Modelling Time-Varying Exchange Rate Dependence using the Conditional Copula

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A Quotation

Problems involving dependent pairs of variables have been studied most intensively in the case of bivariate Normal distributions… This is due primarily to the importance of [this case] but perhaps also to the fact that they exhibit only a particularly simple form of dependence…

- E. L. Lehmann (1966).

Overview

1. *Intro to / Review of Copulas*
2. Motivation
3. Evaluating copula models
4. Applying copulas to exchange rate modelling
   a) The data
   b) The models
   c) Results
5. Conclusions and potential future work

What is a copula?

- It is a function (not a number)

- It links (or *couples*) marginal distributions together to form a joint distribution

- It contains all of the information in the joint distribution not captured in the marginal distributions
  - $\rightarrow$ that is, all of the *dependence* information.

Deriving a copula

- Suppose $X \sim F$, $Y \sim G$ and $(X,Y) \sim H$.
- Let $U \equiv F(X)$ and $V \equiv G(Y)$.
  - (Aside: Then (Pearson, 1933) $U \sim \mathrm{Unif}(0,1)$ and $V \sim \mathrm{Unif}(0,1)$. U and V are the *probability integral transforms* of X and Y.)
- Let $(U,V) \sim C$. By standard theory on the distribution of transformations of random variables we obtain:

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- $$c(\, F(X)\, ,\, G(Y)\,) = h(X,Y) \cdot \begin{vmatrix} dX/dU & dX/dV \\ dY/dU & dY/dV \end{vmatrix}$$

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$$= h(X,Y) \cdot \begin{vmatrix} f(X)^{-1} & 0 \\ 0 & g(Y)^{-1} \end{vmatrix}$$

Deriving a copula (cont'd)

- $c(\ F(X)\ ,\ G(Y)\ ) = h(X,Y) \cdot \begin{vmatrix} f(X)^{-1} & 0 \\ 0 & g(Y)^{-1} \end{vmatrix}$
- $c(\ F(X)\ ,\ G(Y)\ ) = \dfrac{h(X,Y)}{f(X) \cdot g(Y)}$, or

Deriving a copula (cont'd)

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- $c(\,F(X)\,,\,G(Y)\,) = \dfrac{h(X,Y)}{f(X) \cdot g(Y)}$, or
- $h(X,Y) = f(X) \cdot g(Y) \cdot c(\,F(X)\,,\,G(Y)\,)$

Joint density

Marginal densities

Copula density

Defining the copula

Definition: A 2-dimensional copula is a function $C:[0,1]\times[0,1]\rightarrow[0,1]$ with the following properties:

1. $C(0,w) = C(w,0) = 0$, and $C(w,1)=C(1,w)=w$, for every $w\in[0,1]$
2. $V_C([u_1,u_2]\times[v_1,v_2]) \equiv \Pr[u_1 \leq U \leq u_2 \cap v_1 \leq V \leq v_2]$
   $= C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$
   for all $u_1,v_1,u_2,v_2 \in [0,1]$ s.t. $u_1\leq u_2$ and $v_1\leq v_2$.

- Alternative definition: if $U\sim Unif(0,1)$ and $V\sim Unif(0,1)$, then C is any function satisfying the properties of being a joint cdf of (U,V); it is a *multivariate cdf with Unif(0,1) margins.*

Sklar's theorem

- Sklar (1959) showed that we may decompose the distribution of (X,Y) into three parts:

$$H(x, y) = C(F(x), G(y)) \quad \forall\, x,y \in \overline{\Re}$$

Sklar's theorem

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$$H(x, y) \Leftrightarrow C(F(x), G(y)) \quad \forall\, x,y \in \bar{\Re}$$

Joint dist'n of X and Y

Copula of X and Y

Marginal dist'n of X

Marginal dist'n of Y

Why should economists care about copulas?

- *Because economists care about dependence.*

- There are two main places in economics where knowledge of the copula would be useful...

