Multi-Factor Models and the Arbitrage Pricing Theory (APT)

Introduction

- The empirical failures of the CAPM is not really that surprising
  - ↪ We had to make a number of strong and pretty unrealistic assumptions to arrive at the CAPM
  - ↪ All investors are rational, only care about mean and variance, have the same expectations, ...
  - ↪ Also, identifying and measuring the return on the market portfolio of *all* risky assets is difficult, if not impossible (Roll Critique)

- In this lecture series we will study an alternative approach to asset pricing called the Arbitrage Pricing Theory, or APT
  - ↪ The APT was originally developed in 1976 by Stephen A. Ross
  - ↪ The APT starts out by specifying a number of "systematic" risk factors
  - ↪ The only risk factor in the CAPM is the "market"

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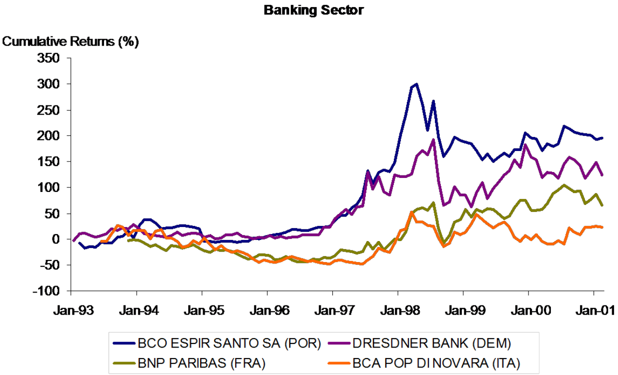
Introduction: Multiple Risk Factors

- Stocks in the same industry tend to move more closely together than stocks in different industries

  ↪ European Banks (some old data):

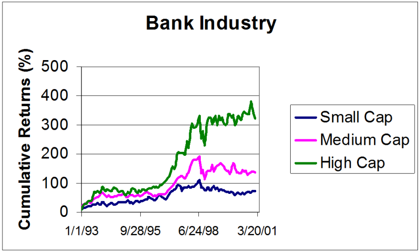


Source: BARRA

Introduction: Multiple Risk Factors

- Other common factors might also affect stocks within the same industry

  ↪ The size effect at work within the banking industry (some old data):



Source: BARRA

  ↪ How does this compare to the CAPM tests that we just talked about?

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Multiple Factors and the CAPM

- Suppose that there are only two fundamental sources of systematic risks, “technology” and “interest rate” risks
- Suppose that the return on asset i follows the equation:

$$r_i = f_T + f_I + \varepsilon_i$$

  ↪ Of course, all stocks do not necessarily respond the same to technological and interest rate risk

- Suppose also that the CAPM is true:

$$E[r_i] = r_f + \beta_i(E(r_m) - r_f)$$

$$\beta_i \equiv \frac{cov(r_i, r_m)}{var(r_m)} = \frac{cov(f_T, r_m) + cov(f_I, r_m)}{var(r_m)}$$

  ↪ The two different risks (covariances) would be priced the same

  ↪ Is that reasonable?

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Arbitrage Pricing Theory

- The APT explicitly allows for multiple sources of systematic risks and different pricing of these risks
- The APT provides a framework for determining asset values based on the law of one price and no arbitrage
- The APT is derived from a *statistical* model for the returns, whereas the CAPM is an *equilibrium* based model
- Unlike the CAPM, we only need some fairly weak additional assumptions to arrive at the APT
- In particular, the APT doesn't require that *everyone* is optimizing in a rational fashion

APT Assumptions

- The basic assumptions necessary for the APT are:
  - ↪ All securities have finite expected returns and variances
  - ↪ Some agent(s) can form well diversified portfolios
  - ↪ There are no market “frictions” (taxes, transaction costs, etc.)
- These assumptions are considerably weaker than what we needed for the CAPM
- The central idea behind the APT is to price assets *relative* to one another
  - ↪ The resulting restrictions on the prices will be based on *no-arbitrage*
  - ↪ Similar approximate results hold true if we exclude “near-arbitrage,” or extremely “good deals”

No Arbitrage

- *Absence of arbitrage* in financial markets precludes the existence of any security with a zero price and a strictly positive payoff
  - ↪ Also, no portfolio can be created with this property
  - ↪ This implies that two securities, or portfolios, with the same payoffs must have the same price
  - ↪ No "free lunch"
- In an efficiently functioning financial market arbitrage opportunities should not exist
  - ↪ At least not for very long ...
  - ↪ This same no-arbitrage principle is also used extensively in the pricing of options and other derivative instruments
- Unlike the equilibrium arguments underlying the CAPM, this no-arbitrage rule only requires *one* smart investor

Arbitrage - Example

- Suppose that there are only two possible and equally likely *states of nature* for inflation and real interest rates: high or low

- Suppose that the four securities A, B, C and D are all currently selling for $100, and that the known payoffs in each of the four possible states are:

State/ Stock	High Real Int. Rates		Low Real Int. Rates	
	High Infl.	Low Infl.	High Infl.	Low Infl.
Int. Rate	5%	5%	0%	0%
Inflation	10%	0%	10%	0%
Prob.	0.25	0.25	0.25	0.25
Apex (A)	-20	20	40	60
Bull (B)	0	70	30	-20
Crush (C)	90	-20	-10	70
Dreck (D)	15	23	15	36

Arbitrage - Example

- Using the standard formulas, the expected returns, standard deviations, and correlations are:

Stock	Current Price	Expected Return(%)	Standard Dev. (%)	Correlation Matrix A	B	C	D
A	100	25.00	29.58	1.00	-0.15	-0.29	0.68
B	100	20.00	33.91	-0.15	1.00	-0.87	-0.38
C	100	32.50	48.15	-0.29	-0.87	1.00	0.22
D	100	22.25	8.58	0.68	-0.38	0.22	1.00

- Everything looks "fine"

  ↪ But there is a simple arbitrage opportunity lurking in these numbers ...

Arbitrage - Example

- Consider the return/payoff of an equally weighted portfolio of A, B and C, and compare this with the return/payoff of D:

State/ Port.	High Real Int. Rates		Low Real Int. Rates	
	High Infl.	Low Infl.	High Infl.	Low Infl.
Portfolio	23.33	23.33	20.00	36.67
D	15	23	15	36

  ↪ The return/payoff on the portfolio made up of A, B and C is higher than D in *all* states of nature

  ↪ This is an *arbitrage opportunity*

  ↪ What would happen to the price of D in a well functioning market?

