



Testing for jumps when asset prices are observed with noise – a “swap variance” approach

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ABSTRACT

This paper proposes a new test for jumps in asset prices that is motivated by the literature on variance swaps. Formally, the test follows by a direct application of Itô's lemma to the semi-martingale process of asset prices and derives its power from the impact of jumps on the third and higher order return moments. Intuitively, the test statistic reflects the cumulative gain of a variance swap replication strategy which is known to be minimal in the absence of jumps but substantial in the presence of jumps. Simulations show that the jump test has nice properties and is generally more powerful than the widely used bi-power variation test. An important feature of our test is that it can be applied – in analytically modified form – to noisy high frequency data and still retain power. As a by-product of our analysis, we obtain novel analytical results regarding the impact of noise on bi-power variation. An empirical illustration using IBM trade data is also included.

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1. Introduction

Discontinuous price changes or “jumps” are believed to be an essential component of financial asset price dynamics. The arrival of unanticipated news or liquidity shocks often result in substantial and instantaneous revisions in the valuation of financial securities. As emphasized by Aït-Sahalia (2004), relative to continuous price changes which are often modeled as a diffusive process, jumps have distinctly different implications for the valuation of derivatives (e.g. Merton (1976a,b)), risk measurement and management (e.g. Duffie and Pan (2001)), as well as asset allocation (e.g. Jarrow and Rosenfeld (1984)). The importance of jumps is also clear from the empirical literature on asset return modeling where the focus is often on decomposing the total asset return variation into a continuous diffusive component and a discontinuous pure jump component.¹

In many applications, specific knowledge about the properties of the jump process may be required and a variety of formal tests

have been developed for this purpose. For instance, Aït-Sahalia (2002) exploits the transition density derived from diffusion processes to test the presence of jumps using discrete financial data. Carr and Wu (2003) examine the impact of jumps on option prices and use the decay of time-value with respect to option maturity to test the existence of jumps. Johannes (2004) proposes non-parametric tests of jumps in a time-homogeneous jump diffusion process. Other tests include the parametric particle filtering approach of Johannes et al. (2006) and the wavelet approach of Wang (1995). Even though the above mentioned procedures vary widely from a methodological perspective, a shared feature is that they are typically designed for the analysis of low frequency data. Yet, the most natural and direct way to learn about jumps is by studying high frequency or intra-day data instead. Such an approach has rapidly gained momentum in recent years, as it opens up many new and interesting avenues for exploring the empirical jump process. The earliest contributions to this stream of literature include Barndorff-Nielsen and Shephard (2004, 2006) who developed a jump robust measure of integrated variance called bi-power variation (BPV) that, when compared to realized variance (RV), can be used to test for jumps over short time intervals. Exploiting the properties of BPV, Lee and Mykland (in press) developed an alternative non-parametric test that allows for identification of the exact timing of the jump. Aït-Sahalia and Jacod (in press) build on the concept of power variation to derive a family of jump tests that can be conducted under both the null

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¹ See for instance Andersen et al. (2002), Bates (2000), Chernov et al. (2003), Das (2002), Eraker et al. (2003), Garcia et al. (in press), Ho et al. (1996), Maheu and McCurdy (2004), Pan (2002) and Schaumburg (2004).

and the alternative hypothesis and may be applied to cases where jumps have finite or infinite activity. Other approaches include the threshold technique of Mancini (2006) and the wavelet approach of Fan and Wang (2007). Tests for jumps in a multivariate setting have been recently proposed by Bollerslev et al. (2007), Gobbi and Mancini (2007), and Jacod and Todorov (2007).

This paper contributes to the existing literature by developing a new jump test that is similar in purpose to the bi-power variation test of Barndorff-Nielsen and Shephard (2006), but with distinctly different underlying logic and properties. Intuitively, while the BPV test learns about jumps by comparing RV to a jump robust variance measure, our test does so by comparing RV to a jump sensitive variance measure involving higher order moments of returns, making it more powerful in many circumstances. Our test builds on the insight that, in the absence of jumps, the accumulated difference between the simple return and the log return captures one half of the integrated variance in the continuous-time limit. This relation is well known in the finance literature and forms the basis of a variance swap replication strategy (see Neuberger (1994)): a short position in a so-called “log contract” plus a continuously re-balanced long position in the asset underlying the swap contract. The profit/loss of such replication strategy will accumulate to a quantity that is proportional to the realized variance and, as such, allows for perfect replication of the swap contract. However, with jumps, such a strategy fails and the replication error is fully determined by the realized jumps. Our proposed jump test is based on precisely this insight. Specifically, we compute the accumulated difference between simple returns and log returns – a quantity we call “Swap Variance” or SwV given the above interpretation – and compare this to RV. When jumps are absent the difference will be indistinguishable from zero, but when jumps are present it will reflect the replication error of the variance swap which, in turn, lends it power to detect jumps. It is important to emphasize that the proposed SwV test is fully non-parametric and its implementation requires no other data than high frequency observations of the asset price process (specifically, it does not require the trading of a log contract or data on illiquid OTC variance swap prices).

The motivation for exploiting the wealth of high frequency data for jump detection is undisputed, so are the complications that arise in practical implementation due to market microstructure effects present in the data sampled at high frequencies. In the realized variance literature, the focus has almost exclusively been on the development of noise robust estimators.² In a recent paper, Fan and Wang (2007) propose methods to estimate both integrated volatility and jump variation from the data containing jumps in the price and contaminated with the market microstructure noise. To our knowledge, at present there has been no formal development of jump tests to specifically deal with high frequency data observed with noise. A distinguishing feature of our suggested swap variance jump test is that it can be applied, in analytically modified form, to noisy high frequency data. Moreover, we show that the test retains the power to detect jumps in empirically realistic scenarios. As a by-product of our analysis, we obtain novel analytical results regarding the impact of i.i.d. microstructure noise on bi-power variation. Although not pursued in this paper, these results may be used to adapt the tests proposed by Barndorff-Nielsen and Shephard (2006) and Lee and Mykland (in press) to a setting with noise.

The paper conducts extensive simulations to examine the performance of the proposed test. Throughout, we compare

results to those of the bi-power variation test because it has been widely used in literature. Overall, our findings suggest that the proposed SwV jump test performs well and constitutes a useful complement to the widely used bi-power variation test. An empirical implementation, using high frequency IBM trade data over a 5 year period, is also included and serves to highlight some of the empirical properties of the swap variance test and to further expand on the behavior of the bi-power variation test in the presence of noise.

The remainder of the paper is organized as follows. In Section 2 we develop the swap variance jump test and state its asymptotic distribution. We discuss the feasible implementation of the test and report extensive simulation results regarding its size and power. Section 3 derives an adjusted test statistic that can be applied to noisy high frequency data. Again, simulations are performed to examine the performance of the test. Section 4 contains an empirical illustration using IBM trade data, and Section 5 concludes.

2. Testing for jumps in asset returns: The “swap variance” test

Let $y_t = \ln S_t$, $t \geq 0$, be the logarithmic asset price, and (Ω, \mathcal{F}, P) a probability space with information filtration $(\mathcal{F}_t) = \{\mathcal{F}_t : t \geq 0\}$. The logarithmic asset price is specified as an Itô semimartingale relative to (\mathcal{F}_t) as follows:

$$dy_t = \alpha_t dt + V_t^{1/2} dW_t + J_t dq_t \quad (1)$$

where α_t is the instantaneous drift, V_t is the instantaneous variance when there is no jump, J_t is a random variable representing jumps in the asset price, W_t is an (\mathcal{F}_t) -standard Brownian motion, and q_t is a (\mathcal{F}_t) -counting process with finite instantaneous intensity λ_t .

The jump diffusion model in Eq. (1) is a very general representation of the asset return process. Since the demeaned asset price process is a local martingale, it can be decomposed into two canonical orthogonal components, namely a purely continuous martingale and a purely discontinuous martingale (see Jacod and Shiryaev (2003, Theorem 4.18)). In addition, there are no functional specifications on the dynamics of α_t , V_t , J_t , and q_t . In this sense, our jump test is developed in a model-free setting. We further note that our test is developed under the null hypothesis of no jumps. As further elaborated upon below, to our knowledge, the only tests developed in a model-free framework under both alternatives (jumps and no jumps) are those proposed by Aït-Sahalia and Jacod (in press).³

Applying Itô’s lemma to Eq. (1), we obtain the corresponding dynamics of the price process in levels S_t :

$$dS_t/S_t = (\alpha_t + \frac{1}{2}V_t)dt + V_t^{1/2}dW_t + (\exp J_t - 1)dq_t. \quad (2)$$

Combining Eqs. (1) and (2), over the unit time interval, we have:

$$2 \int_0^1 (dS_t/S_t - dy_t) = V_{(0,1)} + 2 \int_0^1 (\exp J_t - J_t - 1)dq_t. \quad (3)$$

This expression forms the basis for our jump test. In particular, we introduce a quantity “SwV” – the choice of terminology will become clear momentarily – defined as the discretized version of the left-hand side of Eq. (3) based on returns sampled with step size $1/N$ over the interval $[0, 1]$, i.e.

$$\text{SwV}_N = 2 \sum_{i=1}^N (R_i - r_i) \quad (4)$$

² See for instance, Aït-Sahalia et al. (2005a), Bandi and Russell (2006), Barndorff-Nielsen et al. (in press), Christensen et al. (in press), Large (2005), Oomen (2005, 2006b), Zhang (2006), Zhang et al. (2005) and Zhou (1996).

³ Lee and Mykland (in press) derive some properties of their jump test under the alternative of jumps being present, including the probability of spurious detection of jumps and failure to detect jumps.

where $R_i = S_{i/N}/S_{(i-1)/N} - 1$, i.e. the simple return, and $r_i = \ln S_{i/N}/S_{(i-1)/N}$, i.e. the continuously compounded or log return. Now, by construction, we have that:

$$\begin{aligned} & \text{plim}_{N \rightarrow \infty} (\text{SwV}_N - \text{RV}_N) \\ &= \begin{cases} 0 & \text{if no jumps in } [0, 1] \\ 2 \int_0^1 (\exp(J_t) - J_t - 1) dq_t - \int_0^1 J_t^2 dq_t & \text{if jumps in } [0, 1] \end{cases} \end{aligned} \quad (5)$$

where realized variance is defined as:

$$\text{RV}_N = \sum_{i=1}^N r_i^2. \quad (6)$$

In the above, we use the fact that RV_N , converges to the total variation of the process $V_{(0,1)} + \int_0^1 J_t^2 dq_t$ as $N \rightarrow \infty$ (see Jacod, 1994). Thus, from Eq. (5) it is clear that the difference between the SwV and RV quantities can be used to detect the presence of jumps. If the continuous sample path is observed, then we know with certainty that there are jumps if and only if $\text{SwV} \neq \text{RV}$. On the other hand, if we observe the price process only at discrete time points, then we can devise a statistical test based on the difference between SwV and RV to judge whether or not jumps have occurred. This is precisely what we do in this paper.

To provide some intuition for the suggested test statistics in Eq. (5), we point out that Eq. (3) and its discretized counterpart in Eq. (4) are deeply rooted in the literature on variance swaps (see e.g. Carr and Madan (1998), Demeterfi et al. (1999), Dupire (1993) and Neuberger (1994)). A variance swap is a forward contract on the realized variance of an asset price over a fixed time horizon. Specifically, a variance swap pays its holder the difference between an asset's ex-post realized variance – defined as the sum of squared returns at a pre-specified frequency (e.g. hourly) and over a pre-specified horizon (e.g. 3 months) – and the strike price on the notional value of the contract. Thus, a variance swap allows investors to manage volatility risk much more directly and effectively than using for instance a position in a standard put or call option where volatility exposure is diluted by price and interest rate exposure. To price and hedge a variance swap, Neuberger (1994) proposes a replication strategy using the so-called “log contract”: a contract with price equal to the logarithmic asset price, i.e., $\ln S_t$. Since at any given time t the delta of such a contract is equal to $\partial \ln S_t / \partial S_t = 1/S_t$, a delta-hedging on a short position of the log contract thus involves taking a long position in the underlying asset with number of shares equal to $1/S_t$. The payoff of a continuously re-balanced delta-hedging strategy for a short position in two log contracts is equal to:

$$2 \int_0^1 \left(\frac{1}{S_t} dS_t - d \ln S_t \right) \quad (7)$$

where $-d \ln S_t$ measures the instantaneous change in value for the short position of the log contract, and $\frac{1}{S_t} dS_t$ the instantaneous change in value for the long position in the underlying asset. From Eq. (3), it is clear that when there are no jumps, this payoff perfectly replicates the integrated variance $V_{(0,1)}$. Yet, when there are discontinuities or jumps in the price process, the position will be subject to a stochastic and unhedgeable replication error, i.e.

$$\text{P\&L due to jumps} = 2 \int_0^1 (\exp(J_t) - J_t - 1) dq_t.$$

In practice, rebalancing of the replication portfolio is of course done at discrete intervals instead of continuously, and the strike of the variance swap contract is not the latent integrated variance but its discretized counterpart realized variance. From Eq. (7) we can

see that the previously defined SwV quantity measures the payoff of a variance swap replication strategy using a *discretely* delta-hedging on a short position in two log contracts. The jump test developed in this paper is based on the difference between the SwV and RV quantities which, by the same argument, can be interpreted as the cumulative replication error of a discretely hedged variance swap. In the absence of jumps, the replication error is due to discretization only and will therefore be relatively small. In the presence of jumps, the replication strategy fails and the difference between SwV and RV is likely to be large. This logic forms the basis for our test and hence the terminology “Swap Variance” or SwV.⁴

Further intuition about the swap variance test – from a statistical viewpoint – can be gained by considering the following Taylor series expansion:

$$\text{SwV}_N - \text{RV}_N = \frac{1}{3} \sum_{i=1}^N r_i^3 + \frac{1}{12} \sum_{i=1}^N r_i^4 + \dots \quad (8)$$

From this it is clear that the swap variance test exploits the impact of jumps on the third and higher order moments of asset returns. This is in line with a number of other papers in this area, particularly Ait-Sahalia and Jacod (in press) and Johannes (2004) (see also Bandi and Nguyen (2003)). Moreover, because $\text{SwV}_N - \text{RV}_N = \frac{1}{3} \sum_{i=1}^N \bar{r}_i^3$ where \bar{r}_i is between 0 and r_i , the difference between the swap variance and realized variance measures tends to be positive with positive jumps in the testing interval, and negative with negative jumps. As such, it is a two-sided test.