I. When the joint distribution is required

- There are a number of situations where the first couple of moments are not enough:

1. Pricing options with more than one underlying asset (Rosenberg, 2000)
2. Calculating the Value-at-Risk of a portfolio of assets (Hull and White, 1998)
3. Multivariate density forecasting (Diebold, Hahn and Tay, 1999)
4. Quantile regression

II. When linear correlation isn't enough

- Linear correlation is a nice, simple measure of dependence, but it's not always sufficient:

1. Asymmetric equity correlations: stock returns seem more dependent on down days than on up days (Erb, Harvey and Viskanta, 1994)

2. Financial contagion: international markets seem more dependent in crash states than during 'normal' states

3. Asset pricing when you don't believe that investors have quadratic utility or that returns are multivariate normally distributed, as is assumed in the CAPM

Visualizing copulas

- Copulas themselves aren't that nice to look at

- One way of looking at them is to use them to couple two standard normal margins, and then examine the resulting joint distribution.

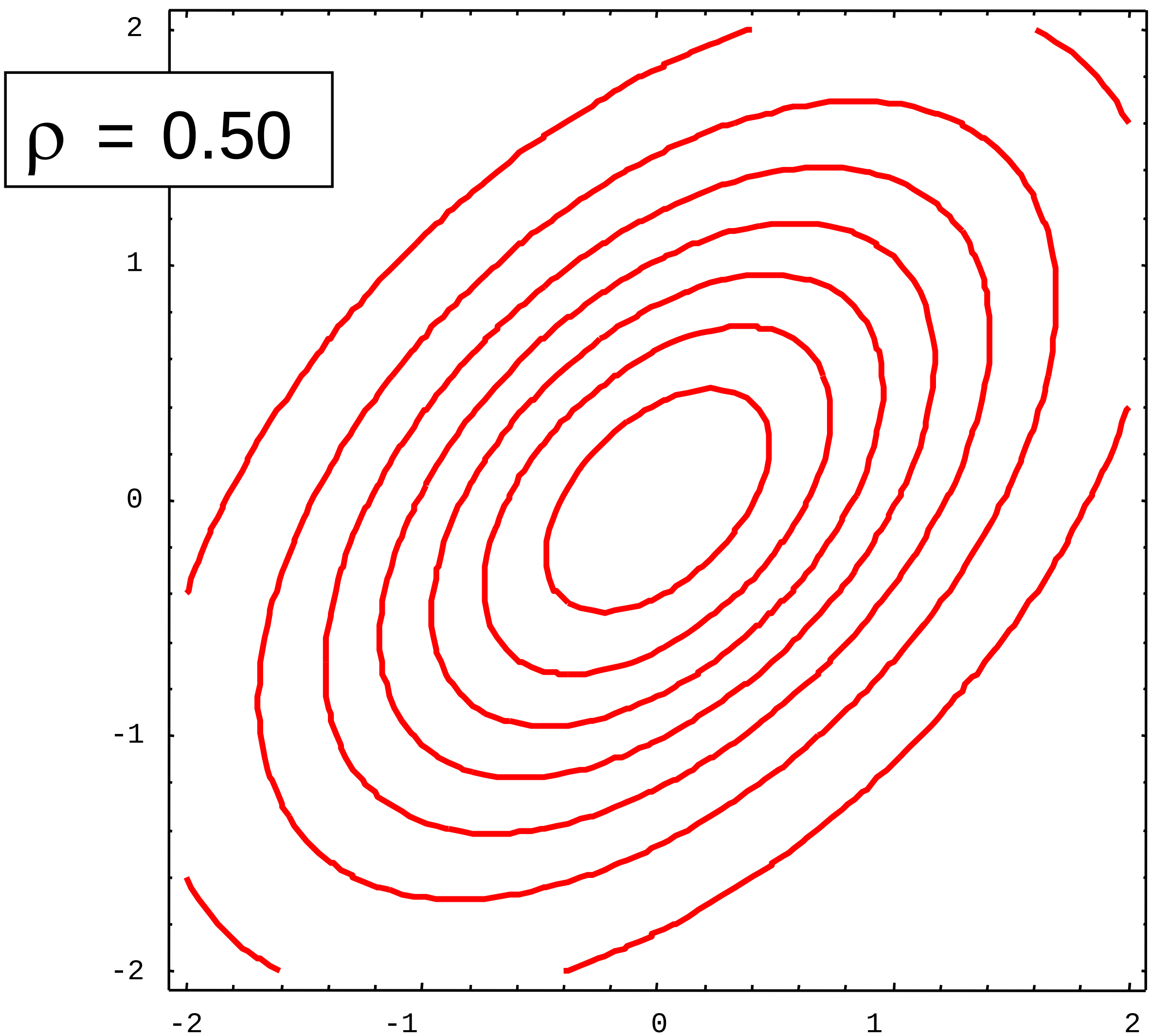
$$C(\ F(x)\ ,\ G(y)\ )\Rightarrow H(\ x\ ,\ y\ )$$