- This numerical example is obviously too simplistic

  ↪ In reality there is a *continuum* of possible states of nature and *multiple* sources of risks

Specifying Risks

- We will assume that we know the probabilities of each of the different *states of nature* that can occur and what will happen in each of these different states
  - ↪ Of course, we don't know which state will actually occur
  - ↪ Factor models provide a convenient framework for formally operationalizing this

- As an aside, this is different from so-called *Knightian uncertainty* in which the risks are immeasurable
  - ↪ *Risk* applies to situations where we do not know the outcome of a given situation, but can accurately assess the odds
  - ↪ *Uncertainty* applies to situations where we don't have enough information to identify the possible outcomes in the first place
  - ↪ Much of behavioral finance (economics) builds on this latter idea
  - ↪ The known, the unknown, and the unknowable ...

Specifying Risks

- Factor models provide a convenient framework for realistically describing how security returns move with economy wide risks, and in turn with one another
  - ↪ A factor model is a multivariate statistical/mathematical model for returns (return generating process)
  - ↪ The sources of co-movement are called *factors* (*systematic risks*)
  - ↪ The sensitivities of the assets to the different factors are called *factor loadings* (*factor betas* or *factor sensitivities*)
  - ↪ The single-index model that we used to simplify the calculation of covariances and correlations is a one-factor model
  - ↪ In the absence of arbitrage, we can price assets relative to one another based on their comovements with the factors
  - ↪ This is the basic idea behind the APT

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Factor Model

- A K-factor model is formally defined by:

$$r_i = b_{i,0} + b_{i,1}f_1 + b_{i,2}f_2 + \ldots + b_{i,K}f_K + e_i$$

↪ The f_j's represent the K common factors that affect most assets

- Examples of macroeconomic factors might be economic growth, interest rates, inflation

↪ $b_{i,j}$ is the factor loading of asset i with respect to the j'th factor

- This tells you how much the asset's return goes up/down when the factor is one unit higher/lower than expected

↪ e_i accounts for the idiosyncratic risk of asset i

- For example, e_i is likely negative when a firm loses a big contract
- The factor model assumes that $cov(e_i, e_h) = 0$ for $i \neq h$
- What does that mean?

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Factor Model

- It is often convenient to write the K-factor model in terms of the factor surprises $\tilde{f}_j$:

$$r_i = a_i + b_{i,1}\tilde{f}_1 + b_{i,2}\tilde{f}_2 + \ldots + b_{i,K}\tilde{f}_K + e_i$$

↪ By definition $E(\tilde{f}_j) = 0$

- Instead of defining a factor directly as economic growth, it is defined as the deviation of economic growth from what was expected

↪ The intercept a_i in this representation is equal to $E(r_i)$

- Why?

↪ Sometimes we will also assume that $cov(\tilde{f}_j, \tilde{f}_k) = 0$ for $j \neq k$

- There are statistical techniques to make this happen

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Factor Model - Example

- An example:

  Suppose that two factors have been identified for the U.S. economy: the growth rate of industrial production (IP) and the inflation rate (Inf). Industrial production is expected to grow at 4%, along with an inflation rate of 6%. A stock with a beta of 1.0 for IP and 0.4 for Inf is currently expected to provide an annual rate of return of 14%. If industrial production actually grows by 5% over the next year, while the inflation rate turns out to be 7%, what is your revised best estimate of the return on the stock?

Factor Model - Example

- We know that $E(IP) = 4\%$, $b_{IP} = 1$, $E(Inf) = 6\%$, $b_{Inf} = 0.4$, and $E(r_i) = 14\%$
- The actual realized factor values and surprises are:

$$\begin{aligned} \tilde{f}_{IP} &= 0.05 - 0.04 = 0.01 \\ \tilde{f}_{Inf} &= 0.07 - 0.06 = 0.01 \end{aligned}$$

- Consequently, our best guess as to the return on the stock *conditional* on the actual realized industrial production growth rate (IP) and the inflation rate (Inf) is:

$$\begin{aligned} E(r_i|\tilde{f}_{IP}, \tilde{f}_{Inf}) &= 0.14 + 1 \cdot 0.01 + 0.4 \cdot 0.01 \\ &= 15.4\% \end{aligned}$$

  - ↪ Is this necessarily what the return on the stock will *actually* turn out to be?
  - ↪ There is still the idiosyncratic risk, e_i

Factor Model

- The factor model has important implications about asset return variances and covariances

- Consider a two-factor model:

$$
\begin{aligned}
var(r_i) &= var(b_{i,1}f_1 + b_{i,2}f_2 + e_i) \\
&= b_{i,1}^2 var(f_1) + b_{i,2}^2 var(f_2) + 2 \cdot b_{i,1} \cdot b_{i,2} \cdot cov(f_1, f_2) + \sigma_{e,i}^2
\end{aligned}
$$

- If the factors are *uncorrelated*:

$$
var(r_i) = b_{i,1}^2 var(f_1) + b_{i,2}^2 var(f_2) + \sigma_{e,i}^2
$$

↪ $b_{i,1}^2 var(f_1) + b_{i,2}^2 var(f_2)$ represents the *systematic* variance

↪ $\sigma_{e,i}^2$ is the *idiosyncratic* variance

↪ How does this expression compare to that for the single-index model?