As already mentioned above, with discretely sampled data, we require a distribution theory on the proposed jump test in order to establish significance. The theorem below provides various versions of the swap variance jump test statistic as well as their asymptotic distributions.

Theorem 2.1 (Swap variance jump tests). *For the price process specified in Eq. (1) with the assumptions that (a) the drift α_t is a predictable process of locally bounded variation, and (b) the instantaneous variance V_t is a well-defined strictly positive càdlàg semimartingale process of locally bounded variation with $\int_0^T V_t dt < +\infty, \forall T > 0$, and under the null hypothesis of no jumps, i.e. $H_0 : \lambda_t = 0$ for $t \in [0, T]$, we have as $N \rightarrow \infty$*

(i) the difference test:

$$\frac{N}{\sqrt{\Omega_{\text{SwV}}}} (\text{SwV}_N - \text{RV}_N) \xrightarrow{d} \mathcal{N}(0, 1) \quad (9)$$

(ii) the logarithmic test:

$$\frac{V_{(0,1)} N}{\sqrt{\Omega_{\text{SwV}}}} (\ln \text{SwV}_N - \ln \text{RV}_N) \xrightarrow{d} \mathcal{N}(0, 1) \quad (10)$$

(iii) the ratio test:

$$\frac{V_{(0,1)} N}{\sqrt{\Omega_{\text{SwV}}}} \left(1 - \frac{\text{RV}_N}{\text{SwV}_N} \right) \xrightarrow{d} \mathcal{N}(0, 1) \quad (11)$$

where $\Omega_{\text{SwV}} = \frac{1}{9} \mu_6 X_{(0,1)}$, $X_{(a,b)} = \int_a^b V_u^3 du$, and $\mu_p = E(|x|^p)$ for $x \sim \mathcal{N}(0, 1)$.

Proof. See Appendix A. ■

⁴ This terminology of Swap Variance or SwV is deliberately chosen not to confuse the quantity with the variance swap contract itself and to be in line with the terminologies of RV, BPV, IV, QV, etc.

The assumptions imposed on the price process in Theorem 2.1 ensure local boundedness conditions on the drift and diffusion functions, which are satisfied in all concrete models. The assumptions also ensure that integrals of the drift and diffusion functions are well defined, see, e.g., Jacod and Shiryaev (2003). Further, the assumptions are similar to those in Barndorff-Nielsen et al. (2005). As detailed later, feasible implementation of the SwV test makes use of the multi-power variations developed in Barndorff-Nielsen and Shephard (2004) and Barndorff-Nielsen et al. (2005). In particular, we note that Barndorff-Nielsen et al. (2005) has extended earlier results on BPV in Barndorff-Nielsen and Shephard (2004, 2006) by relaxing the restriction of “leverage effect” on the asset return process. Thus, the SwV test allows for leverage effect or a contemporaneous relation between dy_t and dV_t . The assumptions on instantaneous variance process are similar to those in Ait-Sahalia and Jacod (in press), and accommodate stochastic volatility models commonly specified in the literature including those with jumps.

The three versions of the SwV test mirror those available for the bi-power variation (BPV) test proposed by Barndorff-Nielsen and Shephard (2004, 2006). The motivation for considering the logarithmic- and ratio-type tests is that these are generally found to have better finite sample properties. While the structure of the SwV test is very similar to that of the BPV test, the underlying logic is fundamentally different: the BPV test attempts to detect jumps by comparing RV to the *jump robust* bi-power variation quantity involving the product of contiguous absolute returns, whereas the SwV test does so by comparing RV to the *jump sensitive* swap variance quantity involving cubed returns in the leading term. Put differently, the BPV test is based on second order moments while the SwV is based on the third and higher order moments. As a consequence, the convergence rate of the SwV test is of order N , compared to \sqrt{N} for the BPV tests, and the simulation results below illustrate that the power of SwV generally dominates that of the BPV test.

SwV test statistics in Theorem 2.1 are infeasible because they depend on the latent quantities $V_{(0,1)}$ and $X_{(0,1)}$. Analogous to the BPV test, feasible versions of the SwV test can be obtained by replacing these quantities with consistent and jump robust estimates based on the concept of multi-power variation developed in Barndorff-Nielsen et al. (2005) and Barndorff-Nielsen et al. (in press). In particular, $V_{(0,1)}$ can be estimated using bi-power variation:

$$BPV_N = \mu_1^{-2} \frac{N}{N-1} \sum_{i=1}^{N-1} |r_i r_{i+1}| \tag{12}$$

whereas estimates of Ω_{SwV} can be obtained using multi-power variation:

$$\widehat{\Omega}_{SwV}^{(p)} = \frac{\mu_6}{9} \frac{N^3 \mu_{6/p}^{-p}}{N-p+1} \sum_{i=0}^{N-p} \prod_{k=1}^p |r_{i+k}|^{6/p} \tag{13}$$

for $p \in \{1, 2, \dots\}$. Clearly, $\widehat{\Omega}_{SwV}^{(4)}$ and $\widehat{\Omega}_{SwV}^{(6)}$ are the obvious candidates for the robust estimation of Ω_{SwV} .

To conclude, we point out that the SwV test is developed under the null hypothesis of no jumps. Expressions of the test statistic under the alternative are not readily available. When jumps are realizations of a countable process such as Poisson process, the test statistic is a function of the realized jumps. Specifically, the BPV statistic is a function of jump variance, whereas the SwV statistic is a function of higher order moments of jumps. Thus, the distribution of the test statistic under the alternative is dependent on the jump process. To our knowledge, the family of tests proposed by Ait-Sahalia and Jacod (in press) is the only one in the literature that is developed under both alternatives (i.e. jumps and no jumps).

2.1. Finite sample properties of the SwV jump test

Below, we investigate the finite sample properties of the proposed SwV jump test using simulations. We compare all our results with the BPV jump ratio-test of Barndorff-Nielsen and Shephard (2004, 2006) because this is the natural alternative in the current setting, i.e.

$$\frac{V_{(0,1)} \sqrt{N}}{\sqrt{\Omega_{BPV}}} \left(1 - \frac{BPV_N}{RV_N} \right) \xrightarrow{d} \mathcal{N}(0, 1) \tag{14}$$

where $\Omega_{BPV} = (\pi^2/4 + \pi - 5) Q_{(0,1)}$ and $Q_{(a,b)} = \int_a^b V_u^2 du$. When implementing jump tests, we concentrate exclusively on the *feasible* versions, that is, those evaluated using an asymptotic variance estimate based on observed returns instead of the latent variance path.⁵ To simulate the price process in Eq. (1), we use an Euler discretization scheme and specify the stochastic variance (SV) component as the Heston (1993) square-root process, i.e.

$$dV_t = 20(0.04 - V_t) dt + 0.75 \sqrt{V_t} dW_t^V. \tag{15}$$

The choice of SV parameters is guided by the empirical estimates available in the literature (e.g. Andersen et al. (2002) and Bakshi et al. (1997)). Using VIX data, Bakshi et al. (2006) estimate a mean reversion coefficient of 8 and volatility of volatility coefficient of 0.43. So the values used in this paper generate a somewhat less persistent and more erratic variance process. For simplicity, we set $\alpha_t = 0$ in Eq. (1) because reasonable specifications of the drift component will not have a discernable impact on the test performance, particularly at high intra-day frequencies. In addition, we assume that the Brownian motion driving the variance process is independent of the one driving the returns process: the impact of leverage will be investigated in the robustness analysis below. To gauge the variability of the variance process, we compute the ratio of maximum over the minimum volatility attained within the day based on simulated paths. For the parameter values used here, this ratio is equal to 1.25 and thus comparable in magnitude to the typical diurnal variation of volatility (see for instance Engle (2000, Figure 4)). In the robustness analysis below, we consider alternative variance dynamics where this ratio is about 3 reflecting a substantially more volatile process.

All simulation results reported below are based on 100,000 replications to ensure high accuracy.

2.1.1. Size of the SwV jump test

To examine the size of the SwV test, we simulate the price process as discussed above, with $J_t = 0$ for $t \in [0, 1]$. With regard to the sampling frequency we consider three scenarios, namely $N = \{26, 78, 390\}$ corresponding to 15-, 5-, and 1-min data over a 6.5 h trading day respectively. Table 1 reports the standard deviation, skewness, and kurtosis of the jump test distributions under the null hypothesis of no jumps. Fig. 1 contains the QQ plots for the ratio-test. For comparison, analogous results for BPV test are also reported.

A number of observations can be made. For small sample sizes the SwV test distribution is heavy tailed and has a variance greater than 1. Both findings are not surprising given that the calculation of the feasible test statistic involves division by integrated sixthicity, a quantity that is difficult to estimate. Based on so few observations, there is likely to be a substantial amount of measurement error so that, by Jensen’s inequality, we expect all even moments such

⁵ Unreported simulation results show that when the sample size is reasonable and the variance process not overly erratic, the performance of the feasible test is close to that of the infeasible one indicating that asymptotic variance estimation is not an impediment.

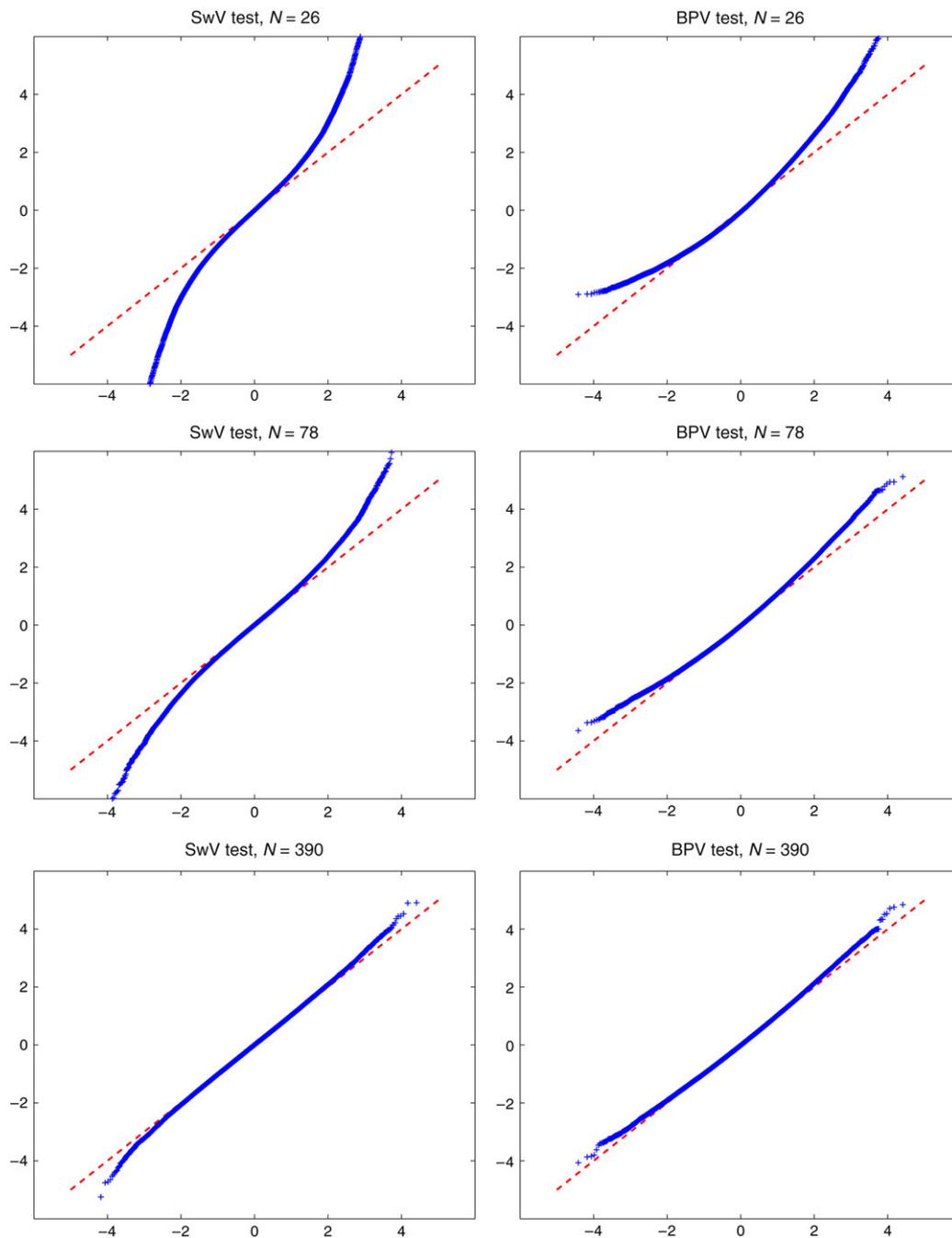


Fig. 1. QQ-plots of the feasible SwV and BPV jump ratio-test statistics under the null of no jumps.