Copula of interest

N(0,1)

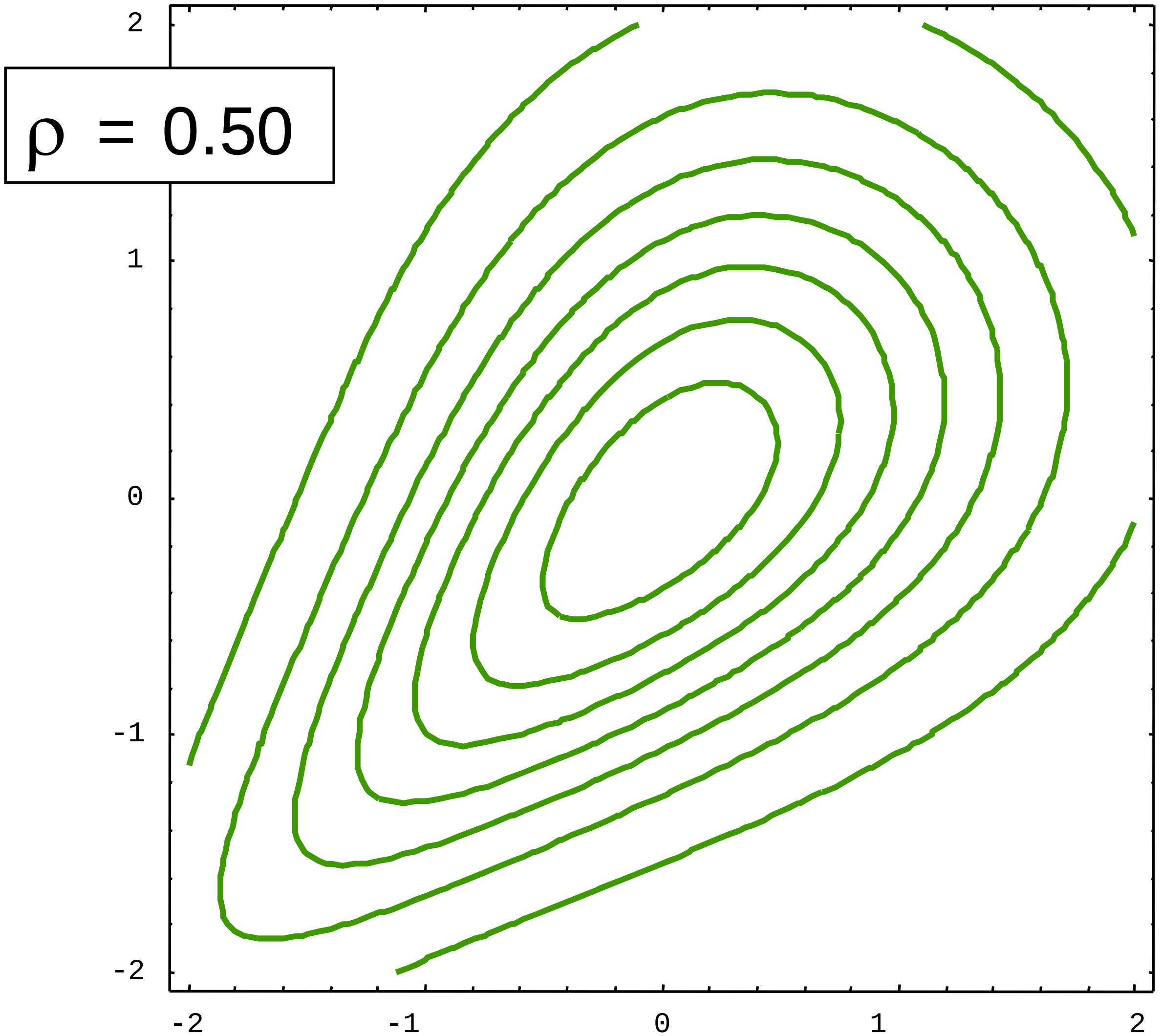
Resulting joint dist'n