Factor Model

- The general formula with K factors:

$$var(r_i) \;=\; \sum_{j=1}^{K}\sum_{k=1}^{K} b_{i,j}\cdot b_{i,k}\cdot \sigma_{j,k} \;+\; \sigma_{e,i}^2$$

  - $\hookrightarrow$ $\sigma_{j,k}$ for $j \neq k$ denotes the covariance between the j'th and k'th factors
  - $\hookrightarrow$ $\sigma_{j,j} \equiv \sigma_j^2$ denotes the variance of the j'th factor
  - $\hookrightarrow$ The *systematic* variance is given by $\sum_{j=1}^{K}\sum_{k=1}^{K} b_{i,j}\cdot b_{i,k}\cdot \sigma_{j,k}$
  - $\hookrightarrow$ The *idiosyncratic* variance is $\sigma_{e,i}^2$

- If the factors are *uncorrelated* the formula simplifies to:

$$var(r_i) \;=\; \sum_{k=1}^{K} b_{i,k}^2\cdot \sigma_k^2 \;+\; \sigma_{e,i}^2$$

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- Now consider the covariance between stocks i and j for the two-factor model:

$$
\begin{aligned}
cov(r_i, r_j) &= cov(b_{i,1} f_1 + b_{i,2} f_2 + e_i,\ b_{j,1} f_1 + b_{j,2} f_2 + e_j) \\
&= b_{i,1}\, b_{j,1} var(f_1) + b_{i,2}\, b_{j,2} var(f_2) + \\
&\quad + (b_{i,1}\, b_{j,2} + b_{j,1}\, b_{i,2})\, cov(f_1, f_2)
\end{aligned}
$$

- If the factors are *uncorrelated*:

$$
cov(r_i, r_j) = b_{i,1}\, b_{j,1} var(f_1) \ + \ b_{i,2}\, b_{j,2} var(f_2)
$$

↪ How does this expression compare to that for the single-index model?

↪ What about a K-factor model?

Diversified Portfolios

- The APT implies that only *systematic* risk should be rewarded
  - ↪ The *idiosyncratic* (*non-systematic*) risk can be diversified away

- A *diversified portfolio* is a portfolio that carries no idiosyncratic risk
  - ↪ For a K-factor model the return on a *well diversified portfolio* is given by:

$$r_p = E(r_p) + b_{p,1}\tilde{f}_1 + \ldots + b_{p,K}\tilde{f}_K$$

  - ↪ Note the actual return depends on the specific factor model and the corresponding factor surprises $\tilde{f}_j$
  - ↪ We will assume that investors can form such well diversified portfolios

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Diversified Portfolios

- Returns for a one-factor ($K = 1$) model:

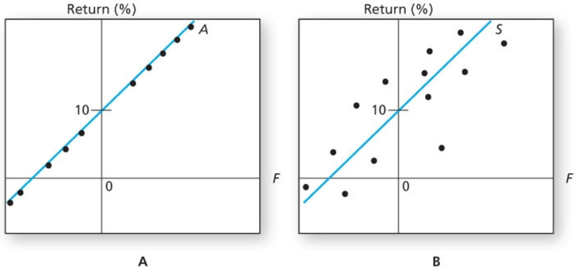


Figure 10.1 Returns as a function of the systematic factor.
Panel A, Well-diversified portfolio *A*. ***Panel B,*** Single stock (*S*).

$\hookrightarrow$ Remember the logic behind the portfolio-based tests of the CAPM that we talked about

APT Pricing Equation

- The APT *pricing equation* (which we will develop both intuitively and more formally) states that:

$$E(r_i) = \lambda_0 + \lambda_1 b_{i,1} + \ldots + \lambda_K b_{i,K}$$

  - ↪ The λ_js are called *factor risk premia*
  - ↪ They represent the extra return for an extra unit of the j'th risk
  - ↪ There is one λ_j for each of the systematic risk factors, plus one additional λ_0
  - ↪ If there is a risk-free asset, then $\lambda_0 = r_f$
  - ↪ If this equation is not satisfied for *all* well diversified portfolios, there is an arbitrage opportunity

APT Pricing Equation - One Factor

- APT for single factor (K=1) model and well diversified portfolios:

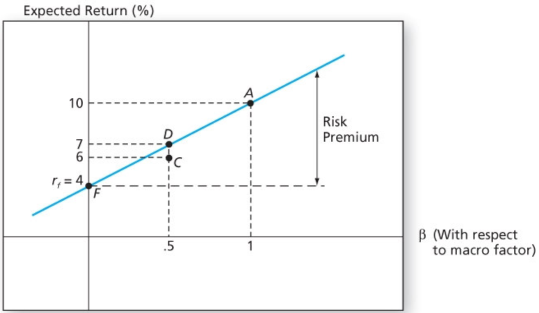
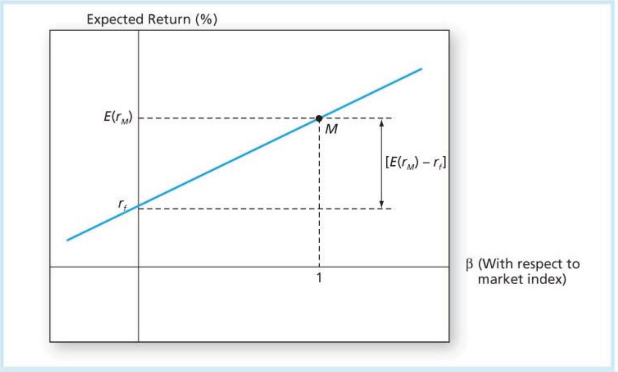


Figure 10.3 An arbitrage opportunity

APT Pricing Equation - One Factor Equal to r_M

- APT for single factor equal to the market:



$\hookrightarrow$ APT implies the CAPM-SML as a special case

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Arbitrage - Example

- Suppose that Apple (AAPL) is currently (time 0) selling for $100 per share
- Suppose also that Apple is going to pay a liquidating dividend in exactly one year from now (time 1), and this is the only future payment that Apple will ever make
- The dividend that Apple will pay is uncertain, and depends on how well the economy is doing
  - ↪ If the economy is in an expansion the dividend will be $140
  - ↪ If the economy is in a recession the dividend will only be $100
  - ↪ Assume that the two states are equally likely, so that the expected cash flow from Apple equals $E(CF_1^{AAPL}) = \$120$
- Note this setup corresponds to our previous definition of *risk*
  - ↪ We know exactly what will happen to Apple in each *scenario*, but we don't know which scenario will actually occur

	AAPL
Boom Payoff (Pr=0.5)	140
Bust Payoff (Pr=0.5)	100
$E(CF_1)$	120
Time 0 Price	100
Discount Rate	20%

- Given Apple's current price of $100, investors are applying a *discount rate* of 20% to Apple's expected cash-flows
  - ↪ The rate that equates the time 1 expected cash flow $E(CF_1^{AAPL}) = \$120$ to the current price of $100
  - ↪ Alternatively, we may say that the expected return on Apple is 20%
- In general, the *expected return* on an asset is equivalent to the *discount rate* that investors are applying to the expected future cash-flows
  - ↪ Expected returns, or equivalently discount rates, are determined by market forces and supply and demand

Arbitrage - Example

- Now let's consider a second stock, Starbucks (SBUX), which like Apple, is going to pay a liquidating dividend exactly one year from now:

	AAPL	SBUX
Boom Payoff (Pr=0.5)	140	160
Bust Payoff (Pr=0.5)	100	80
$E(CF_1)$	120	120
Time 0 Price	100	?
Discount Rate	20%	?