Table 1
Summary statistics of the SwV and BPV test distribution under the null hypothesis of no jumps

	Standard deviation			Skewness			Kurtosis		
	diff	log	ratio	diff	log	ratio	diff	log	ratio
Panel A: SwV test									
N = 26	1.77	1.48	1.48	-0.37	-0.11	-0.12	20.23	9.42	9.42
N = 78	1.22	1.16	1.16	-0.01	0.00	-0.00	4.84	3.91	3.91
N = 390	1.04	1.03	1.03	0.00	0.00	-0.00	3.28	3.16	3.16
Panel B: BPV test									
N = 26	1.37	1.21	1.12	1.93	1.16	0.61	10.19	5.44	3.58
N = 78	1.11	1.07	1.04	0.94	0.63	0.35	4.59	3.65	3.13
N = 390	1.02	1.01	1.01	0.42	0.29	0.17	3.32	3.15	3.05

as the variance and kurtosis to be overestimated. The log- and ratio-tests partially alleviate this. When the sample size grows,

both the variance and the kurtosis rapidly converge to values consistent with the asymptotic standard normal distribution. This

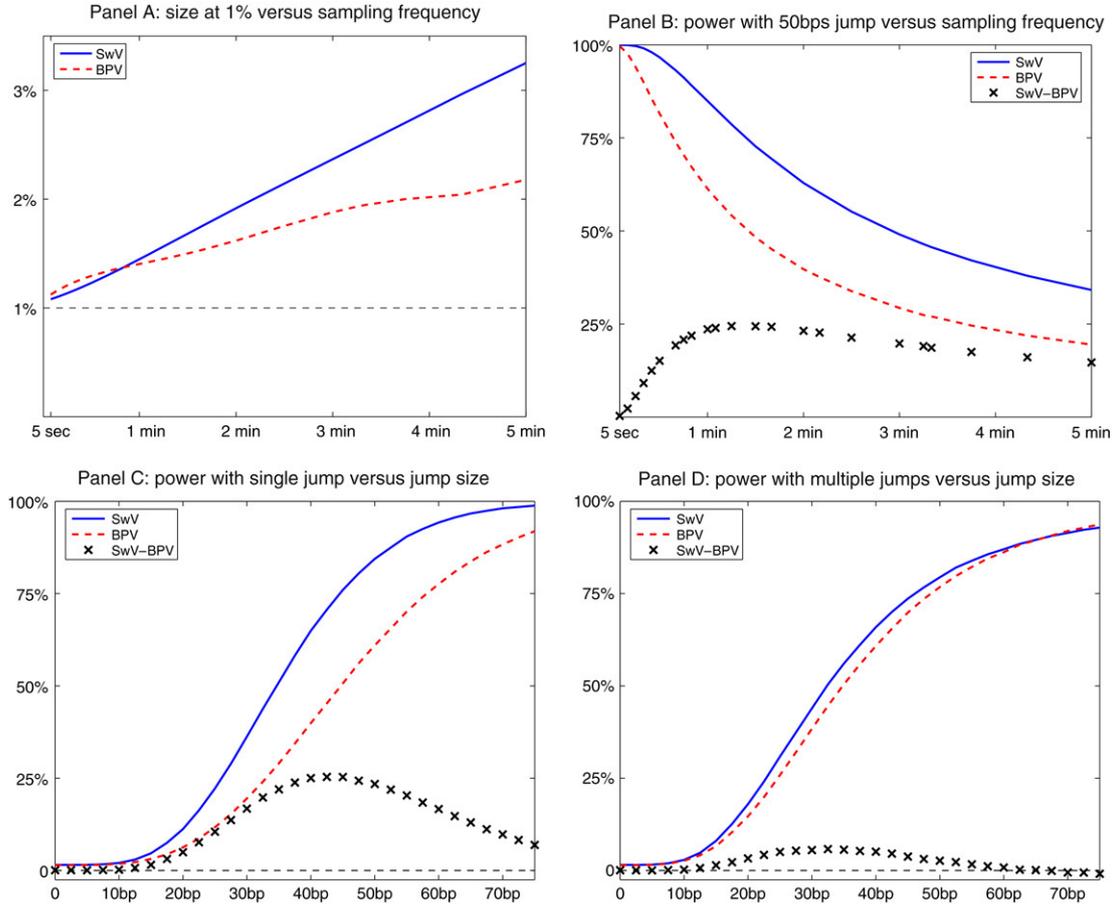


Fig. 2. Size and power of the feasible SwV jump test.

is confirmed by the QQ plots in Fig. 1. In comparison, the BPV test shows similar distortions for small sample size, albeit of lesser magnitude. Consistent with the simulation results of Huang and Tauchen (2005) for the BPV test, we also find that the logarithmic- and ratio-versions of the test have better finite sample properties than the difference test. Importantly, at a one-minute frequency or above, both the SwV and BPV test distributions are remarkably close to their asymptotic counterparts. Motivated by this, we exclusively focus on the ratio tests in the remainder of this paper.

To get a better idea of the magnitude of size distortion in finite sample, Panel A of Fig. 2 plots the 1% size of the feasible jump ratio-tests for sampling frequencies between 5 seconds (i.e. $N = 4680$) and 5 min (i.e. $N = 78$). Both tests are somewhat oversized: the SwV has a larger distortion than the BPV at low frequencies but also converges more rapidly so that at sampling frequencies of 1 minute and up both tests have similar size properties.

2.1.2. Power of the SwV jump test

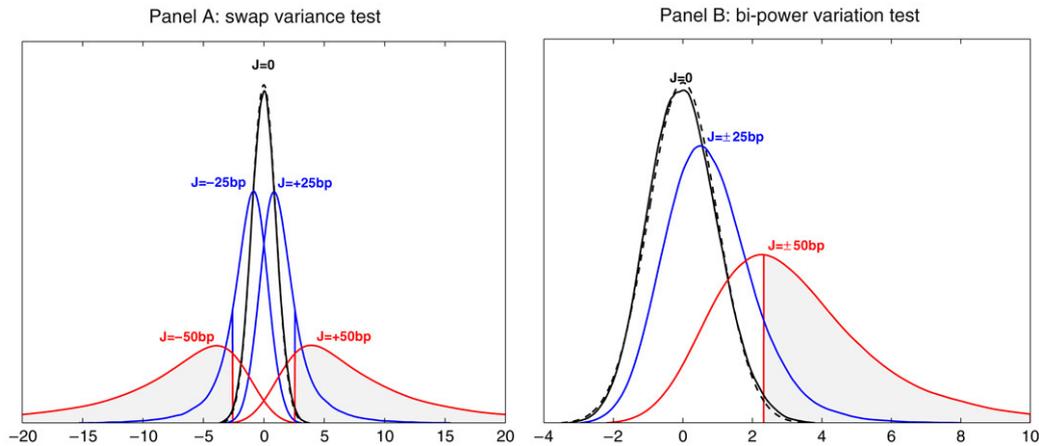
To examine the properties of the SwV ratio-test under the alternative hypothesis we simulate the price process but now add jumps to the simulated price path. Here, we consider three different simulation scenarios, namely

- (i) A single jump with random sign and fixed size of 50 basis points (bps), randomly placed in the sample. The sampling frequency is varied between 5 min ($N = 78$) and 5 s ($N = 4680$).
- (ii) A single jump with random sign and size varying between 0 and 75 bps, randomly placed in the sample. The sampling frequency is fixed at one-minute ($N = 390$).

- (iii) A random number of jumps, with random sign and size, randomly placed in the sample. The sampling frequency is fixed at one-minute ($N = 390$), the expected number of jumps is 2 (with a variance of 1), while the jump size $|J| = \mu(1 + \varepsilon/4)$ where ε is a standard normal random variable and μ is varied between 0 and 75 bps.

In the presence of jumps, robust estimation of the asymptotic variance is key: the conventional integrated sixticity estimator involves sixth powers of returns that makes it upward biased so that the power of the test can deteriorate substantially. Thus, in our simulations we implement the feasible jump test using the jump robust estimator in Eq. (13) with $p = 6$. A similar issue arises for the BPV test so we estimate the integrated quarticity using quad-power variation. Unreported simulation results indicate that (i) if non-robust estimators are used, the power is virtually zero for both tests, (ii) using a different robust estimator, e.g. $\hat{\Omega}_{SwV}^{(4)}$ or tri-power variation for integrated quarticity, makes little difference to the performance of the tests, (iii) deterioration in power associated with the feasible test, relative to the infeasible one, is limited and minimal with realistic sample sizes.

Panels B–D of Fig. 2 plot the power of the feasible SwV ratio test for the three different jump scenarios described above. As a benchmark, the corresponding BPV results are added as well. The results can be summarized as follows. For a single jump with fixed size (scenario (i), Panel B) the SwV test is uniformly more powerful than the BPV test across all sampling frequencies considered. The difference in performance can be quite substantial. At a low 5-min frequency the difference in size distortion between the SwV and BPV test is about 1% but the SwV test has almost 15% more power, detecting about 1 out of every 3 jumps. When the sampling



Note. The dashed line denotes the standard normal distribution. Density estimates are calculated using the Epanechnikov kernel with Silverman optimal bandwidth.

Fig. 3. Distribution of feasible jump ratio-test statistics under null and alternative.

frequency increases, the absolute gain in power of the SwV test grows and peaks at about 25% at a sampling frequency between 1 and 2 min. Beyond this, the power of both tests rapidly converge to unity. To further illustrate the above, Fig. 3 plots the distribution of the SwV and BPV tests in the absence and presence of jumps. The two-sided nature of the SwV test is evident, taking on negative values with negative jumps and vice versa. More importantly, the SwV test is much more sensitive to the presence of jumps than BPV, with the test statistic taking on larger values and a larger fraction exceeding the critical value of the test. As already discussed above, this can be understood by noting from Eq. (8) that the SwV test primarily uses third order moments that are more sensitive to jumps than the second order moments exploited by the BPV test.⁶

Considering the case with a single jump at a fixed sampling frequency of one-minute (scenario (ii), Panel C), we again find that the SwV test is uniformly more powerful than the BPV test across jump sizes. The difference in power between the two tests is often considerable: with a jump of 40 bps, the power of the SwV test is 65%, compared to 40% for BPV. Even with large jumps of 75 bps, the BPV test misses about 1 in 10 jumps whereas the SwV test detects virtually each one of them.

Finally, with multiple random jumps (scenario (iii), Panel D) the power of the SwV test is comparable to that of BPV across the expected jump size. In simulations, the average number of jumps is equal to 2. If we further increase this then the BPV test becomes more powerful than the SwV test. This can be understood by observing that in the presence of jumps the power of the SwV test primarily comes from the leading term in Eq. (5) which is proportional to $\int_0^t J_t^3 dq_t$, compared to $\int_0^t J_t^2 dq_t$ for the bi-power variation test. Thus, with multiple jumps of differing sign, the SwV test loses power because the cubed terms will, at least partially, offset each other thereby reducing the value of the test statistic. It is noted, however, that it is widely believed that jumps are a rare occurrence and thus from a practical viewpoint the scenario with multiple jumps over relatively short time horizons as considered here is of limited interest.

2.2. Robustness analysis

To assess the robustness of the SwV test performance, we consider (i) the “leverage effect” and (ii) alternative variance dynamics. For leverage, we introduce a correlation of -75% between Brownian motions driving the variance and return dynamics, i.e. $E(dW_t dW_t^v) = -0.75dt$ in Eqs. (2) and (15). This level of correlation is in line with empirical estimates and close to the value used in the simulation study by Huang and Tauchen (2005). For the alternative variance specification, we follow Lee and Mykland (in press) and adopt the general SEV-ND model introduced by Ait-Sahalia (1996) which accommodates stochastic elasticity of variance and non-linear drift and take parameter values from Bakshi et al. (2006, Table 2):

$$dV_t = (-0.554 + 21.322V_t - 209.348V_t^2 + 0.005V_t^{-1})dt + \sqrt{0.017V_t + 53.973V_t^{2.882}}dW_t. \quad (16)$$

Fig. 4 reports the size and power of the feasible SwV jump test as a function of the sampling frequency. The results are compared to the benchmark case, i.e. SV process as in Eq. (15) with no “leverage effect”. Confirming the theoretical results in Theorem 2.1, we find that inclusion of leverage has no noticeable impact on the size or power of the SwV test. With alternatively variance dynamics as specified by the SEV-ND model we observe a substantial deterioration of size and limited deterioration of power. The specification in Eq. (16) produces sample paths of the variance process that are much more erratic than those of the SV model used previously. Because the SwV test requires an estimate of integrated sixthicity, which is very challenging in this setting, the observed deterioration of performance is perhaps not that surprising. Importantly, however, at empirically reasonable sampling frequencies of 1 min or so, the size distortion is less than 2% and the power more than 75%.

3. The SwV jump test in the presence of market microstructure noise

In practice, an important complication that arises with the use of high frequency data for the purpose of realized variance calculation, or indeed jump identification, is the emergence of market microstructure noise. Niederhoffer and Osborne (1966) is one of the first studies to recognize that the existence of a bid-ask spread leads to a negative first order serial correlation in observed

⁶ Extending this logic, one might be tempted to construct supposedly even more powerful tests using say the sixth order moment. But in doing so one has to keep in mind that the variance of the test statistic will include a term involving the integrated variance process raised to the power six. Thus, the feasible implementation of such a test will be extremely challenging and the power gain may be offset by deterioration in the estimate of the asymptotic variance of the test statistic.

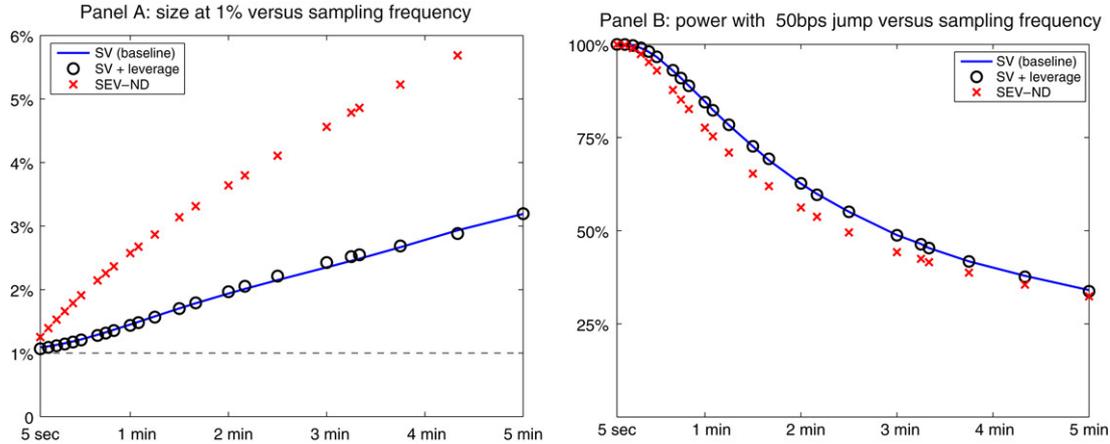


Fig. 4. Robustness analysis for feasible SwV test.

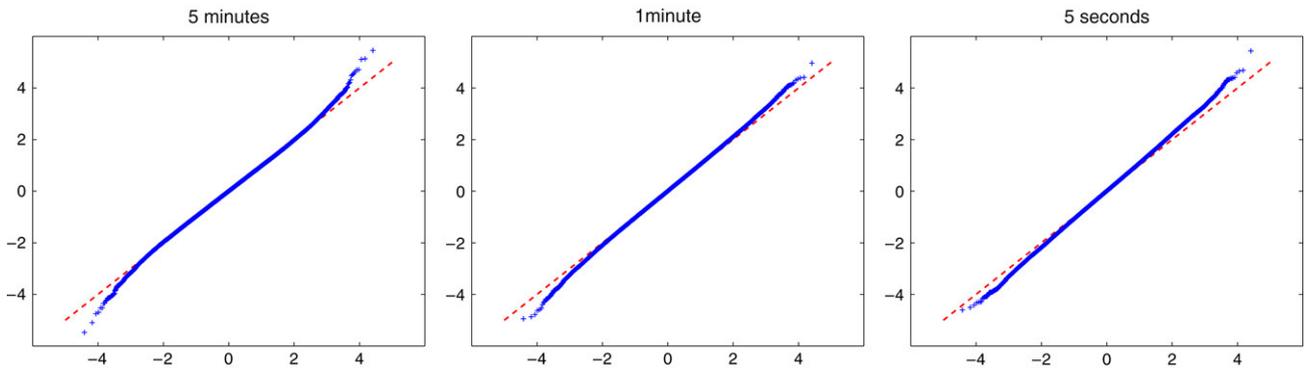


Fig. 5. QQ-plots of the feasible noise-corrected SwV jump ratio-test statistics with i.i.d. microstructure noise.

