributed random, so

$$\mathrm{E}\left(e^{iuY} | N = k\right) = \mathrm{E}\left(e^{iu \sum_{j=1}^{k} X_j}\right) = \phi(u)^k \tag{5}$$

Unconditionally, we get

$$\begin{aligned}\sum_{k=0}^{\infty} \Pr(N=k)\,\phi(u)^k &= \sum_{k=0}^{\infty} \Pr(N=k)\,\phi(u)^k \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda}\phi(u)^k \\ &= \sum_{k=0}^{\infty} \frac{(\lambda\phi(u))^k}{k!} e^{-\lambda} \\ &= e^{\lambda\phi(u)-\lambda} \\ &= e^{\lambda\int e^{iux} f(x)\ dx-\lambda} \\ &= e^{\lambda\int\left(e^{iux}-1\right) f(x)\ dx}\end{aligned} \tag{6}$$

To summarize the characteristic function of Y is

$$\mathrm{E}\left(e^{iuY}\right) = e^{\lambda\int\left(e^{iux}-1\right) f(x)\ dx} \tag{7}$$

2 Compound Poisson processes as building blocks for pure jump Lévy processes

Compound Poisson processes for $t \in [0,T]$ are fundamental to Lévy processes and financial econometrics.

Let $\lambda > 0$ and $N \sim \text{Poisson}(\lambda T)$. N is the total finite number of jumps on $[0,T]$. Let $\{X_i\}_1^N$ be independent and identically distributed random variables independent of N. All that needs to be done is to scatter the jumps uniformly over $[0,T]$. To this end, Let $\{U_i\}_1^N$ be independent Uniform$([0,T])$ on $[0,T]$ variables, and set

$$Y_t = \sum_{i=1}^{N} \mathbf{1}_{\{U_i \le t\}} X_i, \quad 0 \le t \le T.$$

Then Y_t is a compound Poisson process with intensity parameter λ and jump pdf $f(x)$. The characteristic function of Y_1 is

$$\mathrm{E}(e^{iuY_1}) = e^{\lambda\int(e^{ix}-1)f(x)dx}$$

and that of Y_t is

$$\mathrm{E}(e^{iuY_t}) = e^{t\lambda\int(e^{ix}-1)m(x)dx}, \quad t \ge 0.$$

where we write

$$m(x) = \lambda f(x)$$

for the intensity density. If $a < b$, then the expected number of jumps in Y_t of size between a and b is

$$t \int_a^b m(x)dx.$$

Since $f(x)$ is a density that integrates to 1, then $m(x)$ integrates to λ:

$$\int m(x)dx = \lambda < \infty,$$

which is why we call $m(x)$ the intensity density. We can always compute the relative frequency of jumps in $[a, b]$ relative to $[c, d]$ as the ratio

$$0 \leq \frac{\int_a^b m(x)dx}{\int_c^d m(x)dx} < \infty$$

where $c < d$.

3 Example: Exp(ϕ) Jump Density

An interesting special case is when the jumps are exponentially distributed:

$$f(x) = \phi e^{-\phi x}, x \geq 0, \ \phi > 0. \tag{8}$$

Just to note, $\mathrm{E}(X) = \frac{1}{\phi}$, $\mathrm{Var}(X) = \frac{1}{\phi^2}$. The intensity density is

$$m(x) = \lambda \phi e^{-\phi x}, x \geq 0, \ \phi > 0. \tag{9}$$

The two parameters are positive. The process is

$$Y_t = \sum_{i=1}^{N} \mathbf{1}_{\{U_i \leq t\}} X_i, \quad 0 \leq t \leq T. \tag{10}$$

The compensator is $\frac{\lambda}{\phi} t$ so the compensated process is

$$\tilde{Y}_t = \sum_{i=1}^{N} \mathbf{1}_{\{U_i \leq t\}} X_i \ - \ \frac{\lambda}{\phi} t \qquad 0 \leq t \leq T. \tag{11}$$

Figure 1 below shows in the top panel a realization of Y_t for $\lambda = 6$, $\phi = 0.75$, over $T = 10$ periods and the compensated process in the bottom panel.