↪ Since Apple and Starbucks have the same expected cash-flows, one might naturally think that the price for Starbucks would also be $100

↪ But, that is not necessarily the case ...

Arbitrage - Example

- Looking a bit closer at the cash-flows for Starbucks, we see that even though the expected value is the same as for Apple, the distribution of the cash-flows for Starbucks is arguably worse
  - ↪ Starbucks' payoff is lower in the bust/recession state, and higher in the boom/expansion state
  - ↪ The rest of our portfolio is likely to do poorly during the recession (job prospects are likely to be poor as well), so we need the cash more then
  - ↪ In the expansion the rest of our portfolio will probably do well (job opportunities are also likely to be better), so an extra dollar isn't worth as much then
  - ↪ If Apple and Starbucks were selling at the same price, we would therefore want to buy Apple
  - ↪ Consequently, to induce investors to buy all of the outstanding shares, the current price of Starbucks must be *lower* than $100

Arbitrage - Example

- Let's assume that investors are only willing to buy up all of Starbucks' shares if the current price of Starbucks is \$90

  ↪ The discount rate (or equivalently the expected return) for Starbucks is therefore:

$$E(r_{SBUX}) = \frac{E(CF_1^{SBUX}) - P_{SBUX}}{P_{SBUX}} = \frac{120 - 90}{90} = 0.333$$

	AAPL	SBUX
Boom Payoff (Pr=0.5)	140	160
Bust Payoff (Pr=0.5)	100	80
$E(CF_1)$	120	120
Time 0 Price	100	90
Discount Rate	20%	33.3%

- Now let's see how all of this relates to the APT equations

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Arbitrage - Example

- Let's begin by quantifying the business cycle factor f_{BC}
  - ↪ Let's rely on the same idea behind the NBER (National Bureau of Economic Research) business cycle indicator to construct our factor
  - ↪ The NBER indicator is one if the economy is in an expansion, and zero if the economy is in a recession
  - ↪ Recall that we need the *unexpected component* of the business cycle for the factor
  - ↪ Following our example, let's assume that there is a 50/50 chance that at time 1 we will be in an expansion/recession, so that the expected value of the indicator is 0.5
  - ↪ This means that the surprise in the business-cycle factor has a value of $0.5 = 1 - 0.5$ in expansions, and $-0.5 = 0 - 0.5$ in recessions

Arbitrage - Example

- Let's now calculate the factor loadings $b_{AAPL,BC}$ and $b_{SBUX,BC}$
  - ↪ Run a regression of Apple returns on the surprise in the factor:

$$r_{AAPL,t} = a_{AAPL} + b_{AAPL,BC} \cdot \tilde{f}_{BC,t} + e_{AAPL,t}$$

  - ↪ Since we only have two data points, we can fit the line perfectly:

$$\begin{aligned} 0.40 &= a_{AAPL} + b_{AAPL,BC} \cdot 0.5 \qquad \text{(boom)} \\ 0.00 &= a_{AAPL} + b_{AAPL,BC} \cdot -0.5 \qquad \text{(bust)} \end{aligned}$$

  - ↪ Solving these two equations yields $a_{AAPL} = 0.20$ and $b_{AAPL,BC} = 0.4$
  - ↪ Solving the corresponding two equations for Starbucks yields $a_{SBUX} = 0.33$ and $b_{SBUX,BC} = 0.89$
  - ↪ The factor loadings $b_{AAPL,BC}$ and $b_{SBUX,BC}$ tell us how much *systematic* business cycle risk Apple and Starbucks are exposed to
  - ↪ The intercepts correspond to the expected returns $E(r_{AAPL}) = a_{AAPL} = 0.20$ and $E(r_{SBUX}) = a_{SBUX} = 0.33$

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Arbitrage - Example

- Finally, let's calculate the factor risk premia λ_0 and λ_{BC}
  - ↪ To determine how investors price the risks we need the APT *pricing equation*:

$$E(r_i) = \lambda_0 + \lambda_{BC} \cdot b_{i,BC}$$

  - ↪ λ_{BC} represents the price of business cycle risk
    - How much more investors discount the cash flows as a result of having one extra unit of business cycle factor risk
  - ↪ λ_0 is the required return for a security with no risk
    - The time value of money
  - ↪ Solving these two equations based on our previous estimates for $b_{AAPL,BC}$ and $b_{SBUX,BC}$, yields $\lambda_0 = 0.0909$ and $\lambda_{BC} = 0.2727$
    - Note, with two equations in two unknowns, we can solve this *exactly*

Arbitrage - Example

- Arbitrage opportunities arise when the price of risk isn't consistent across all assets
  - ↪ Specifically, if the APT pricing equation:

$$E(r_i) = \lambda_0 + \lambda_{BC} \cdot b_{i,BC}$$

    isn't satisfied for all assets

- To illustrate, let's augment the previous example to include a *risk-free* asset with a return of 5%
  - ↪ Since $\lambda_0 = 0.0909$ based on the equations for Apple and Starbucks, we know that it is possible to combine Apple and Starbucks to create a *synthetic* risk-free portfolio with a return of 9.09%
  - ↪ Borrowing money at 5% to invest in this synthetic risk-free portfolio therefore represents an arbitrage opportunity

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Arbitrage - Example

- To determine how much of Apple and Starbucks we have to buy/sell to construct this *risk-free* portfolio, we need to solve:

$$b_{p,BC} = w_{AAPL} \cdot b_{AAPL,BC} + (1 - w_{AAPL}) \cdot b_{SBUX,BC} = 0$$

  ↪ Why is this portfolio risk-free?