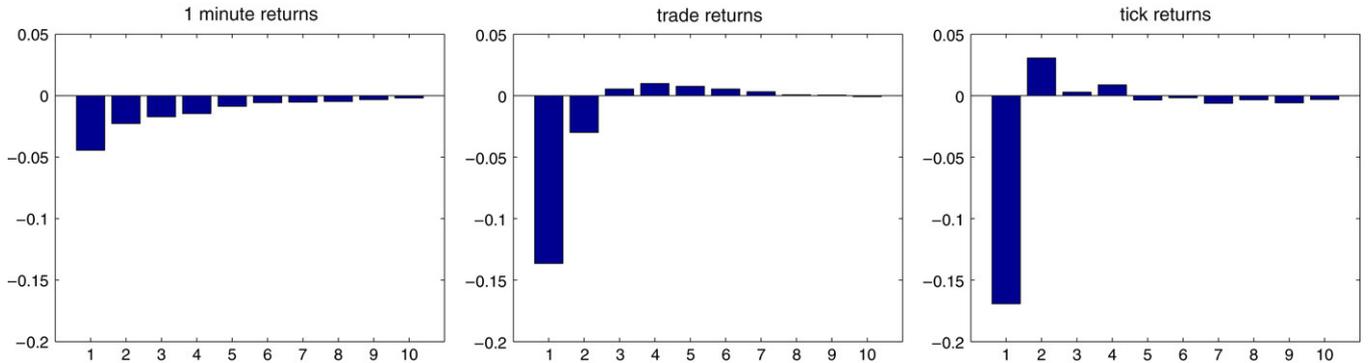


Fig. 6. Autocorrelation of IBM returns on different time scales.

returns (see also Roll (1984)). The impact that these and other microstructure effects have on realized variance has recently been studied in detail and is now well understood (see for instance Aït-Sahalia et al. (2005a,b), Bandi and Russell (2006), Barndorff-Nielsen et al. (in press), Hansen and Lunde (2006), Christensen et al. (in press), Large (2005), Oomen (2005, 2006b), Zhang (2006), Zhang et al. (2005) and Zhou (1996)). However, the impact of market microstructure noise on the BPV jump test is, as pointed out by Barndorff-Nielsen and Shephard (2006), currently an open question.⁷ Also, the recently developed jump tests by Aït-Sahalia and Jacod (in press) and Lee and Mykland (in press) have not yet considered for microstructure effects. In this section, we show that

the SwV test proposed in this paper can be applied, in analytically modified form, to high frequency data contaminated with i.i.d. market microstructure noise and still retains good power. As a by-product of our analysis, we obtain novel analytical results regarding the impact of i.i.d. noise on bi-power variation. Although not pursued here, these results may be used to adapt the tests of Barndorff-Nielsen and Shephard (2006) and Lee and Mykland (in press) to a setting with noise.

Regarding the noise specification, we consider the case where the observed price y_t^* can be decomposed into an “efficient” price component y_t and an i.i.d. market microstructure noise component ε_t , i.e.

$$y_{i/N}^* = y_{i/N} + \varepsilon_i, \quad (17)$$

for $i = 0, 1, \dots, N$ and $\varepsilon_i \sim \text{i.i.d. } \mathcal{N}(0, \omega^2)$. Consistent with the presence of a bid-ask spread, Eq. (17) implies an MA(1) dependence

⁷ See Huang and Tauchen (2005) for some exploratory analysis of this issue.

structure on observed returns:

$$r_i^* = r_i + \varepsilon_i - \varepsilon_{i-1},$$

where $r_i^* = y_{i/N}^* - y_{(i-1)/N}^*$. It is noted that while the i.i.d. assumption on ε_i can be restrictive, it is widely used in the literature and provides a reasonable approximation to reality in many situations (see Hansen and Lunde (2006) for further discussion).

Theorem 3.1 (Swap Variance Test in the Presence of i.i.d Market Microstructure Noise). For the price process specified in Eq. (1) with assumptions as stated in Theorem 2.1, and in the presence of i.i.d. market microstructure noise as in Eq. (17) with $\omega^2 \ll V_{(0,1)}$, then under the null hypothesis of no jumps, i.e. $H_0 : \lambda_t = 0$ for $t \in [0, 1]$, the following test statistics have approximately zero mean and unit variance for large but finite N :

(i) the difference test:

$$\frac{SwV_N^* - RV_N^*}{\sqrt{\Omega_{SwV}^*}} \tag{18}$$

(ii) the logarithmic test:

$$\frac{V_{(0,1)}^*}{\sqrt{\Omega_{SwV}^*}} (\ln SwV_N^* - \ln RV_N^*) \tag{19}$$

(iii) the ratio test:

$$\frac{V_{(0,1)}^*}{\sqrt{\Omega_{SwV}^*}} \left(1 - \frac{RV_N^*}{SwV_N^*} \right) \tag{20}$$

where $V_{(0,1)}^* = V_{(0,1)} + 2N\omega^2$, $\Omega_{SwV}^* = 4N\omega^6 + 12\omega^4V_{(0,1)} + 8\omega^2\frac{1}{N}Q_{(0,1)} + \frac{5}{3}\frac{1}{N^2}X_{(0,1)}$, and SwV_N^* and RV_N^* are computed using the contaminated prices y^* .

Proof. See Appendix A. ■

In the proof we show that $\text{plim}_{N \rightarrow \infty} (SwV_N^* - RV_N^*)/N \rightarrow \omega^4$ which illustrates that, in the limit, the test statistic diverges. The more interesting case, however, is as described in Theorem 3.1. Here N is large but finite – it is explicitly not an asymptotic result – and the noise has an impact on the test statistic but it does not dominate it. In particular, considering the Taylor series expansion of the SwV test in Eq. (8), the impact of noise is primarily on the second order term involving quadratic returns. Finite sample adjustment essentially accounts for this. The assumption that the noise variance ω^2 is of smaller magnitude than the integrated variance $V_{(0,1)}$ allows us to drop a number of terms that are not important in practice and obtain the relatively compact expression for Ω_{SwV}^* . We will show below that, with these adjustments, the test retains good power to detect jumps in empirically realistic scenarios.

3.1. Feasible implementation of the SwV* test

The critical issue for the implementation of the feasible noise adjusted SwV jump test is to obtain a good estimate of Ω_{SwV}^* , i.e. one that is robust to jumps and incorporates the impact of market microstructure noise correctly at the same time. A natural way of estimating Ω_{SwV}^* is to estimate each of its components separately, i.e. ω^2 , $V_{(0,1)}$, $Q_{(0,1)}$, and $X_{(0,1)}$.

Estimates of the market microstructure noise variance ω^2 can be relatively straightforward to obtain. For instance, Bandi and Russell (2006) propose $RV_N^*/(2N)$ as a consistent estimator of the noise variance. However, in finite sample this estimator can be

severely biased. Thus, in this paper we use the autocovariance-based noise variance estimator proposed by Oomen (2006b):

$$\hat{\omega}^2 = -\frac{1}{N-1} \sum_{i=1}^{N-1} r_i^* r_{i+1}^* \tag{21}$$

It is easy to see that this estimator is unbiased with i.i.d. noise and robust to jumps in the same way that the BPV quantity is (see Oomen (2006a) for further discussion). Here, returns at the highest sampling frequency can be used to maximize estimation accuracy.

Computing robust but accurate estimates of the integrated variance $V_{(0,1)}$ (and $Q_{(0,1)}$, and $X_{(0,1)}$ alike) is much more challenging because we need to avoid, or correct for, the impact of jumps as well as market microstructure noise. In a related context, Bandi and Russell (2006) suggest the use of realized variance computed using data at sampled at low frequency to obtain estimates of the integrated variance free of noise. In principle a similar approach could be taken here, with the only difference that since we require robustness to jumps, bi-power variation should be used instead of realized variance. In this paper we propose an alternative approach that makes more efficient use of the available data. In particular, we first compute the bi-power variation using noisy data at the highest frequency, i.e. BPV_N^* to get an estimate of $V_{(0,1)}$. This estimate is robust to jumps but remains biased as it is based on noise contaminated returns. In the second step, we then correct for this bias based on the following result regarding the impact of i.i.d. market microstructure noise on bi-power variation quantity.

Proposition 3.2 (Bias Correction for BPV in the Presence of i.i.d Market Microstructure noise). Under the conditions as specified in Theorem 3.1, and with constant return variance \bar{V} over the interval $[0, 1]$, we have:

$$E [BPV_N^*] = (1 + c_b(\gamma))E [BPV_N], \tag{22}$$

where

$$c_b(\gamma) = (1 + \gamma) \sqrt{\frac{1 + \gamma}{1 + 3\gamma}} + \gamma \frac{\pi}{2} - 1 + 2 \frac{\gamma}{(1 + \lambda) \sqrt{2\lambda + 1}} + 2\gamma\pi\kappa(\lambda), \tag{23}$$

with $\gamma = N\omega^2/\bar{V}$, $\lambda = \frac{\gamma}{1+\gamma}$, $\kappa(\lambda) = \int_{-\infty}^{\infty} x^2 \Phi(x\sqrt{\lambda})(\Phi(x\sqrt{\lambda}) - 1)\phi(x) dx$, and $\Phi(\cdot)$ and $\phi(\cdot)$ are the CDF and PDF of the standard normal respectively. The expectation in Eq. (22) is conditional on \bar{V} and γ . BPV_N^* and BPV_N denote bi-power variation computed from noise contaminated and clean return data respectively.

Proof. See Appendix A. ■

In the above, the function $c_b(\gamma)$ in Eq. (23) measures the impact of i.i.d. market microstructure noise on BPV and, as such, provides the bias correction for the bi-power variation calculated from market microstructure noise contaminated returns.⁸

For the estimation of $Q_{(0,1)}$ and $X_{(0,1)}$ in a jump-robust and noise-adjusted fashion, we may take a similar approach and bias correct quad-power variation and six-power variation. In particular, under the assumptions specified in Proposition 3.2 it

⁸ It is noted that the above results rely on the assumption that the return variance (and hence the noise ratio γ) is constant over the interval of interest. It is obvious from the proof that the results can be generalized to the case where the noise ratio is constant but both return variance and noise are time varying. However, when return variance varies over time and the noise is constant, the impact is more complicated and the bias correction becomes much more cumbersome.

Table 2
Properties of the jump statistics with i.i.d. market microstructure noise

Sampling frequency	Panel A: Standard statistic				Panel B: Noise corrected statistic			
	Feasible SwV		Feasible BPV		Infeasible SwV*		Feasible SwV*	
	Size (%)	Power (%)	Size (%)	Power (%)	Size (%)	Power (%)	Size (%)	Power (%)
5 s	0.00	8.76	0.00	0.00	2.04	60.60	1.92	58.50
15 s	0.02	32.05	0.00	0.00	1.79	71.62	1.81	70.24
1 min	0.33	41.64	0.16	4.20	1.47	58.17	1.52	55.95
5 min	2.41	23.10	1.81	10.81	1.24	25.32	1.09	19.97

can be shown that when quad-power variation is calculated from noisy data as an estimate of integrated quarticity its bias is $c_q(\gamma) \approx 5.46648\gamma^2 + 4\gamma$. Similarly, for six-power variation as an estimate of integrated sexticity the bias is $c_x(\gamma) \approx 13.2968\gamma^3 + 14.4255\gamma^2 + 6\gamma$. Because, particularly with noisy data, the quality of quarticity and sexticity estimates may be poor, in this paper we estimate $Q_{(0,1)}$ and $X_{(0,1)}$ simply as the squared and cubed estimate of the integrated variance described above. Unreported simulation results indicate that for reasonable parameter values and sample sizes this ad hoc approach works well.

3.2. Finite sample properties of the noise adjusted SwV jump test

To gauge the finite sample properties of the noise adjusted SwV test proposed in Theorem 3.1, with a feasible implementation relying on noise corrected BPV estimates, we conduct further simulation experiments. The volatility process is specified as in Eq. (15). For simplicity, we rule out leverage here since the test has been shown to be robust to this and the effect of noise will dominate in any case. The noise variance parameter ω^2 is set equal to $\frac{0.2}{2 \times 78} \frac{0.04}{252}$, corresponding to a noise volatility of about 4.5 bps. With such noise levels, we expect a 20% bias in realized variance, or roughly a 45% bias in realized volatility, calculated from 5 minute returns.

Table 2 reports the size and power for the various jump tests in the presence of noise. First, consider the scenario where we apply the unadjusted jump tests to noisy data (panel A). The size of the SwV test tends to zero as does the power, albeit that at moderate frequencies the SwV test still has some ability to detect jumps. For the BPV test both the size and the power rapidly vanish. This can be understood better from the results presented above. Note from Eq. (23) that for low sampling frequencies (or small values of γ), BPV “behaves” like RV since the slope of $c_b(\gamma)$ is close to 2. However, when the sampling frequency increases (and the noise ratio grows) we have:

$$\lim_{N \rightarrow \infty} \frac{BPV_N^*}{\gamma} = \frac{2}{\sqrt{3}} + \frac{\pi}{2} + 2\pi\kappa(1) \approx 2.2556,$$

compared to

$$\lim_{N \rightarrow \infty} \frac{RV_N^*}{\gamma} = 2.$$

This illustrates that BPV is slightly more sensitive to i.i.d. market microstructure noise⁹ than RV. As a consequence, when computing the BPV test on noisy data, we see the power disappear because the statistic diverges (i.e. for large N we have $RV_N^* - BPV_N^* \approx -0.25\gamma = -0.25N\omega^2/V_{(0,1)}$).