(12)

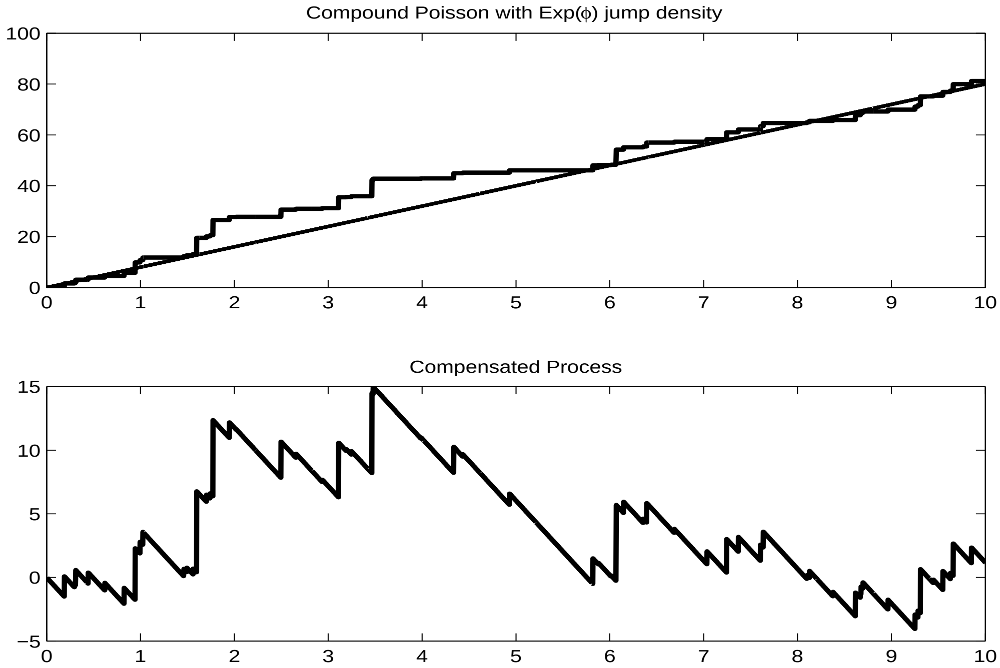

Figure 1: Compound Poisson

4 Small Jumps

Note that jumps of size 0 make no sense so we can redefine the jump intensity at 0 as $m(0) = 0$. If $a < 0$ and $b > 0$ then strictly we write the expected number of jumps of Y_t in the interval $[a, b]$ as

$$t \int_a^b m(x)dx$$

with the understanding that the above integral skips over the point $x = 0$. We will see later that the behavior of the intensity function around 0, i.e., for very small positive or negative jumps is delicate, and very important for studying a Lévy process.

5 A Brief Look at the α-stable

The prototypical pure jump process is the α–stable. The jump density for the stable is

$$m(x) = \frac{c}{x^{1+\alpha}}, \quad 0 \leq \alpha < 2 \tag{13}$$

Figure 2 shows $m(x)$ for different values of α.