  ↪ Solving this equation yields:

$$w_{AAPL} = 1.8182 \qquad w_{SBUX} = (1 - w_{AAPL}) = -0.8182$$

- We can create an *arbitrage portfolio* by investing \$1.8182 in Apple, shorting \$0.8182 worth of Starbucks, and borrowing \$1 (or equivalently shorting \$1 worth of the risk-free asset)

  ↪ This portfolio requires *zero* initial investment, but it has a *positive payoff* in *all* states

- To verify that this works, lets look at the return on the stock portion of the portfolio in the two possible states:

$$w_{AAPL} \cdot (140/100) + (1 - w_{AAPL}) \cdot (160/90) = 1.0909 \quad \text{(boom)}$$

$$w_{AAPL} \cdot (100/100) + (1 - w_{AAPL}) \cdot (80/90) = 1.0909 \quad \text{(bust)}$$

- The payoff from the zero-investment *arbitrage portfolio* is therefore $1.0909 - 1.05 = 0.0409$ in *both* the boom and bust states
  - ↪ So this portfolio is indeed risk-free
  - ↪ We can scale this up as much as we like
  - ↪ For a $1 million investment in this long-short portfolio, we would get a *risk-free* payoff of $40,900
  - ↪ We have created a "money pump" ...

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Arbitrage - Example

- This works because the systematic risk isn't priced consistently across the different assets

- Proceeding as before, we could have calculated λ_0 and λ_{BC} for different pairs of the securities:

Security 1	Security 2	λ_0	λ_{BC}
AAPL	SBUX	0.0909	0.2727
AAPL	RF	0.05	0.3750
SBUX	RF	0.05	0.3187

↪ We used the first set of premia to find the arbitrage

↪ We could have:

- Calculated the λ's using *any* pair of securities
- Calculated the expected return (or discount rate) for the third security
- Bought the high return and sold the low return security/portfolio

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What Have We Learned?

- Investors generally require different rates of return for different securities

- The required rates of returns depend on the "risk profiles" of the different securities

  ↪ If the risks are not priced consistently across all securities, there will be arbitrage opportunities

- The idea behind the APT is that investors will take advantage of these arbitrage opportunities, in turn pushing prices back "in line"

  ↪ This implicitly assumes that the arbitrageurs have unlimited capital (and patience)

  ↪ In some cases this may not be true, and there may in fact be *limits to arbitrage*

- Accordingly, the APT may be used as a systematic framework for spotting "good deals"

APT: A Formal Derivation

- Start out with a K-factor model for the N assets:

$$r_i = a_i + b_{i,1}\tilde{f}_1 + ... + b_{i,K}\tilde{f}_K + e_i \qquad i = 1, 2, ..., N$$

  ↪ where the factors have been normalize so that $E(\tilde{f}_j) = 0$

- Construct an *arbitrage portfolio*:

  ↪ Zero cost:

$$w_1 + w_2 + ... + w_N = 0$$

  ↪ No systematic risk:

$$w_1 b_{1,1} + w_2 b_{2,1} + ... + w_N b_{N,1} = 0$$
$$w_1 b_{1,2} + w_2 b_{2,2} + ... + w_N b_{N,2} = 0$$

$$w_1 b_{1,K} + w_2 b_{2,K} + ... + w_N b_{N,K} = 0$$

  ↪ Note, if $N > K$ we can always construct such a portfolio

APT: A Formal Derivation

- It must be the case that the expected returns/payoffs on *all* such arbitrage portfolios are equal to zero

  ↪ Why?

- From a result in linear algebra this implies the following *pricing equation*:

$$a_i = E(r_i) = \lambda_0 + \lambda_1 b_{i,1} + \lambda_2 b_{i,2} + \ldots + \lambda_K b_{i,K}$$

  ↪ What does this pricing equation look like graphically for $K = 1$?

  ↪ What does this pricing equation look like graphically for $K = 2$?

  ↪ What is the pricing equation for $K = 1$ and $f_1 = r_m$?

APT Glossary

- **Factors** (f_k's): Economy wide, or systematic, risks that impact the returns on most assets
- **Factor loadings** ($b_{i,k}$'s): How much a given asset i moves (on average) when the k'th factor moves by one unit (or 1%)
- **Factor risk-premia** (λ_k's): The effect on the expected return (or discount-rate) of a one unit increase in the sensitivity to the k'th factor
- **Arbitrage opportunity**: Positive payoff at zero cost
- **Arbitrage portfolio**: A well-diversified zero-cost portfolio with no systematic risk
  - $\hookrightarrow$ Such portfolios may be used to spot arbitrage opportunities, and assess whether the systematic risks (factors) are priced consistently across all assets

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APT Summary

- The APT can be used in place of the CAPM for:
  - ↪ Calculating expected returns and cost of capital
  - ↪ Performance evaluation
  - ↪ Risk management

- Unlike the CAPM, the APT does not tell us what the systematic risks that drive the returns are

- The APT relies on a statistical factor model for describing the systematic risks and the co-movements among returns
  - ↪ The usefulness of the APT depends on getting the “right” factors

- So, how do you determine the factors?

APT Practical Implementation: Three Approaches

1) *Macroeconomic Approach*

- ↪ Treat the factors f_j as the primitives
- ↪ The factors might include macroeconomic variables like inflation and GDP growth
- ↪ These variables should be able to capture all the systematic risks in the economy
- ↪ The Chen, Roll and Ross (CRR) model is one of the first examples of this approach

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2) *"Fundamental" Approach*

- ↪ Treat the factor *loadings* $b_{i,k}$'s as the primitives
- ↪ The loadings are inferred from "fundamental" information about the characteristics of the securities
- ↪ The corresponding factors must be constructed from indices based on these characteristics
- ↪ The Fama-French 3-factor model is an example of this approach

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APT Practical Implementation: Three Approaches

3) *Statistical Approach*

- ↪ Treat both the factors and the loadings as unobservable/latent
- ↪ Principal Components Analysis (PCA) provides a statistical procedure for identifying the "best" set of factors and factor loadings based on a sample of historical returns
- ↪ The resulting factors will be portfolios, or linear combinations, of the different assets
- ↪ However, the resulting factors are often hard to interpret from an *economic* perspective
- ↪ This makes forecasting very difficult/impossible