⁹ To mitigate the impact of noise on BPV, Andersen et al. (2007) and Huang and Tauchen (2005) have suggested to use staggered returns, i.e. $\sum |r_t^*| |r_{t-2}^*|$. Based on the results presented here it is easy to see that with this construction of BPV, the bias due to i.i.d. noise is equal to $1 + 2\gamma$, i.e. the same as for RV. This may explain why the results for the BPV ratio test are better when returns are staggered this way.

Turning to the performance of the noise adjusted SwV jump test (panel B of Table 2), we consider both the infeasible and the feasible version. Three observations can be made. Firstly, the size and power of the feasible and infeasible versions are quite close suggesting that the estimation of noise level and integrated variance quantities based on noise adjusted bi-power variation works well. Secondly, we detect a modest size distortion when the sampling frequency is increased. Fig. 5 draws the qq-plots of the test under the null hypothesis of no jumps at different sampling frequencies. We see that although the distribution is close to normal, it has fat tails at low frequency and slightly higher variance at 5 s frequency explaining the size distortion. Thirdly, the power of the test grows with an increase of sampling frequency up to 15 s and then subsequently drops when the sampling frequency is increased further and the noise starts to dominate.

4. An empirical illustration

As an illustration of our proposed SwV jump test, we conduct a small scale empirical exercise using high frequency IBM trade data. Below, we consider the standard SwV jump test, the noise adjusted SwV jump test, as well as the BPV jump test for comparison. We start by applying these tests to sparsely sampled data, i.e. data aggregated to a frequency where the impact of microstructure noise is limited and the BPV test is still valid. The results here will provide insights into the performance of SwV relative to BPV. Next, we apply the jump tests to returns sampled at the highest available frequency where noise is pervasive. The results here illustrate the performance of the noise adjusted SwV test compared to its unadjusted counterpart.

The IBM data used below is extracted from the TAQ database and consists of all trades that took place on the primary exchange (NYSE) over the period January 2002 through December 2006. We also retain all trades executed through NYSE Direct+ (indicated by sale condition “E”). Towards the end of the sample period, these latter trades constitute 30% of trading volume. We apply the following filtering rules, (i) remove all trades with a time stamp before 9:45 am and after 4:00 pm leaving us with a trading day of 375 minutes, (ii) remove all trades with a non-zero correction indicator, (iii) remove all trades with a non-empty sale condition different from “E”. The resulting data set contains more than 5 million observations, i.e. an average of 4661 trades per day for 1259 trading days.

4.1. Jump detection using sparsely sampled returns

To mitigate the impact of noise at this stage, we construct the equivalent of 1 minute returns in trade time, i.e. each day we sample 376 prices equally spaced in the sequence of trades. The left panel of Fig. 6 plots the autocorrelation function of returns, pooled across days. We find significantly negative first order serial correlation, but the magnitude is relatively small indicating that the level of noise in this data is limited. For each day, we compute the three feasible jump ratio-tests (i.e. SwV, SwV*, and BPV) and report the jump detection frequencies in Panel A of Table 3.

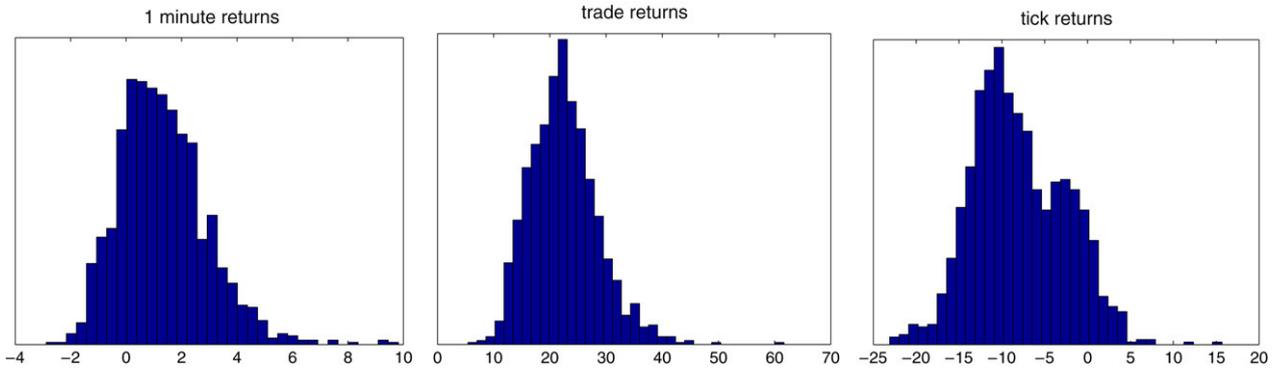


Fig. 7. Histogram of BPV ratio-test statistic based on returns of different time scales.

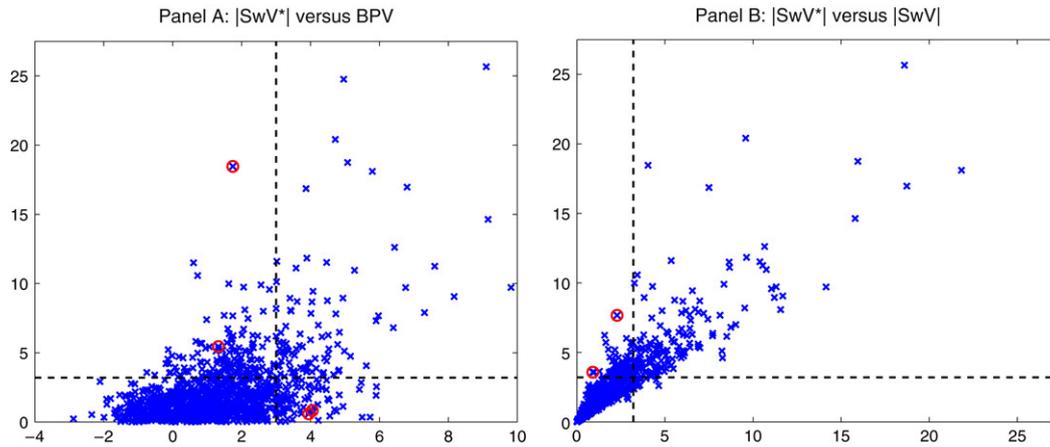


Fig. 8. Jump detection for 1 min IBM data (2002–2006).

Table 3
Jump detections for IBM stock prices (2002–2006, 1259 days)

	Panel A: 1 minute returns		Panel B: Tick returns	
	Cutoff = 3	Cutoff = 4	Cutoff = 3	Cutoff = 4
BPV	179 (0.14)	69 (0.05)	27 (0.02)	12 (0.01)
SwV	173 (0.14)	118 (0.09)	242 (0.19)	152 (0.12)
SwV*	245 (0.19)	150 (0.12)	493 (0.39)	370 (0.29)

Note. This table reports the number of days (fraction in parenthesis) identified as having jumps by the respective tests. The critical value or “cutoff” level applies to the BPV test. Because the SwV test is two-sided we use the corresponding value given by $\Phi^{-1}((1 + \Phi(c))/2)$ where Φ is the cumulative normal distribution (i.e. 3.21 and 4.16 respectively).

With a commonly used critical value equal to three (e.g. Andersen et al. (2007) and Huang and Tauchen (2005)), we find that the BPV, SwV, SwV* tests detect 179, 173, and 245 days as having a jump in the price process. With a critical value of four – focusing mainly on the large jumps – we find that the BPV, SwV, SwV* tests detect jumps roughly once a month, once every two weeks, and once every 8 days respectively. This pattern is consistent with the simulation results above, where we found that SwV* is more powerful than SwV, and SwV is more powerful than BPV. Of course, looking at the detection frequency alone is not sufficient because with spurious detection of jumps a particular test may appear more powerful than it really is. With this in mind, consider Fig. 8. In panel A, we plot the (absolute) value of the SwV* test statistic as a function of the BPV test statistic for each day in the sample. If we take the origin to be (3, 3) then the observations in the first and third quadrants of the graph indicate instances where both tests detect the presence of jumps. More importantly, the second (fourth) quadrant contains the instances where only the

SwV* (BPV) test detects jumps but the BPV (SwV*) test does not. These are days of particular interest because they provide insights into the relative properties of the competing jump tests.

In Fig. 9 we present a representative sample of such days. First consider Panel A, i.e. days where only SwV* detects a jump. On 2002/12/27 we observe multiple contiguous jumps around the 25th price observation. Such a price path violates the requirement of the BPV test for jumps to be preceded and succeeded by small “diffusive” returns. As a result, bi-power variation loses robustness to jumps in this case and the test statistic does not pick up the jump. The SwV* test, on the other hand, picks up the jump: even though there are multiple jumps, the power does not deteriorate in this case because they are of the same sign. On 2003/07/10 we observe a smallish 50 bps jump around the 120th price observation. Again, the SwV* test picks it up while the BPV test does not, reflecting the difference in power. Next, we consider some examples of days where the BPV test picks up a jump but SwV* does not (Panel B of Fig. 9). On 2002/02/27 we observe a very volatile price path with a range of almost 4% but no single clear large jump. Yet, it is conceivable that a number of small jumps may have occurred and this is clearly a scenario where the BPV test has an edge over the SwV test. Recall from the discussion towards the end of Section 2.1.2 that the SwV test suffers from a deterioration of power when the cubed jump terms (partially) offset each other. A similar pattern is observed on 2005/03/18 where numerous positive and negative jumps occur. The SwV* test statistic is 0.59 but the BPV test, not surprisingly, detects the presence of jumps.

Although the level of noise in the 1 min data is limited, we still observe a difference in jump detection frequencies between the SwV and SwV* tests. Panel B of Fig. 8 plots the (absolute) value

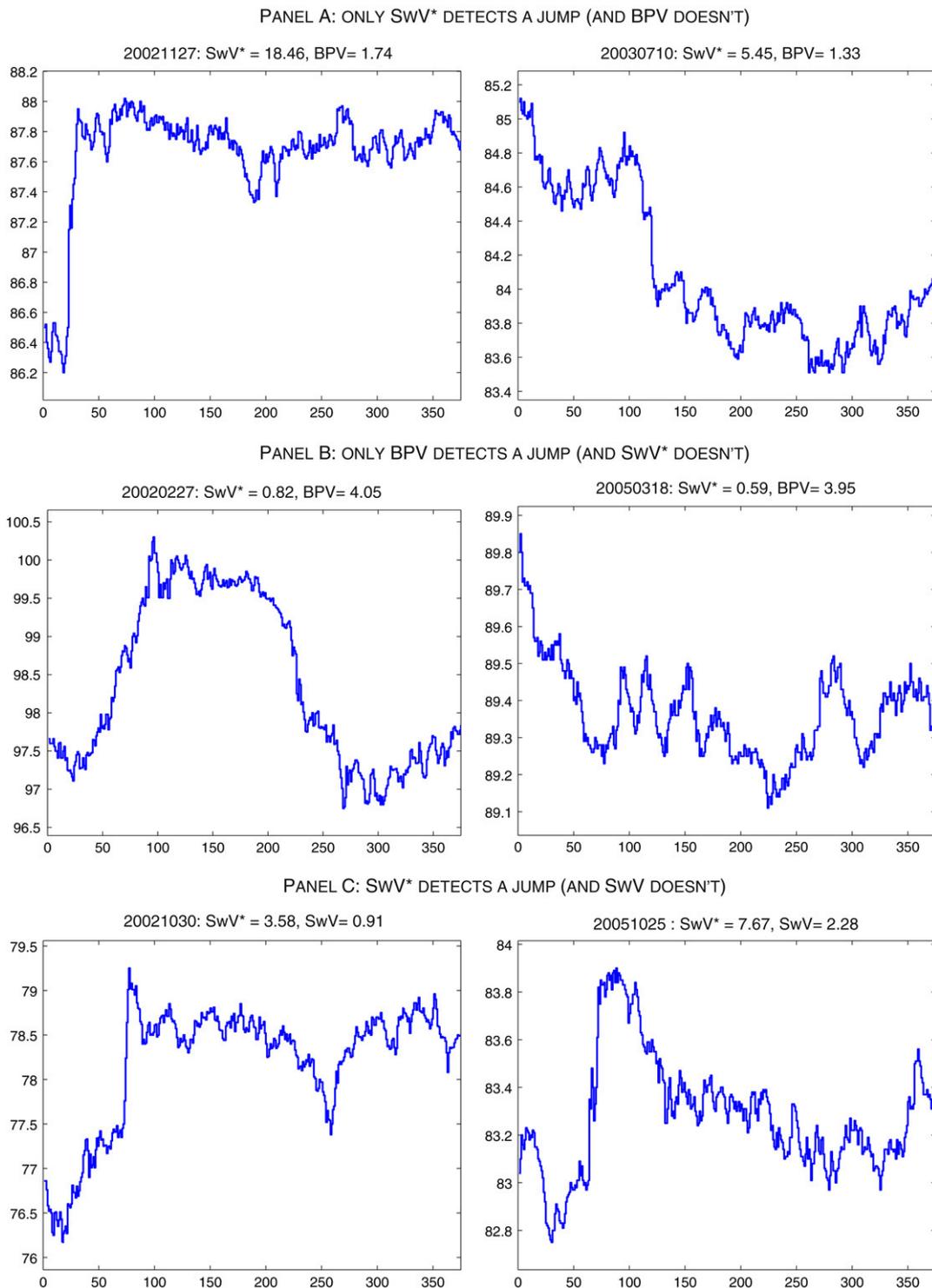


Fig. 9. Examples of jump detection discrepancies for 1 min IBM data (2002 – 2006).

of the SwV^* test as a function of the SwV test for each day in the sample. Interestingly, there are few observations (deep) in the fourth quadrant suggesting that when SwV detects a jump, SwV^* does as well. Yet, the reverse is not true. There are numerous days where the unadjusted test does not detect jumps whereas the noise adjusted test does. To illustrate that this is not due to spurious detection of jumps, we present two representative examples of two such days in Panel C of Fig. 9. On 2002/10/30 and 2005/10/25 we clearly observe large upward jumps that only the noise adjusted test manages to pick up.