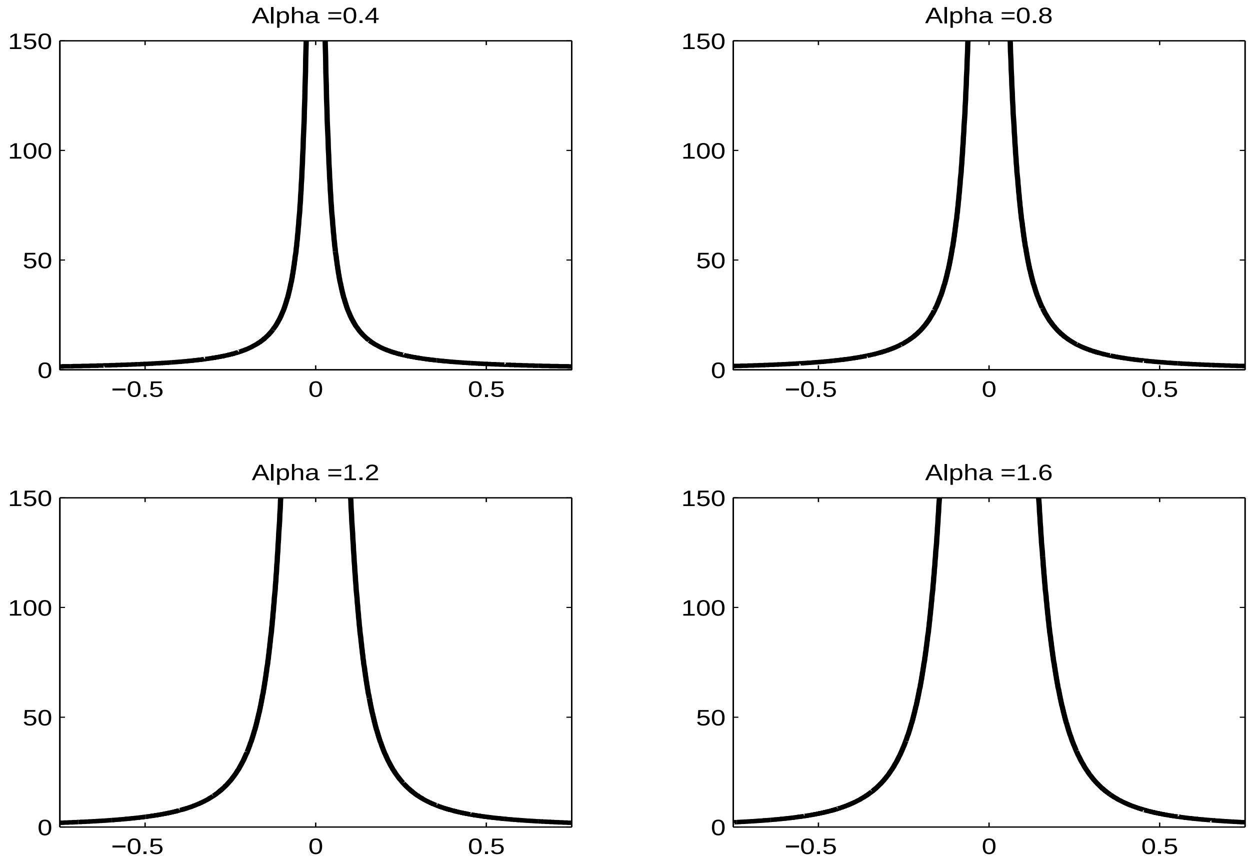

Figure 2: α-Stable Jump Density

6 The Compound Poisson Approximation

Let's look at the compound Poisson approximation to the α–stable. Let $a > 0$ be a positive number and later we will study $a \to 0$ so think of a as a small positive number, at least for now. Now consider the compound Poisson process defined by the intensity density

$$m(x) = \frac{c}{x^{1+\alpha}} 1_{\{x \geq a\}}, \qquad 0 \leq \alpha < 2, \, c > 0 \tag{14}$$

A little calculus gives

$$\int_a^b \frac{c}{x^{1+\alpha}} \, dx = \frac{c}{\alpha} \left[\left(\frac{1}{a} \right)^{\alpha} - \left(\frac{1}{b} \right)^{\alpha} \right] \tag{15}$$

With $b \to \infty$ we get

$$\lambda = \frac{c}{\alpha} \left(\frac{1}{a} \right)^{\alpha} \tag{16}$$

for the intensity of this process. Clearly, for small a the intensity will be very high,which is the whole idea. The jump density is $m(x)/\lambda$:

$$f(x) = \alpha a^{\alpha} \frac{1}{x^{1+\alpha}} 1_{\{x \geq a\}} \tag{17}$$

and you should check $\int_a^\infty f(x)\ dx = 1$ here.

Now let's think about a realization Y_t, $t \in [0, T]$, of the continuous time process with this intensity and jump density:

$$Y_t = \sum_{i=1}^{N} X_i 1_{\{U_i \leq t\}}, \quad t \in [0, T], \tag{18}$$

where N is a Poisson random variable Compound Poisson(λT) and $\{U_i\}_1^N$ are independent and identically distributed Uniform([0,T]) random variables (The U_i sre just the jump times), and $X \sim f(x)$ with $f(x)$ given by (17). Note that Y_t is nondecreasing because all of the jumps are positive.