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Macroeconomic Factors

- This approach requires us to specify the factors a priori
- Most investment firms have their own preferred set of macro factors and corresponding factor sensitivities
  - ↪ Investment firms put *a lot* of effort into identifying "good" factors
- Some commonly used macro-economic risk factors:
  - ↪ GDP growth
  - ↪ Inflation
  - ↪ Interest Rates
  - ↪ Sentiment
  - ↪ Business Cycle Indicators

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Macroeconomic Factors

- Examples of business cycle indicators:

A. Leading indicators

1. Average weekly hours of production workers (manufacturing)
2. Initial claims for unemployment insurance
3. Manufacturers' new orders (consumer goods and materials industrie
4. Vendor performance—slower deliveries diffusion index
5. New orders for nondefense capital goods
6. New private housing units authorized by local building permits
7. Yield curve slope: 10-year Treasury minus federal funds rate
8. Stock prices, 500 common stocks
9. Money supply (M2)
10. Index of consumer expectations

B. Coincident indicators

1. Employees on nonagricultural payrolls
2. Personal income less transfer payments
3. Industrial production
4. Manufacturing and trade sales

C. Lagging indicators

1. Average duration of unemployment
2. Ratio of trade inventories to sales
3. Change in index of labor cost per unit of output
4. Average prime rate charged by banks
5. Commercial and industrial loans outstanding
6. Ratio of consumer installment credit outstanding to personal income
7. Change in consumer price index for services

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Macroeconomic Factors

- In order to obtain the factor surprises you need to subtract the market expectations
- To calculate the expectations of the factors you could use
  - ↪ A statistical forecasting model
  - ↪ Analyst surveys/expectations

This Week's Calendar

Date	ET	Release	For	Actual	Briefing.com	Consensus	Prior	Revised From
Feb 21	10:00	Leading Indicators	Jan	1.1%	0.6%	0.5%	0.3%	0.1%
Feb 21	14:00	FOMC Minutes	Jan 31					
Feb 22	08:30	Core CPI	Jan	0.2%	0.2%	0.2%	0.1%	0.2%
Feb 22	08:30	CPI	Jan	0.7%	0.4%	0.5%	-0.1%	
Feb 23	08:30	Initial Claims	02/18	278K	285K	300K	298K	297K
Feb 23	10:00	Help-Wanted Index	Jan	37	40	40	38	39
Feb 23	10:30	Crude Inventories	02/17	1121K	NA	NA	4853K	
Feb 24	08:30	Durable Orders	Jan	-10.2%	-4.0%	-2.0%	2.5%	1.3%

Macroeconomic Factors

- After having chosen the factors and constructed the factor surprises, $\tilde{f}_k = f_k - E(f_k)$, you need to estimate the factor loadings, $b_{i,k}$

- For each of the $i = 1, ..., N$ securities, run the time-series regression:

$$r_{i,t} = a_i + b_{i1}\tilde{f}_{1,t} + b_{i2}\tilde{f}_{2,t} + \ldots + b_{iK}\tilde{f}_{K,t} + \varepsilon_{i,t}$$

  - ↪ The estimated loading $\hat{b}_{i,k}$ for security i with respect to factor k measures how that security moves with that factor *on average*
  - ↪ $\varepsilon_{i,t}$ represents the idiosyncratic risk for security i
  - ↪ The APT assumes that $\varepsilon_{i,t}$ and $\varepsilon_{j,t}$ for assets i and j are (approximately) uncorrelated, and therefore can be diversified away

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Macroeconomic Factors

- The market prices of the risks associated with each of the factors, $\lambda_1, \ldots, \lambda_K$, may be estimated from a second-step cross-sectional regression:

$$\bar{r}_i = \lambda_0 + \lambda_1 \hat{b}_{i1} + \lambda_2 \hat{b}_{i2} + \ldots + \lambda_K \hat{b}_{iK} + u_i$$

where $\bar{r}_i$ denotes the average sample return on asset i

↪ This is similar to the two-step Fama-MacBeth procedure that we used in testing the CAPM

↪ $\hat{\lambda}_k$ provides an estimate of the difference in the expected returns between two otherwise identical securities except for a one unit difference in their exposure to factor k

↪ Note, if the APT held perfectly, all the u_i's should be equal to zero

Macroeconomic Factors - CRR

Chen, Roll and Ross (1986) specify the following five factors:

1. Unanticipated growth in industrial production (IP)
2. Changes in expected inflation, as measured by the change in r_{TBill} (EI)
3. Unexpected inflation (UI)
4. Unanticipated changes in bond return spread, as measured by $r_{Baa} - r_{AAA}$ (CG)
   - ↪ This is often called the “Default Spread”
5. Unanticipated changes in the slope of the term structure, as measured by $r_{TBond} - r_{TBill}$ (GB)
   - ↪ This is often called the “Term Spread”
6. They also include the return on the equal-weighted (EWNY) and value-weighted (VWNY) NYSE market portfolio when estimating the pricing equation
   - ↪ Why might you want to do that?

Macroeconomic Factors - CRR

- Estimated pricing equation (λs):

A	EWNY	IP	EI	UI	CG	GB	Constant
	5.021	14.009	−0.128	−0.848	0.130	−5.017	6.409
	(1.218)	(3.774)	(−1.666)	(−2.541)	(2.855)	(−1.576)	(1.848)
B	**VWNY**	**IP**	**EI**	**UI**	**CG**	**GB**	**Constant**
	−2.403	11.756	−0.123	−0.795	8.274	−5.905	10.713
	(−0.633)	(3.054)	(−1.600)	(−2.376)	(2.972)	(−1.879)	(2.755)

Table 13.4

Economic variables and pricing (percent per month × 10), multivariate approach

VWNY = Return on the value-weighted NYSE index; EWNY = Return on the equally weighted NYSE index; IP = Monthly growth rate in industrial production; EI = Change in expected inflation; UI = Unanticipated inflation; CG = Unanticipated change in the risk premium (Baa and under return – Long-term government bond return); GB = Unanticipated change in the term structure (long-term government bond return – Treasury-bill rate); Note that *t*-statistics are in parentheses.

Source: Modified from Nai-Fu Chen, Richard Roll, and Stephen Ross, "Economic Forces and the Stock Market," *Journal of Business* 59 (1986). Reprinted by permission of the publisher, The University of Chicago Press.