Overall, the empirical results are consistent with theoretical and simulation results and agree with intuition. In particular, the SwV test appears more powerful than the BPV test in situations with a single jump or multiple jumps of the same sign due to its reliance on higher order moments that are more sensitive to jump than those employed by the BPV test. On the other hand, in scenarios with many jumps of differing sign the BPV test has an advantage over the SwV test because the power of the latter is compromised due to the cubed jump terms that appear in the leading term partially cancelling out and reducing the value of the

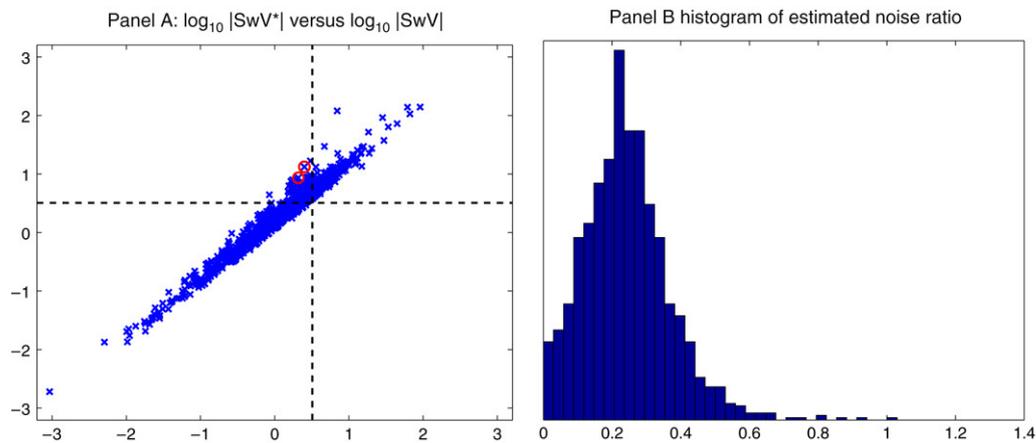


Fig. 10. Jump detection in tick time using noise adjusted SwV.

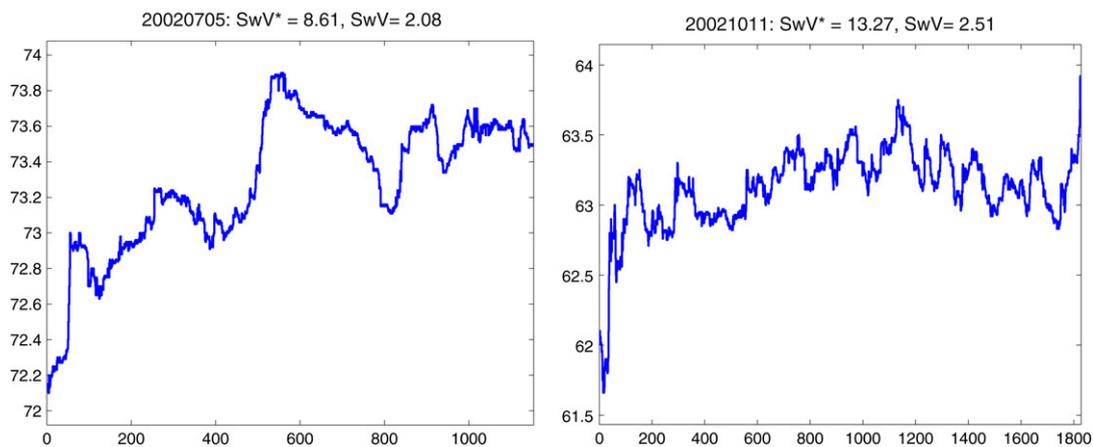


Fig. 11. Examples of jump detection discrepancies for IBM tick data (2002–2006).

test statistic. Finally, even at relatively low sampling frequencies with little noise in the data, the noise adjustment to the SwV test still appears important to retain power.

4.2. Jump detection using tick-by-tick returns

We now consider returns sampled at the highest sampling frequency, i.e. every trade. From the autocorrelations in Fig. 6 it is clear that data is contaminated by a substantial amount of noise as indicated by the highly significant and large negative first order autocorrelation coefficient. Given the discussion above, with high noise levels, we expect the BPV test to tend to take on large negative values. Surprisingly, however, the opposite is true judging from Fig. 7. Here we plot the histogram of the daily BPV test statistics for the full sample and find that the minimum value is around 5 with a mean around 25. This observation can be explained as follows. Out of an average of 4661 trade returns per day, 2323 are zero reflecting flat pricing. Thus, computing bi-power variation on such data will cause the quantity to be heavily downward biased because the multiplication of contiguous returns will be zero in about half the cases. It is therefore quite intuitive that $RV_N^* - BPV_N^*$ tends to be large and positive, and even more so in the presence of jumps. Because the implementation of the SwV* test requires reliable estimates of bi-power variation, the swap variance test on such data won't perform well either. Therefore, instead of sampling in trade time, we now sample in "tick time", i.e. we sample all observations that constitute a price change. From Fig. 6 we can see that the autocorrelation of tick returns is similar to that of trade returns, with the only qualitative difference being that the

sign of the second order autocorrelation has flipped (see Griffin and Oomen (2008) for an explanation of why this happens). More importantly, we see that – as predicted by our results above – the BPV test is now taking on large negative values and consequently its power to detect jumps vanishes.

Turning to the results for the SwV test in Table 3, we observe that the noise adjusted test identifies almost twice as many days with jumps as its unadjusted counterpart does. Intuitively, with high levels of noise in tick data, noise correction becomes increasingly important and the power gain of the SwV* test increases. Panel A of Fig. 10 draws the cross plot of SwV* test realizations as a function of the SwV test statistic for all days in the sample. Interestingly, there is not a single observation in the fourth quadrant indicating that on all days that the SwV test detects jumps, SwV* does so as well. To illustrate that spurious jump detection is not of prime concern, Fig. 11 presents two examples of typical days where the SwV* test detects a jump and the SwV test does not. On 2002/07/05 and 2002/10/11, clear jumps can be observed and only after applying the noise correction does the SwV test pick it up. Again, the results here indicate the importance of a noise adjustment when applying the jump test, particularly when data is sampled at high frequency with noise.

5. Conclusion

This paper develops a new test for the presence of jumps. The proposed test is easy to implement, is designed for use with high frequency data, exploits the third and higher order return moments making it more powerful than the bi-power variation test in many circumstances, can be applied in analytically modified

form to microstructure noise contaminated data, and has a nice interpretation in the context of the literature on variance swaps - hence the name “Swap Variance” test. Simulations as well as empirical results show that the test performs well and is able to detect jumps even when data is sampled at the highest available frequency where noise is pervasive.

Throughout the paper, we have compared our results to the widely used bi-power variation test of Barndorff-Nielsen and Shephard (2004, 2006). Recently, however, a number of alternative jump tests have been proposed in the literature (e.g. Ait-Sahalia and Jacod (in press), Lee and Mykland (in press), Mancini (2006) and Fan and Wang (2007)) and a comprehensive comparison would be interesting. In particular, it is important to understand the relative performance of these jump tests when applied to noisy data as well as in scenarios with finite and infinite activity jumps. Such an analysis is well beyond the scope of the current paper and we leave it for future research.

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Appendix. Proofs

Proof of Theorem 2.1. We first show that under the null hypothesis of no jumps in the price path, the difference between the SwV and the RV converges to zero in probability, i.e.,

$$\text{plim}_{N \rightarrow \infty} (\text{Sw}V_N - RV_N) = 0. \quad (24)$$

From the definition of swap variance in Eq. (4), we have

$$\lim_{N \rightarrow \infty} \text{Sw}V_N = 2 \int_0^1 (dS_u/S_u - dy_u) = V_{(0,1)}.$$

This result only requires the application of Itô’s lemma (i.e. see Eq. (3)). Further, under regularity conditions as specified in Jacod (1994), it also follows that

$$\text{plim}_{N \rightarrow \infty} RV_N = V_{(0,1)}$$

for continuous semimartingales. It is emphasized that the convergence of the SwV measure is non-stochastic and that convergence of RV only requires the absence of jumps and no restrictions on the variance process or the correlation between variance and return processes, such as the leverage effect.

To derive the asymptotic distribution of the SwV test, we use a Taylor series expansion to obtain:

$$\text{Sw}V_N - RV_N = \frac{1}{3} \sum_{i=1}^N r_{\delta,i}^3 + \frac{1}{12} \sum_{i=1}^N r_{\delta,i}^4 + \dots \quad (25)$$

where $r_{\delta,j} = y_{j\delta} - y_{(j-1)\delta}$. In addition, we derive all asymptotic properties based on the discretized process since our tests are

built on discretely observed asset prices. Note that the following Milstein scheme discretization of the process in Eq. (1) with jump intensity $\lambda_t = 0$ has almost sure (a.s.) convergence to the continuous sampling path (see Talay (1996)):

$$r_{\delta,i} = \mu_{(i-1)\delta} \delta + \sqrt{V_{(i-1)\delta}} (W_{i\delta} - W_{(i-1)\delta}) + \frac{1}{2} V_{(i-1)\delta} ((W_{i\delta} - W_{(i-1)\delta})^2 - \delta)$$

as $\delta \rightarrow 0$, where $\mu_t = \alpha_t - \frac{1}{2} V_t$. Note that almost sure convergence is the notion of convergence used in the strong law of large numbers, and ensures that $\sum_{i=1}^N f(r_{\delta,i}) \xrightarrow{a.s.} \int_0^1 f(dy_t)$ where $f(\cdot)$ is a continuous and twice differential function, see, e.g., Grimmett and Stirzaker (1992).

Examining each of the components, we have the following properties as $\delta \rightarrow 0$:

$$\begin{aligned} \mu_{(i-1)\delta} \delta &= O(\delta) \\ \sqrt{V_{(i-1)\delta}} (W_{i\delta} - W_{(i-1)\delta}) &= O_p(\delta^{1/2}) \\ \frac{1}{2} V_{(i-1)\delta} ((W_{i\delta} - W_{(i-1)\delta})^2 - \delta) &= O_p(\delta). \end{aligned}$$

Since the drift term is of the highest order with a deterministic rate of convergence, for simplicity of notation we assume that $\mu_t = 0$. We note that relaxing this assumption does not affect the results, except making the notations more cumbersome.

First, we determine the convergence rate of the test statistic based on $\text{Sw}V_N - RV_N$. We start with the term $\frac{1}{3} \sum_{i=1}^N r_{\delta,i}^3$, where based on the discretized process with the assumption of $\mu_t = 0$, we have the following expression for $r_{\delta,i}^3$:

$$\begin{aligned} r_{\delta,i}^3 &= (\sqrt{V_{(i-1)\delta}} (W_{i\delta} - W_{(i-1)\delta}))^3 \\ &+ \left(\frac{1}{2} V_{(i-1)\delta} ((W_{i\delta} - W_{(i-1)\delta})^2 - \delta) \right)^3 \\ &+ 3(\sqrt{V_{(i-1)\delta}} (W_{i\delta} - W_{(i-1)\delta})) \\ &\times \left(\frac{1}{2} V_{(i-1)\delta} ((W_{i\delta} - W_{(i-1)\delta})^2 - \delta) \right)^2 \\ &+ 3(\sqrt{V_{(i-1)\delta}} (W_{i\delta} - W_{(i-1)\delta}))^2 \\ &\times \left(\frac{1}{2} V_{(i-1)\delta} ((W_{i\delta} - W_{(i-1)\delta})^2 - \delta) \right). \end{aligned}$$

Taking expectation of $r_{\delta,i}^3$ conditional on $\mathcal{F}_{(i-1)\delta}$, the last term has the slowest convergence rate of $O_p(\delta^2)$. Further results for the variance of $\frac{1}{3} \sum_{i=1}^N r_{\delta,i}^3$ (see below for details) show that the term $\frac{1}{3} \sum_{i=1}^N r_{\delta,i}^3$ has a convergence rate no lower than δ or $1/N$. Now turn to the term $\frac{1}{12} \sum_{i=1}^N r_{\delta,i}^4$ where a similar expansion gives:

$$\begin{aligned} r_{\delta,i}^4 &= (\sqrt{V_{(i-1)\delta}} (W_{i\delta} - W_{(i-1)\delta}))^4 \\ &+ \left(\frac{1}{2} V_{(i-1)\delta} ((W_{i\delta} - W_{(i-1)\delta})^2 - \delta) \right)^4 \\ &+ 4(\sqrt{V_{(i-1)\delta}} (W_{i\delta} - W_{(i-1)\delta})) \\ &\times \left(\frac{1}{2} V_{(i-1)\delta} ((W_{i\delta} - W_{(i-1)\delta})^2 - \delta) \right)^3 \\ &+ 4(\sqrt{V_{(i-1)\delta}} (W_{i\delta} - W_{(i-1)\delta}))^3 \\ &\times \left(\frac{1}{2} V_{(i-1)\delta} ((W_{i\delta} - W_{(i-1)\delta})^2 - \delta) \right)^2 \\ &+ 6(\sqrt{V_{(i-1)\delta}} (W_{i\delta} - W_{(i-1)\delta}))^2 \\ &\times \left(\frac{1}{2} V_{(i-1)\delta} ((W_{i\delta} - W_{(i-1)\delta})^2 - \delta) \right). \end{aligned}$$

Taking expectation of $r_{\delta,i}^4$, conditional on $\mathcal{F}_{(i-1)\delta}$, the first term has the slowest convergence rate of $O_p(\delta^2)$. Note that we have $r_{\delta,i}^{2n+3} = r_{\delta,i}^3 \times r_{\delta,i}^{2n}$ for any odd power terms in Eq. (25), and similarly we have $r_{\delta,i}^{2n+4} = r_{\delta,i}^4 \times r_{\delta,i}^{2n}$ for any even power terms in Eq. (25) with $n = 1, 2, \dots$. This ensures that all higher order terms have faster convergence rate. That is, as $\delta = 1/N \rightarrow 0$, the variance swap test statistic has a convergence of δ or $1/N$.