We can easily simulate from the symmetric two-sided version, and thus make Y_t driftless, by just randomly changing the signs of the jumps. If X is a draw from (17) and U is an independent draw from Uniform([0, 1]), then, since $W = 2 \times \mathbf{1}_{\{U>.5\}} - 1$ is ± 1 with probability 1/2 the process defined by

$$Y'_t = \sum_{i=1}^{N} X'_i 1_{\{U_i \leq t\}}, \quad X'_i = W_i X_i \tag{19}$$

where W_i are independent and identically distributed sign change random variables, is a compound Poisson process with jump density

$$m'(x) = \frac{c'}{|x|^{1+\alpha}} 1_{\{|x| \geq a\}}, \qquad 0 \leq \alpha < 2,\ c' > 0. \tag{20}$$

7 Generating the Y' Process and its Limit

Does the Y' process exist in a way that we could simulate it (easily) in, say, Matlab? Sure. First, drawing an $N \sim$ Poisson(λT) is easy. Also, generating the positive jumps is easy using the probability integral transform. Recall that if U is independent Uniform([0, 1]), $X \sim F(x)$ where $F(x)$ is the cumulative distribution function, then $X = F^{-1}(U)$ follows the distribution F. All we need to do is find the cumulative distribution function of f(x) in (17) above. Using a little more calculus, we get that

$$F(x) \equiv \int_a^x f(x)\ dx = a^{\alpha} \left[\left(\frac{1}{a}\right)^{\alpha} - \left(\frac{1}{x}\right)^{\alpha} \right] \tag{21}$$

or more simply

$$F(x) = 1 - \left(\frac{a}{x}\right)^{\alpha}. \tag{22}$$

$F^{-1}(u)$ is obtained by solving for x in the equation $u = 1 - \left(\frac{a}{x}\right)^{\alpha}$:

$$F^{-1}(u) = a\,(1-u)^{-\frac{1}{\alpha}} \tag{23}$$

Since U and $1-U$ have the same distribution we can also generate the X via

$$X = aU^{-\frac{1}{\alpha}}, \quad U \sim \text{Uniform}([0,1]). \tag{24}$$

The above is about as simple as it gets.

The top panel of Figure 3 shows a realization of this compound Poisson process when $0 < \alpha < 1$, and the top panel of Figure 4 shows a realization when $1 < \alpha < 2$. For technical reasons we actually need to separate the process into the sum of the small jumps and sum of the big jumps, where the threshold for small and big is always 1. The small jump process is

$$Y_t^{''} = \sum_{i=1}^{N} \left(X_i' \mathbf{1}_{\{a \le X_i' \le 1\}}\right) \mathbf{1}_{\{U_i \le t\}} \tag{25}$$

and the big jump process is

$$Y_t^{'''} = \sum_{i=1}^{N} \left(X_i' \mathbf{1}_{\{X_i' > 1\}}\right) \mathbf{1}_{\{U_i \le t\}} \tag{26}$$

The middle panels of the figures show the small jump processes and the bottom the big jump processes. Of course

$$Y_t^{'} = Y_t^{'''} \; + \; Y_t^{''} \tag{27}$$

Using advanced methods one can show that as $a \downarrow 0$ The Y' process converges to a symmetric α–stable process S_t on $[0,T]$ with intensity density

$$m(x) = \frac{c}{x^{1+\alpha}}, \quad 0 \le \alpha < 2 \tag{28}$$

and characteristic function

$$\mathrm{E}\left(e^{iuS_t}\right) = e^{-c|u|^{\alpha}} \tag{29}$$

The technical details are very difficult (See Bertoin (1996)) but the result intuitive given the various figures.