- The market portfolio isn't priced
  - ↪ What do you make of that?

- How do you interpret the sign of the λs?

"Fundamental" Approach

- Instead of directly identifying the systematic risk factors, the "fundamental" approach seeks to identify firm-characteristics that might proxy for different sensitivities to the underlying (latent) systematic risks
  - ↪ Differences in the characteristic should be associated with differences in expected returns
  - ↪ Firms that have similar characteristics should move together
  - ↪ We can then form portfolios of stocks sorted on these characteristics and use these as proxies for the factors

- When we looked at tests of the CAPM, we found that *small* firms and *value* firms had higher returns than predicted by the CAPM
  - ↪ The 3-factor Fama-French model is based on these findings

"Fundamental" Approach: The Fama-French Model

- The Fama-French 3-factor model is now used *extensively* by finance practitioners as an alternative to the CAPM:

$$r_i = r_f + \beta_{i,M} \cdot r_M + \beta_{i,SMB} \cdot r_{SMB} + \beta_{i,HML} \cdot r_{HML} + e_i$$

  - ↪ A stock's systematic risk is summarized by *three* betas
  - ↪ The usual market beta together with a *size* and a *value* beta
  - ↪ SMB and HML are "factor representing portfolios"
    - ▸ SMB: small minus big
    - ▸ HML: high minus low book-to-market

- The Fama-French-Carhart 4-factor model adds an additional momentum factor

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"Fundamental" Approach: The Fama-French Model

- To construct the SML and HML portfolios, Fama and French split the universe of stocks into 6 portfolios based on size (market capitalization) and value/growth (book-to-market value):

	Book-to-market		
Market Cap	**Low**	**Medium**	**High**
Small	Portfolio 1: Small growth	Portfolio 2: Small core	Portfolio 3: Small value
Large	Portfolio 4: Large growth	Portfolio 5: Large core	Portfolio 6: Large value

  - ↪ Small minus Big (SMB): $(1/2SG + 1/2SV) - (1/2LG + 1/2LV)$
  - ↪ High minus Low (HML): $(1/2SV + 1/2LV) - (1/2SG + 1/2LG)$
  - ↪ Note, these are both *zero-cost* portfolios

- Similar zero-cost portfolios are now commonly used in accounting for other characteristics and "fundamental" risks
  - ↪ Carhart momentum factor (MOM)

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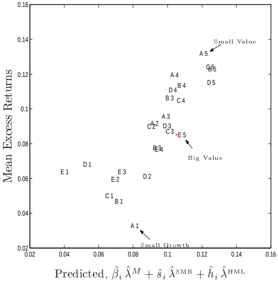

- The Fama-French model does a good job explaining the value and size anomalies
  - ↪ This is not purely mechanical, but works because there are strong comovements among value/growth and small/large stocks
  - ↪ The Fama-French model also help explain some of the other anomalies that we noted in our discussion of tests of the CAPM

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Statistical Approach

- Principal Component Analysis (PCA) is an advanced statistical technique for extracting common factors from a panel of historical returns:

$$r_{i,t} = b_{i,0} + b_{i,1} f_{1,t} + b_{i,2} f_{2,t} + \ldots b_{i,K} f_{K,t} + \varepsilon_{i,t}$$

  - ↪ PCA aims to describe the cross-sectional variation in the returns as accurately as possible, using as small a number of factors K as possible
  - ↪ PCA extracts the "most relevant" information from the data, classifying the remainder $\varepsilon_{i,t}$ as "noise," or non-systematic risk
  - ↪ When applying principal components to a large cross-section of U.S. stocks you typically end up with 4-8 "significant" factors
    - ▸ The first most important principal component is typically highly correlated with the market portfolio
  - ↪ When applied to interest rates, you typically end up with 3 factors
    - ▸ "Level," "slope," and "curvature"

Factor Tilting

- Factor Tilting is designed to take advantage of supposedly superior forecasts, or "views," about the systematic risk factors
  - ↪ This mirrors the idea of "market timing" within the context of the CAPM
  - ↪ By contrast to the CAPM, the APT allows us to express more specific factor "views," rather than only expressing a "view" about the aggregate market
- Given superior forecasting ability, it is possible to earn superior returns by varying the factor betas/loadings in accordance with these "views"
  - ↪ Increase the loading when you think that a factor is likely to be greater than consensus opinion
  - ↪ Decrease the loading when you think that a factor is likely to be less than what is generally expected

Factor Tilting

- Recall the return generating process:

$$r_{p,t} = E[r_{p,t}] + b_{p,1}\tilde{f}_{1,t} + \cdots + b_{p,n}\tilde{f}_{n,t} + e_{p,t}$$

where the $\sim$'s refer to the factor surprises

  - $\hookrightarrow$ The "market" believes $E[\tilde{f}_{k,t}] = 0$
  - $\hookrightarrow$ Suppose that you have superior information leading you to believe that $\tilde{f}_{1,t} \neq 0$
  - $\hookrightarrow$ You know something about the first factor that the "market" doesn't
  - $\hookrightarrow$ To take advantage of this you would want to "tilt" your portfolio and increase/decrease $b_{p,1}$

- Let's look at a specific example

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Factor Tilting - Example

- Suppose that you have estimated the following 2-factor model for the three securities A, B and C:

$$\begin{aligned} r_A &= 0.12 + 1 \cdot \tilde{f}_1 + 1 \cdot \tilde{f}_2 + e_A \\ r_B &= 0.12 + 1 \cdot \tilde{f}_1 + 2 \cdot \tilde{f}_2 + e_B \\ r_C &= 0.12 + 3 \cdot \tilde{f}_1 + 2 \cdot \tilde{f}_2 + e_C \end{aligned}$$

  where factor 1 is a foreign income factor, and factor 2 is an interest rate factor

  $\hookrightarrow$ What are the expected returns for each of the three securities

  $\hookrightarrow$ How would you find the factor risk premia?

Factor Tilting - Example

- You believe that:
  - ↪ Europe and Japan will finally come out of their economic slump, and exports of U.S. produced goods will therefore rise more than the market expects
  - ↪ Analysts' expectations about U.S. interest rates are generally correct

- Using the estimated factor model, how might you take advantage of your supposedly superior forecast?
  - ↪ You want to construct a portfolio with "a lot" of factor 1 risk, and not "too much" factor 2 risk
  - ↪ Intuitively, in which of the three stocks would you want to invest more heavily?
  - ↪ Why?