Next, we derive the asymptotic variance of the variance swap test statistic after first adjusting for the convergence rate, that is:

$$N(\text{Sw}V_N - RV_N) = \frac{N}{3} \sum_{i=1}^N r_{\delta,i}^3 + \frac{N}{12} \sum_{i=1}^N r_{\delta,i}^4 + \dots \quad (26)$$

Continuity of the sampling path implies that $|r_{\delta,i}| \xrightarrow{a.s.} 0$ as $\delta \rightarrow 0$, or for any $\epsilon > 0$, there is a $\delta > 0$ such that $|r_{\delta,i}| < \epsilon$. As a matter of fact, from the Milstein scheme discretization of the process, it is easy to see that $|r_{\delta,i}| = O_p(\delta^{1/2})$. It is thus sufficient to only consider the leading term in Eq. (26). That is, the asymptotic variance can be derived as follows:

$$\begin{aligned} \text{var} \left[\frac{N}{3} \sum_{i=1}^N r_{\delta,i}^3 \right] &= \frac{N^2}{9} \sum_{i=1}^N \text{var}[r_{\delta,i}^3] \\ &+ \frac{2N^2}{9} \sum_{j<i}^N \sum_i \text{cov}[r_{\delta,i}^3, r_{\delta,j}^3]. \end{aligned} \quad (27)$$

Note that the element of the first term is determined by $E_{(i-1)\delta}[(r_{\delta,i}^3 - E_{(i-1)\delta}[r_{\delta,i}^3])^2]$, $i = 1, \dots, N$. Throughout the derivation, the conditional expectation is taken with respect to the filtration $\mathcal{F}_{(i-1)\delta}$ with given path for the instantaneous variance process V_t . Using the fact that $E_{(i-1)\delta}[r_{\delta,i}^3] = 3V_{(i-1)\delta}^2\delta^2 + o_p(\delta^2)$, and based on the expansion of $(r_{\delta,i}^3 - E_{(i-1)\delta}[r_{\delta,i}^3])^2$, the term with the lowest convergence rate is $V_{(i-1)\delta}^3(W_{i\delta} - W_{(i-1)\delta})^6$ with a convergence rate of σ^3 . Ignoring higher order terms, we have

$$\frac{N^2}{9} \sum_{i=1}^N \text{var}[r_{\delta,i}^3] = \frac{N^2}{9} \sum_{i=1}^N E_{(i-1)\delta}[V_{(i-1)\delta}^3(W_{i\delta} - W_{(i-1)\delta})^6] + o_p(\delta).$$

Taking limit as $N \rightarrow \infty$ or $\delta \rightarrow 0$, for the price process defined in Eq. (1) with assumptions listed in the theorem, it follows directly from Barndorff-Nielsen and Shephard (2004) as well as Barndorff-Nielsen et al. (2005, Theorem 2.2) that:

$$\text{plim}_{\delta \rightarrow 0} \frac{N^2}{9} \sum_{i=1}^N \text{var}[r_{\delta,i}^3] = \frac{\mu_6}{9} \int_0^1 V_t^3 dt. \quad (28)$$

Note that Barndorff-Nielsen and Shephard (2004) propose the following consistent estimator of integrated power function of variance that is robust to the presence of jumps for appropriate integer values of r and s , i.e.

$$\text{plim}_{\delta \rightarrow 0} \delta^{1-(r+s)/2} \sum_{j=1}^{N-1} |r_{\delta,i}|^r |r_{\delta,(i+1)}|^s = \mu_r \mu_s \int_0^1 V_t^{(r+s)/2} dt \quad (29)$$

with $r, s \geq 0$. Asymptotic properties for realized power variation such as $\sum_{i=1}^N |r_{\delta,i}|^r$ are also thoroughly investigated in Jacod (2006).

Thus, by setting $r + s = 6$ we can obtain a consistent estimator of $\int_0^1 V_t^3 dt$. In particular, when $r = 6, s = 0$, we essentially use the sixthity as the estimate of this asymptotic variance component.

Now we turn to the second term in Eq. (27) that involves $\text{cov}[r_{\delta,i}^3, r_{\delta,j}^3]$ where $j < i$, for $i, j = 1, \dots, N$. By the iteration of

expectation, we have $\text{cov}[r_{\delta,i}^3, r_{\delta,j}^3] = E_{(j-1)\delta}[r_{\delta,i}^3 \cdot r_{\delta,j}^3] - E_{(i-1)\delta}[r_{\delta,i}^3] \times E_{(j-1)\delta}[r_{\delta,j}^3]$. Note again that $E_{(i-1)\delta}[r_{\delta,i}^3] = 3V_{(i-1)\delta}^2\delta^2 + o_p(\delta^2)$, hence the term $E_{(i-1)\delta}[r_{\delta,i}^3] \times E_{(j-1)\delta}[r_{\delta,j}^3]$ is order of δ^4 . That is,

$$E_{(i-1)\delta}[r_{\delta,i}^3] \times E_{(j-1)\delta}[r_{\delta,j}^3] = 9V_{(j-1)\delta}^2 V_{(i-1)\delta}^2 \delta^4 + o_p(\delta^4).$$

We show that the term is negligible. Applying the double summation as in Eq. (27) and taking limit as $\delta \rightarrow 0$ to the above equation, we have

$$\begin{aligned} \text{plim}_{\delta \rightarrow 0} \frac{2N^2}{9} \sum_{j<i}^N \sum_i E_{(i-1)\delta}[r_{\delta,i}^3] \times E_{(j-1)\delta}[r_{\delta,j}^3] \\ = 2 \int_0^1 V_u^2 \left(\int_0^u V_t^2 dt \right) du \leq 2 \left(\int_0^1 V_t^2 dt \right)^2. \end{aligned}$$

Note that from Eq. (29), a consistent estimator of $\int_0^1 V_t^2 dt$ can be obtained by setting $r + s = 4$. Using the fact that $|r_{\delta,i}| = O_p(\delta^{1/2})$, the absolute return term in the consistent estimator of $\left(\int_0^1 V_t^2 dt \right)^2$ is of order $O_p(\delta^4)$. In comparison, the absolute return term in the consistent estimator of Eq. (28) is of order $O_p(\delta^3)$. Relative to Eq. (28), the above term is thus negligible.

In addition, by the iteration of expectation we have $E_{(j-1)\delta}[r_{\delta,i}^3 r_{\delta,j}^3] = E_{(j-1)\delta}[r_{\delta,j}^3 E_{(i-1)\delta}[r_{\delta,i}^3]]$. The case with $j = i - 1$ illustrates the implications of “leverage effect” in the sense that $dW_t dV_t \neq 0$. Specifically, multiplying $E_{(i-1)\delta}[r_{\delta,i}^3]$ to the expansion of $r_{\delta,j}^3$, we need to take into account the correlation between $V_{(i-1)\delta} - V_{(i-2)\delta}$ and $W_{(i-1)\delta} - W_{(i-2)\delta}$ in the expectation. Under assumption (b) that the instantaneous variance process V_t is a well-defined semimartingale such as those considered in Ait-Sahalia and Jacod (in press), we have $dW_t dV_t = O_p(dt)$ when the asset return process specified in Eq. (1) is correlated with the semimartingale process of instantaneous variance V_t . Further, we note that with application of Itô’s lemma to the semimartingale process of V_t , we have $dV_t^2 = 2V_t dV_t + o_p(\delta^{1/2})$. Here, again we focus on terms with the lowest rate of convergence. For example, when $j = i - 1$ the term $E_{(i-2)\delta}[(\sqrt{V_{(i-1)\delta}}(W_{i\delta} - W_{(i-1)\delta}))^3 E_{(i-1)\delta}[r_{\delta,i}^3]]$ has the lowest convergence rate of δ^4 due to the potential “leverage effect”. In general, we have

$$E_{(j-1)\delta}[r_{\delta,(j-1)}^3 E_{(i-1)\delta}[r_{\delta,i}^3]] = 6V_{(j-1)\delta}^{3/2} V_{(i-1)\delta}^3 O_p(\delta^4) + o_p(\delta^4)$$

for $j < i$, with $i, j = 1, \dots, N$. Applying the double summation as in Eq. (27) and taking limit as $\delta \rightarrow 0$ to the above equation, we have

$$\begin{aligned} \text{plim}_{\delta \rightarrow 0} \frac{2N^2}{9} \sum_{j<i}^N \sum_i E_{(j-1)\delta}[r_{\delta,(j-1)}^3] E_{(i-1)\delta}[r_{\delta,i}^3] \\ = O_p(1) \int_0^1 V_u^{3/2} \left(\int_0^u V_t^3 dt \right) du \end{aligned}$$

where $\int_0^1 V_u^{3/2} \left(\int_0^u V_t^3 dt \right) du \leq \left(\int_0^1 V_t^{3/2} dt \right) \cdot \left(\int_0^1 V_t^3 dt \right)$. Based on the same argument using the continuity property of the process under the null of no jumps, this term is also negligible. Thus, the second term in Eq. (27) is negligible. The asymptotic variance of the swap variance test is given in Eq. (28).

The asymptotic distribution of the logarithmic test can be derived using the following expansion:

$$\ln \text{Sw}V_N - \ln RV_N = \frac{\text{Sw}V_N - RV_N}{\text{Sw}V_N} - \frac{1}{2} \left(\frac{\text{Sw}V_N - RV_N}{\text{Sw}V_N} \right)^2 + \dots,$$

where the convergence rate of $\text{Sw}V_N - RV_N$ ensures that the higher order terms are negligible. Thus, the logarithmic test has the same

asymptotic property as $\frac{SwV_N - RV_N}{SwV_N}$ which is essentially the ratio test. From the Slutsky's theorem, it is clear that:

$$\text{plim}_{\delta \rightarrow 0} \frac{SwV_N - RV_N}{SwV_N} = 0.$$

To derive the asymptotic variance of the above ratio test, we explore the insight of the Hausman (1978) test following the idea of Huang and Tauchen (2005). Compared to the swap variance, realized variance only converges to the integrated variance in probability under the assumption of no jumps and is thus a less efficient estimator. Following Hausman (1978), we have that under the null hypothesis of no jumps and conditional on the volatility path, $SwV_N - RV_N$ is asymptotically independent of SwV_N . In other words, the ratio test is asymptotically the ratio of two conditionally independent random variables. As a result, the asymptotic variance can be derived straightforwardly as $\text{var}[SwV_N - RV_N]/V_{(0,1)}^2$, and thus we have the results in Theorem 2.1. Finally, the asymptotic distribution of the swap variance test is determined by $\frac{1}{3} \sum_{i=1}^N r_{\delta,i}^3$ which is well-behaved under the assumptions on the asset return process. The asymptotic normality of the swap variance test follows directly from the standard results by Lipster and Shiryaev (see e.g. Shiryaev (1981)) regarding the central limit properties of the martingale sequences. ■

Proof of Theorem 3.1. We start with the observation that:

$$\begin{aligned} \mathcal{T} \equiv SwV_N - RV_N &= \frac{1}{3} \sum_{i=1}^N r_i^3 + \frac{1}{12} \sum_{i=1}^N r_i^4 + \frac{1}{60} \sum_{i=1}^N r_i^5 \\ &+ \frac{1}{360} \sum_{i=1}^N r_i^6 + \dots = 2 \sum_{i=1}^N \sum_{k=3}^{\infty} \frac{r_i^k}{k!}. \end{aligned} \quad (30)$$

Since here we consider the case with noise, we replace r_i by $r_i + \varepsilon_i - \varepsilon_{i-1}$ where $\varepsilon_i \sim \text{i.i.d. } \mathcal{N}(0, \omega^2)$.

First we derive the expectation of \mathcal{T} . We use the result that for a standard normal random variable x , we have

$$E(|x|^k) = \frac{2^{\frac{1}{2}k}}{\sqrt{\pi}} \Gamma\left(\frac{k+1}{2}\right) \quad \text{for } k > 0,$$

which can be specialized to:

$$E(x^{2k}) = \prod_{m=1}^k (2m-1) = \frac{(2k-1)!}{2^{k-1}(k-1)!} \quad \text{for } k = 1, 2, 3, \dots$$

Because the efficient price return is $O(N^{-1/2})$ compared to the noise that is $O(1)$, and all uneven integer moments of ε_i are zero, we have as $N \rightarrow \infty$:

$$E\left(\frac{\mathcal{T}}{N}\right) \rightarrow 2 \sum_{k=2}^{\infty} \frac{E((\varepsilon_i - \varepsilon_{i-1})^{2k})}{(2k)!} = 2(e^{\omega^2} - 1 - \omega^2) \approx \omega^4. \quad (31)$$

To derive the variance, consider

$$\begin{aligned} E\left(\frac{\mathcal{T}^2}{N^2}\right) &= E\left(2 \sum_{k=3}^{\infty} \frac{1}{N} \sum_{i=1}^N \frac{r_i^k}{k!}\right)^2 = E \sum_{k=3}^{\infty} \left(\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^k}{k!}\right)^2 \\ &+ 2E \sum_{k=3}^{\infty} \sum_{p=k+1}^{\infty} \left(\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^k}{k!}\right) \left(\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^p}{p!}\right). \end{aligned} \quad (32)$$

Let us start with the first term on the right-hand side:

$$\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^{2k}}{(2k)!} \rightarrow 2 \frac{(2k-1)!(2\omega^2)^k}{(2k)!2^{k-1}(k-1)!} = \frac{2\omega^{2k}}{\Gamma(k+1)}$$

for $k = 1, 2, \dots$

and 0 for uneven powers. As a consequence we have:

$$\left(\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^{2k}}{(2k)!}\right)^2 \rightarrow \frac{4\omega^{4k}}{\Gamma^2(k+1)} \quad \text{for } k = 1, 2, \dots,$$

and so:

$$\begin{aligned} \sum_{k=3}^{\infty} \left(\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^k}{k!}\right)^2 &= \sum_{k=2}^{\infty} \left(\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^{2k}}{(2k)!}\right)^2 \rightarrow \omega^8 \\ &+ \frac{\omega^{12}}{9} + \frac{\omega^{16}}{144} + \dots \approx \omega^8. \end{aligned}$$

With regard to the second term on the right-hand side in Eq. (32), we note that

$$\begin{aligned} \left(\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^{2k}}{(2k)!}\right) \left(\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^{2p}}{(2p)!}\right) &\rightarrow \frac{4\omega^{2k+2p}}{\Gamma(p+1)\Gamma(k+1)} \\ &\text{for } k \neq p = 1, 2, \dots \end{aligned}$$

and 0 otherwise (i.e. for uneven powers in either of the terms). Thus:

$$\begin{aligned} 2 \sum_{k=3}^{\infty} \sum_{p=k+1}^{\infty} \left(\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^k}{k!}\right) \left(\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^p}{p!}\right) \\ = 2 \sum_{k=2}^{\infty} \sum_{p=k+1}^{\infty} \left(\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^{2k}}{(2k)!}\right) \left(\frac{1}{N} \sum_{i=1}^N 2 \frac{r_i^{2p}}{(2p)!}\right) \\ = \frac{2}{3}\omega^{10} + \frac{1}{6}\omega^{12} + \frac{4}{45}\omega^{14} + \frac{1}{60}\omega^{16} + \dots \end{aligned}$$