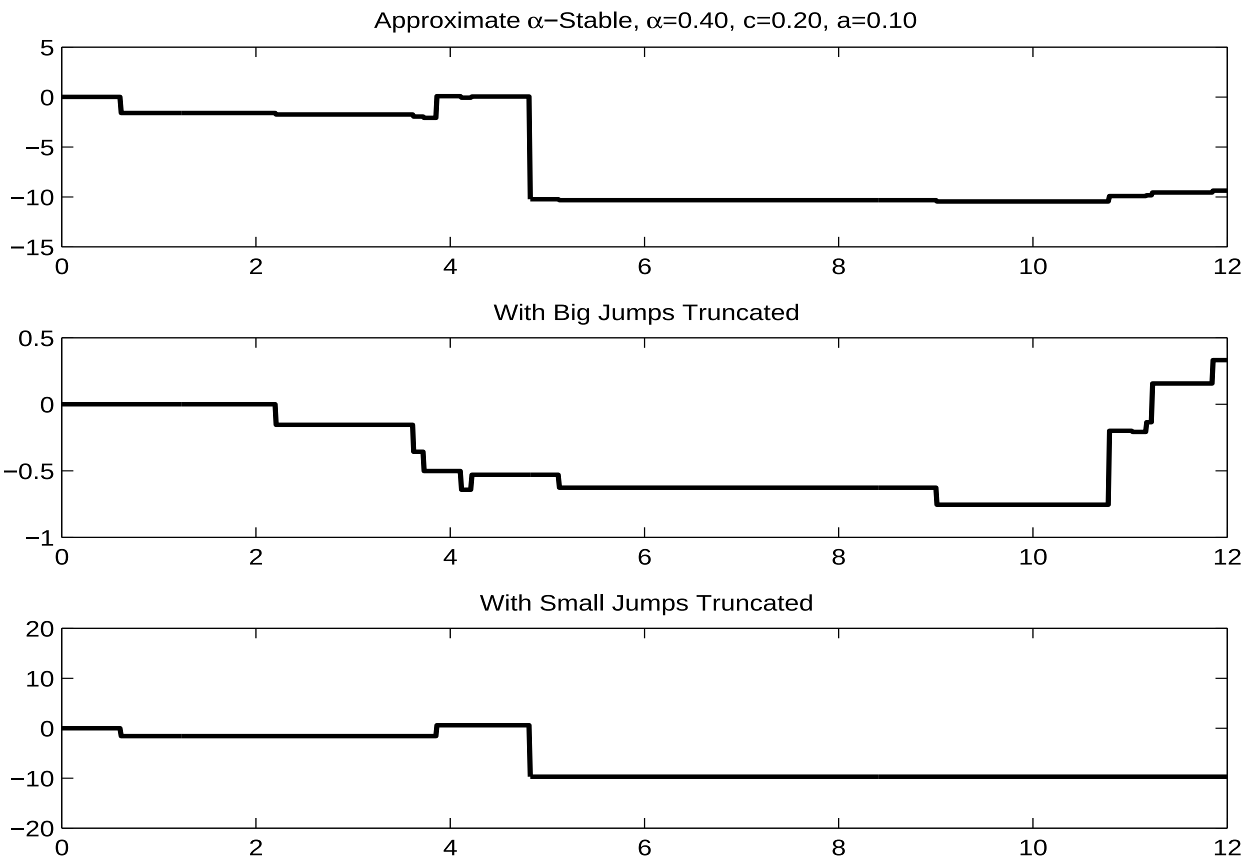

Figure 3: Compound Poisson Approximation $\alpha = 0.40$

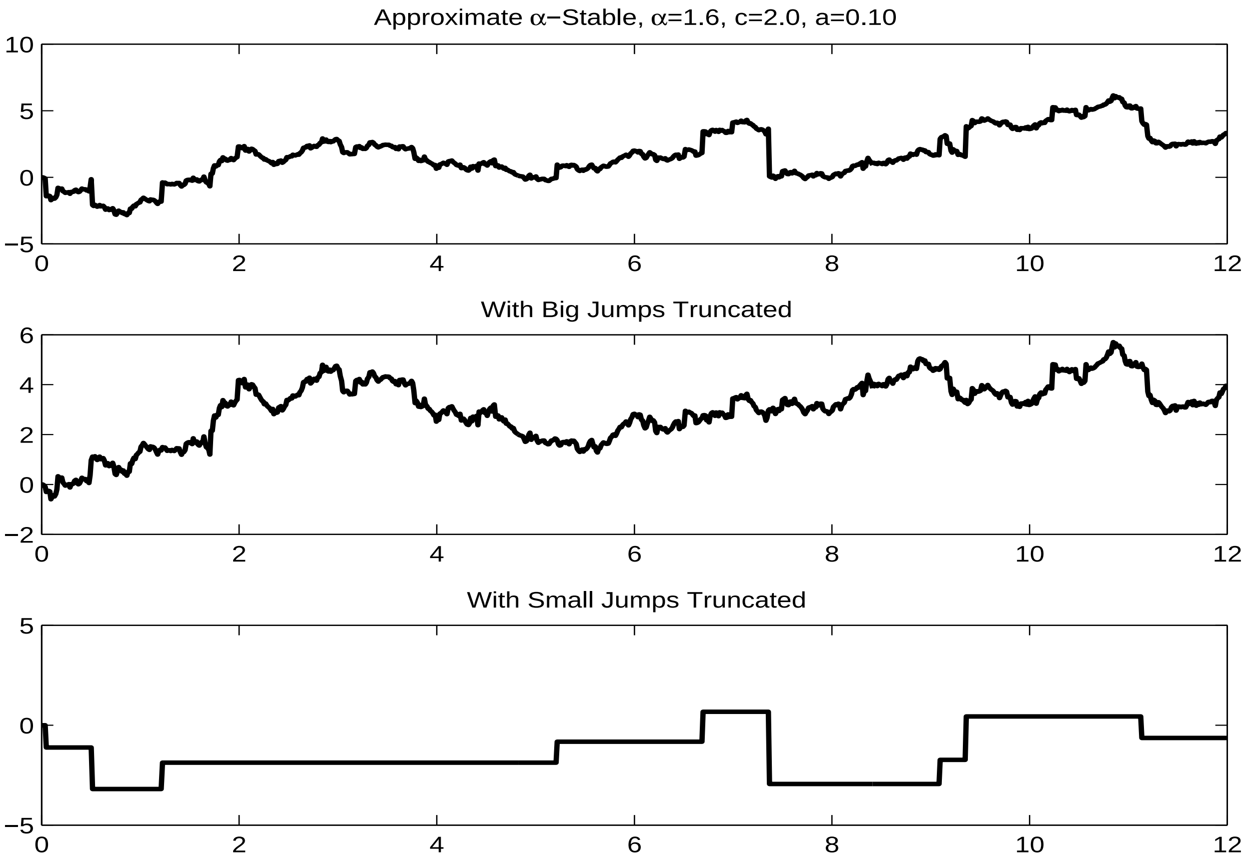

Figure 4: Compound Poisson Approximation $\alpha = 1.60$

8 Some Calculus

8.1 $\int_a^b x^{-\gamma} dx$, $0 < a < b \leq \infty$, $\gamma > 0$

Put

$$\begin{aligned} I(a,b) &= \int_a^b x^{-\gamma} dx \\ &= \frac{1}{1-\gamma} x^{1-\gamma} \Big|_a^b \\ &= \frac{1}{1-\gamma} \left(b^{1-\gamma} - a^{1-\gamma}\right) \quad \text{if } \gamma \neq 1 \\ &= \log(b) - \log(a) \quad \text{if } \gamma = 1 \end{aligned} \tag{30}$$

$\gamma < 1$

$$\begin{aligned} I(a,b) &= \frac{1}{1-\gamma} \left(b^{1-\gamma} - a^{1-\gamma}\right) \\ I(0,b) &= \frac{b^{1-\gamma}}{1-\gamma} \\ I(a,\infty) &= \infty, \quad a \geq 0 \end{aligned} \tag{31}$$

$\gamma > 1$

$$\begin{aligned} I(a,b) &= \frac{1}{\gamma-1} \left(\frac{1}{a^{(\gamma-1)}} - \frac{1}{b^{(\gamma-1)}}\right) \\ I(0,b) &= \infty \\ I(a,\infty) &= \frac{1}{\gamma-1} \left(\frac{1}{a}\right)^{(\gamma-1)} \end{aligned} \tag{32}$$