Factor Tilting - Example

- Specifically, let's assume that we want a loading of 10 for factor 1 and 0 for factor 2:

$$
\begin{aligned}
w_A + w_B + w_C &= 1 \\
1 \cdot w_A + 1 \cdot w_B + 3 \cdot w_C &= 10 \\
1 \cdot w_A + 2 \cdot w_B + 2 \cdot w_C &= 0
\end{aligned}
$$

  ↪ The first equation is the usual restriction that the portfolio weights must sum to one

- Solving these three equations:

  ↪ $w_A = 2$, $w_B = -5.5$, $w_C = 4.5$

  ↪ How do you actually solve this?

Factor Tilting - Example

- If the "surprise" in the foreign income factor is 2%, the best guess for the return on this portfolio is:

$$2 \cdot 0.12 - 5.5 \cdot 0.12 + 4.5 \cdot 0.12 + 10 \cdot 0.02 = 0.32$$

  - ↪ Much larger than the 0.12 without tilting
  - ↪ The portfolio that we just constructed had a loading of 10 on factor 1 and a loading of 0 on factor 2
  - ↪ If instead you believed that you had superior information about U.S. interest rates, you could have constructed a portfolio that only loads on factor 2

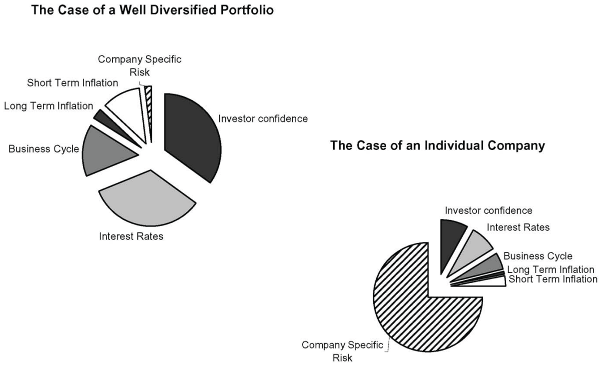
- A portfolio that only loads on one of the factors is called a *factor-mimicking portfolio*
  - ↪ Note, the APT effectively price securities *relative* to a specific set of factor-mimicking portfolios

Factor-Mimicking Portfolios

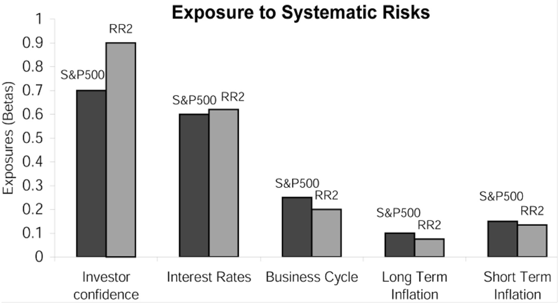
- Factor-mimicking portfolios are used extensively by practitioners in the implementation of portfolio tilting strategies
  - ↪ They can also be used for hedging specific risks
- Sometimes common sense dictates the use of certain assets as factor-mimicking portfolios
  - ↪ Oil futures to mimic an oil price factor
  - ↪ Treasury Inflation-Protected Securities (TIPS) to mimic an inflation factor
  - ↪ Real Estate Investment Trusts (REITs) to mimic a real-estate factor
  - ↪ Commodity futures to mimic a commodity price factor
- A number of different ETFs also naturally serve as factor-mimicking portfolios

APT and Asset Allocation

- The APT may also be used for more nuanced broad based asset allocation decisions
  - ↪ A well-diversified portfolio's long-term return and volatility are (almost) completely determined by its factor loadings



- The factors contribute differently to the aggregate risk and return of different portfolios
  - ↪ S&P 500 versus RR2 (another well diversified portfolio):

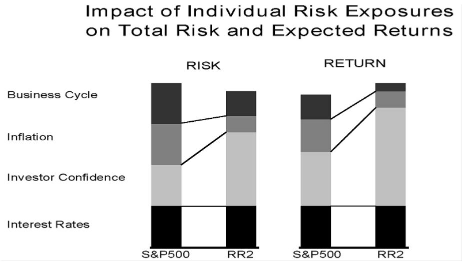


APT and Asset Allocation

- Assuming that the factors are uncorrelated, the Sharpe ratio of a well-diversified portfolio may be expressed as:

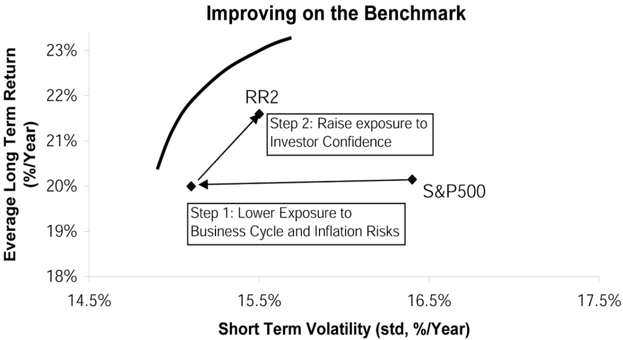
$$SR = \frac{\lambda_1 b_{p,1} + \lambda_2 b_{p,2} + \ldots + \lambda_K b_{p,K}}{\sqrt{\sigma_1^2 b_{p,1}^2 + \sigma_2^2 b_{p,2}^2 + \ldots + \sigma_K^2 b_{p,K}^2}}$$

$\hookrightarrow$ More strings to play on ...



APT and Asset Allocation

- By managing a portfolio's systematic risk exposures (factor loadings) it may be possible to achieve higher Sharpe ratios



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Summary

- The APT framework provides for a very general and rich class of asset pricing models
  - ↪ In contrast to the CAPM, the APT does not specify what the systematic risks are

- Any specific APT model is only as good as the factor model it assumes:

$$r_{i,t} = b_{i,0} + b_{i,1} f_1 + \ldots + b_{i,K} f_K + \varepsilon_{i,t}$$

  - ↪ There are different ways to come up with "good" factor models

- Factor models are widely used in practice as alternatives to the CAPM and the single-index model
  - ↪ MSCI BARRA is one of the leading commercial providers of factor models

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