Collecting all of the above, we have $E(\mathcal{T}/N) = \omega^4 + O(\omega^6)$ and $E(\mathcal{T}^2/N^2) = \omega^8 + O(\omega^{10})$. As a consequence $\text{var}(\mathcal{T}/N) = O(\omega^{10})$ from which it is clear that, in the limit, the expectation of the test statistic swamps its variance. Because this limiting result is not particularly useful in practice, we now derive an approximation to the finite sample mean and variance of the test statistic in the presence of noise. For notational convenience, we define $\kappa_1 = \frac{1}{3} \sum_{i=1}^N (r_i + \varepsilon_i - \varepsilon_{i-1})^3$ and $\kappa_2 = \frac{1}{12} \sum_{i=1}^N (r_i + \varepsilon_i - \varepsilon_{i-1})^4$. Explicitly retaining the efficient return in the calculations, we have:

$$\begin{aligned} E(\mathcal{T}) \approx E(\kappa_2) &= \frac{1}{12} \sum_{i=1}^N E(r_i^4 + 12\omega^4 + 12r_i^2\omega^2) \rightarrow N\omega^4 \\ &+ \omega^2V + \frac{Q}{4N} \end{aligned} \quad (33)$$

where we use that $\sum r_i^2 \rightarrow V$, $\frac{N}{3} \sum r_i^4 \rightarrow Q$. Similarly,

$$\begin{aligned} E(\kappa_1^2) &= \frac{1}{9} E \sum_{i=1}^N (r_i + \varepsilon_i - \varepsilon_{i-1})^6 + \frac{2}{9} E \sum_{i=1}^{N-1} (r_i + \varepsilon_i - \varepsilon_{i-1})^3 \\ &\times (r_{i+1} + \varepsilon_{i+1} - \varepsilon_i)^3, \\ &= \frac{1}{9} E \sum_{i=1}^N (r_i^6 + 30\omega^2 r_i^4 + 180\omega^4 r_i^2 + 120\omega^6) \\ &- \frac{2}{9} E \sum_{i=1}^{N-1} (9\omega^2 r_i^2 r_{i+1}^2 + 36\omega^4 r_i^2 + 42\omega^6), \\ &\rightarrow 4N\omega^6 + \frac{28}{3}\omega^6 + 12\omega^4V + 8\omega^2\frac{Q}{N} + \frac{5}{3}\frac{X}{N^2}, \end{aligned} \quad (34)$$

where we use that $\frac{N^2}{15} \sum r_i^6 \rightarrow X$, and $N \sum r_i^2 r_{i+1}^2 \rightarrow Q$.

Turning to the second order terms, we have:

$$E(\kappa_2^2) = \frac{1}{144} E \sum_{i=1}^N (r_i + \varepsilon_i - \varepsilon_{i-1})^8 + \frac{1}{144} E \sum_{i \neq j} (r_i + \varepsilon_i - \varepsilon_{i-1})^4 (r_j + \varepsilon_j - \varepsilon_{j-1})^4.$$

The first term on the right-hand side can be expressed as:

$$\begin{aligned} E \sum_{i=1}^N (r_i + \varepsilon_i - \varepsilon_{i-1})^8 &= E \sum_{i=1}^N (1680\omega^8 + 3360\omega^6 r_i^2 + 840\omega^4 r_i^4 + 56\omega^2 r_i^6 + r_i^8), \\ &\rightarrow 1680N\omega^8 + 3360\omega^6 V + 2520\omega^4 \frac{Q}{N} + 840\omega^2 \frac{X}{N^2} + O(N^{-3}). \end{aligned}$$

For the second term on the right-hand side, note that for all $i \neq j$ we have:

$$\begin{aligned} E (r_i + \varepsilon_i - \varepsilon_{i-1})^4 (r_{i+1} + \varepsilon_{i+1} - \varepsilon_i)^4 &= E (r_i^4 + 6r_i^2 \varepsilon_i^2 + 6r_i^2 \varepsilon_{i-1}^2 + 6\varepsilon_i^2 \varepsilon_{i-1}^2 + \varepsilon_i^4 + \varepsilon_{i-1}^4) \\ &\quad \times (r_j^4 + \varepsilon_j^4 + \varepsilon_{j-1}^4 + 6r_j^2 \varepsilon_j^2 + 6r_j^2 \varepsilon_{j-1}^2 + 6\varepsilon_j^2 \varepsilon_{j-1}^2) \\ &= E(A_{i,j}) + E(B_{i,i+1}) I_{|j-i|=1} \end{aligned}$$

where $A_{i,j} = 144\omega^8 + 12\omega^4 r_i^4 + 12\omega^4 r_j^4 + 144\omega^6 r_i^2 + 144\omega^6 r_j^2 + 12\omega^2 r_i^4 r_j^2 + 12\omega^2 r_j^4 r_i^2 + 144\omega^4 r_i^2 r_j^2 + r_i^4 r_j^4$, $B_{i,j} = 312\omega^8 + 144\omega^6 r_i^2 + 144\omega^6 r_j^2 + 72\omega^4 r_i^2 r_j^2$. Using the fact that $\frac{1}{N-1} \sum_{i \neq j} r_i^2 r_j^2 \rightarrow V$, $\frac{1}{3} \frac{N}{N-1} \sum_{i \neq j} r_i^4 r_j^4 \rightarrow Q$, $\sum_{i \neq j} r_i^2 r_j^2 = (\sum_{i=1}^N r_i^2)^2 - \sum_{i=1}^N r_i^4 \rightarrow V^2$, $\frac{N}{3} \sum_{i \neq j} r_i^2 r_j^4 = \frac{N}{3} \sum_{i=1}^N r_i^2 \sum_{j=1}^N r_j^4 - \frac{N}{3} \sum_{i=1}^N r_i^6 \rightarrow VQ$, and $\frac{N^2}{9} \sum_{i \neq j} r_i^4 r_j^4 = \frac{N^2}{9} (\sum_{i=1}^N r_i^4)^2 - \frac{N^2}{9} \sum_{i=1}^N r_i^8 \rightarrow Q^2$, we have:

$$\begin{aligned} \sum_{i \neq j} E(A_{i,j}) &= 144\omega^8 N(N-1) + 72\omega^2 Q(\omega^2 + V/N) \\ &\quad + 288\omega^6 V(N-1) + 144\omega^4 V^2 + 9 \frac{Q^2}{N^2} \end{aligned}$$

and

$$\sum_{i \neq j} E(B_{i,i+1}) I_{|j-i|=1} = 624(N-1)\omega^8 + 576\omega^6 V + 144\omega^4 \frac{Q}{N}.$$

Combing all the above, we obtain:

$$\begin{aligned} E(\kappa_2^2) &\rightarrow N^2\omega^8 + 15N\omega^8 + 2\omega^6 V N - \frac{13}{3}\omega^8 \\ &\quad + \frac{76}{3}\omega^6 V + \frac{1}{2}\omega^4 Q + \omega^4 V^2 + \frac{37}{2}\omega^4 \frac{Q}{N} + \frac{1}{2}\omega^2 Q \frac{V}{N} \\ &\quad + \frac{35}{6}\omega^2 \frac{X}{N^2} + \frac{1}{16} \frac{Q^2}{N^2}. \end{aligned} \tag{35}$$

Using Eqs. (31)–(35), we can now approximate the variance of \mathcal{T} , based on the first two terms in Eq. (30) as:

$$\begin{aligned} \text{var}(\mathcal{T}) &\approx 15N\omega^8 + 2\omega^6 V N - \frac{13}{3}\omega^8 + \frac{76}{3}\omega^6 V \\ &\quad + \frac{1}{2}\omega^4 Q + \omega^4 V^2 + \frac{37}{2}\omega^4 \frac{Q}{N} \\ &\quad + \frac{1}{2}\omega^2 Q \frac{V}{N} + \frac{35}{6}\omega^2 \frac{X}{N^2} + \frac{1}{16} \frac{Q^2}{N^2} + 4N\omega^6 + \frac{28}{3}\omega^6 \\ &\quad + 12\omega^4 V + 8\omega^2 \frac{Q}{N} + \frac{5}{3} \frac{X}{N^2}, \end{aligned} \tag{36}$$

where we note that $E(r_i + \varepsilon_i - \varepsilon_{i-1})^3 (r_j + \varepsilon_j - \varepsilon_{j-1})^4 = 0$, for all i, j so that the cross product κ_1 and κ_2 can be ignored. To get a sense for which are the important terms in Eq. (36), we substitute $\omega^2 = \gamma V/N$ noting that in practice, even at the highest available sampling frequency γ is small and typically less than 1. This gives us the following expression:

$$\begin{aligned} &\frac{12\gamma^2(\gamma+3)V^3 + 24\gamma VQ + 5X}{3N^2} \\ &\quad + \frac{8\gamma^2 V^2 Q + 32\gamma^3 V^4 + (Q + 4V^2\gamma)^2}{16N^2} \\ &\quad + \frac{1}{6}\gamma V \frac{90\gamma^3 V^3 + 152\gamma^2 V^3 + 56\gamma^2 V^2 + 111\gamma VQ + 35X}{N^3} \\ &\quad - \frac{13}{3}\gamma^4 \frac{V^4}{N^4}. \end{aligned}$$

In the (empirically relevant) scenario where N is large compared to γ , i.e. $\gamma \ll N$ or equivalently $\omega^2 \ll V$, all terms on the second line are negligible and can thus be ignored. On the first line, the first term is the larger one because here the integrated variation quantities are of lower order. Substituting back γ , and retaining only the first term in the above expression, we have the simplified variance approximation:

$$\text{var}(\mathcal{T}) \approx 4N\omega^6 + 12\omega^4 V + \frac{8}{N}\omega^2 Q + \frac{5}{3} \frac{X}{N^2}. \tag{37}$$

Next, we consider the higher order terms. With a third order approximation, we need to add the following:

$$\frac{1}{3600} E \left(\sum_{i=1}^N (r_i + \varepsilon_i - \varepsilon_{i-1})^5 \right)^2,$$

and

$$\frac{1}{180} \sum_{i=1}^N (r_i + \varepsilon_i - \varepsilon_{i-1})^3 \sum_{j=1}^N (r_j + \varepsilon_j - \varepsilon_{j-1})^5.$$

With regard to the first term, we note that the contribution of returns r^{10} is of order N^{-5} and summing over a maximum of N^2 terms will give a term of order N^{-3} . Substituting $\omega^2 = \gamma V/N$ as above, we see that the noise term $\varepsilon^{10} = (\gamma V/N)^5$ is also of order N^{-5} , and so are all the cross-products of r and ε . With regard to the second term, we note that for $|i-j| > 1$ we have $E(r_i + \varepsilon_i - \varepsilon_{i-1})^3 (r_j + \varepsilon_j - \varepsilon_{j-1})^5 = 0$, so this only leaves us with $2(N-1)$ terms on the off-diagonal, each of order N^{-4} . So also this component is of order N^{-3} . Similar arguments hold for the higher order terms and can thus all be ignored.

Thus, with large but finite N , and $\omega^2 \ll V$, the mean and variance of \mathcal{T} are accurately approximated by Eqs. (33) and (37) above. To understand the magnitude of the mean of the statistic, consider the case with constant volatility (i.e. $Q = V^2$ and $X = V^3$) and note that:

$$\frac{E(\mathcal{T})}{\sqrt{\text{var}(\mathcal{T})}} = \frac{(4\gamma^2 + 4\gamma + 1)}{\sqrt{64\gamma^3 + 192\gamma^2 + 128\gamma + \frac{80}{3}}} \sqrt{V} \approx \frac{2 + 3\gamma}{10} \sqrt{V}.$$

Because γ is typically less than 1, and $V \ll 1$ over short horizons as considered here, it is evident that the magnitude of the mean of the statistic is small and negligible for all practical purposes. ■

Proof of Proposition 3.2. Under the assumptions specified in the proposition, we have that $r_i \sim$ i.i.d. $\mathcal{N}(0, \bar{V}/N)$ so that BPV can be expressed as:

$$\begin{aligned} BPV_N^* &= \frac{\pi}{2} \frac{\bar{V}}{N-1} \sum_{i=2}^N |\tilde{r}_{\delta,i} + \sqrt{\gamma}(\varepsilon_i - \varepsilon_{i-1})| |\tilde{r}_{\delta,i-1} \\ &\quad + \sqrt{\gamma}(\varepsilon_{i-1} - \varepsilon_{i-2})|, \end{aligned}$$

where $\tilde{r}_i = r_i/(\bar{V}/N)^{1/2} \sim \text{i.i.d. } \mathcal{N}(0, 1)$, $\varepsilon_i \sim \text{i.i.d. } \mathcal{N}(0, 1)$, and $\gamma = N\sigma_\varepsilon^2/\bar{V}$. Consequently, we can write:

$$E[BPV_N^*] = \bar{V} \frac{\pi}{2} E \left[|\tilde{r}_2 + \sqrt{\gamma}(\varepsilon_2 - \varepsilon_1)| \left| \tilde{r}_1 + \sqrt{\gamma}(\varepsilon_1 - \varepsilon_0) \right| \right].$$

Conditional on ε_1 , we define the function:

$$\begin{aligned} c_b(\gamma|\varepsilon_1) &\equiv \frac{\pi}{2} E \left[|\tilde{r}_2 + \sqrt{\gamma}(\varepsilon_2 - \varepsilon_1)| \left| \tilde{r}_1 + \sqrt{\gamma}(\varepsilon_1 - \varepsilon_0) \right| \right] \\ &= \frac{\pi}{2} \xi(\sqrt{\gamma}\varepsilon_1, 1 + \gamma) \xi(-\sqrt{\gamma}\varepsilon_1, 1 + \gamma), \end{aligned}$$

with

$$\xi(a, \sigma^2) \equiv E|x - a| = 2\phi(a/\sigma)\sigma + 2a\Phi(a/\sigma) - a,$$

where a is a constant, $x \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2)$ and $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal density and distribution respectively. The expression for $c_b(\gamma)$ is then obtained by integrating ε_1 out of $c_b(\gamma|\varepsilon_1)$ and subtracting 1, i.e.

$$\begin{aligned} c_b(\gamma) &= \int_{-\infty}^{\infty} c_b(\gamma|x) \phi(x) dx - 1 \\ &= (1 + \gamma) \sqrt{\frac{1 + \gamma}{1 + 3\gamma}} + \gamma \frac{\pi}{2} - 1 + 2 \frac{\gamma}{(1 + \lambda) \sqrt{2\lambda + 1}} \\ &\quad + 2\gamma\pi\kappa(\lambda) \end{aligned}$$

with λ and $\kappa(\lambda)$ as defined in the Proposition. ■

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