Bivariate normal distribution



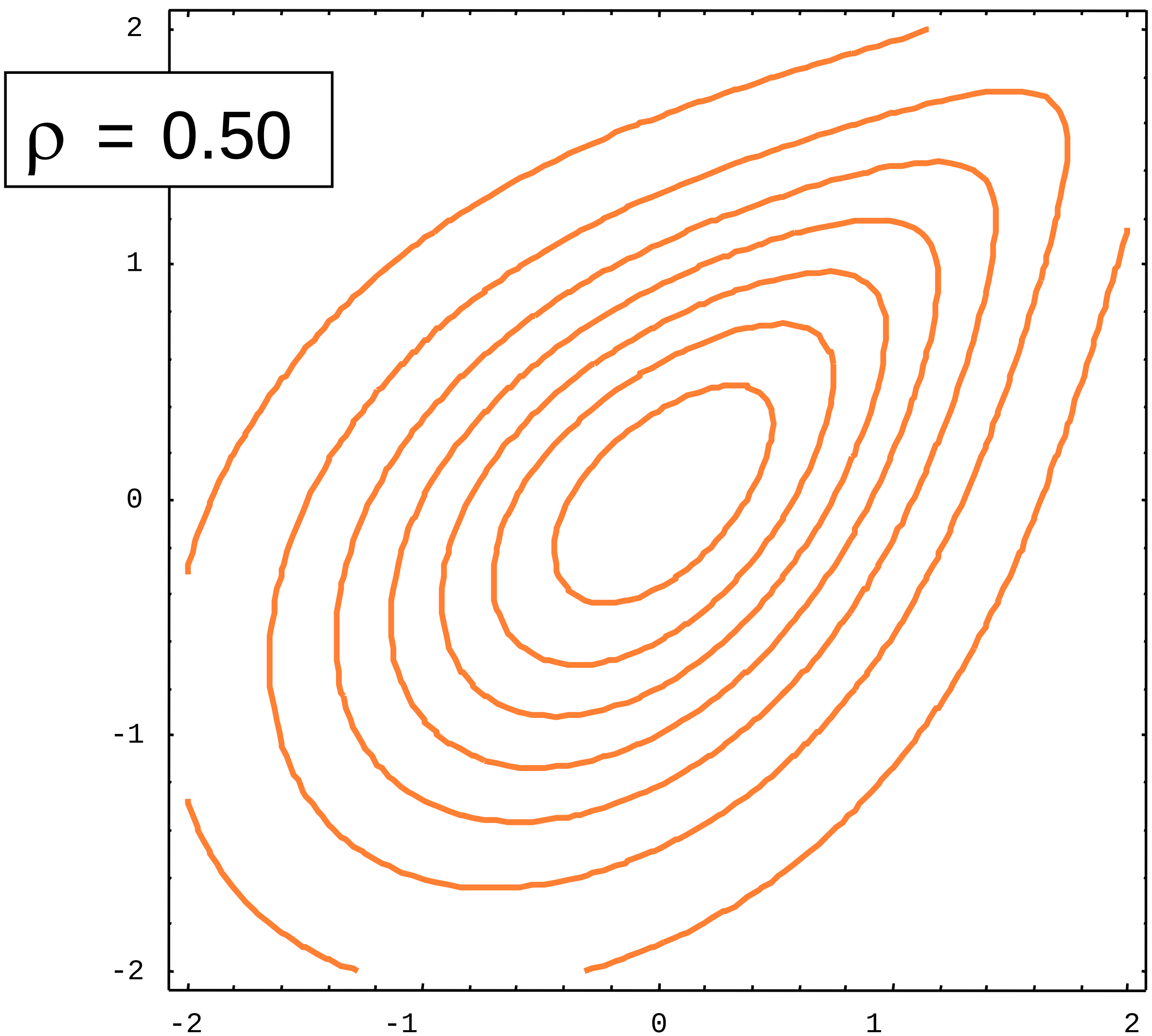
ρ=0.5

Clayton's (1978) copula



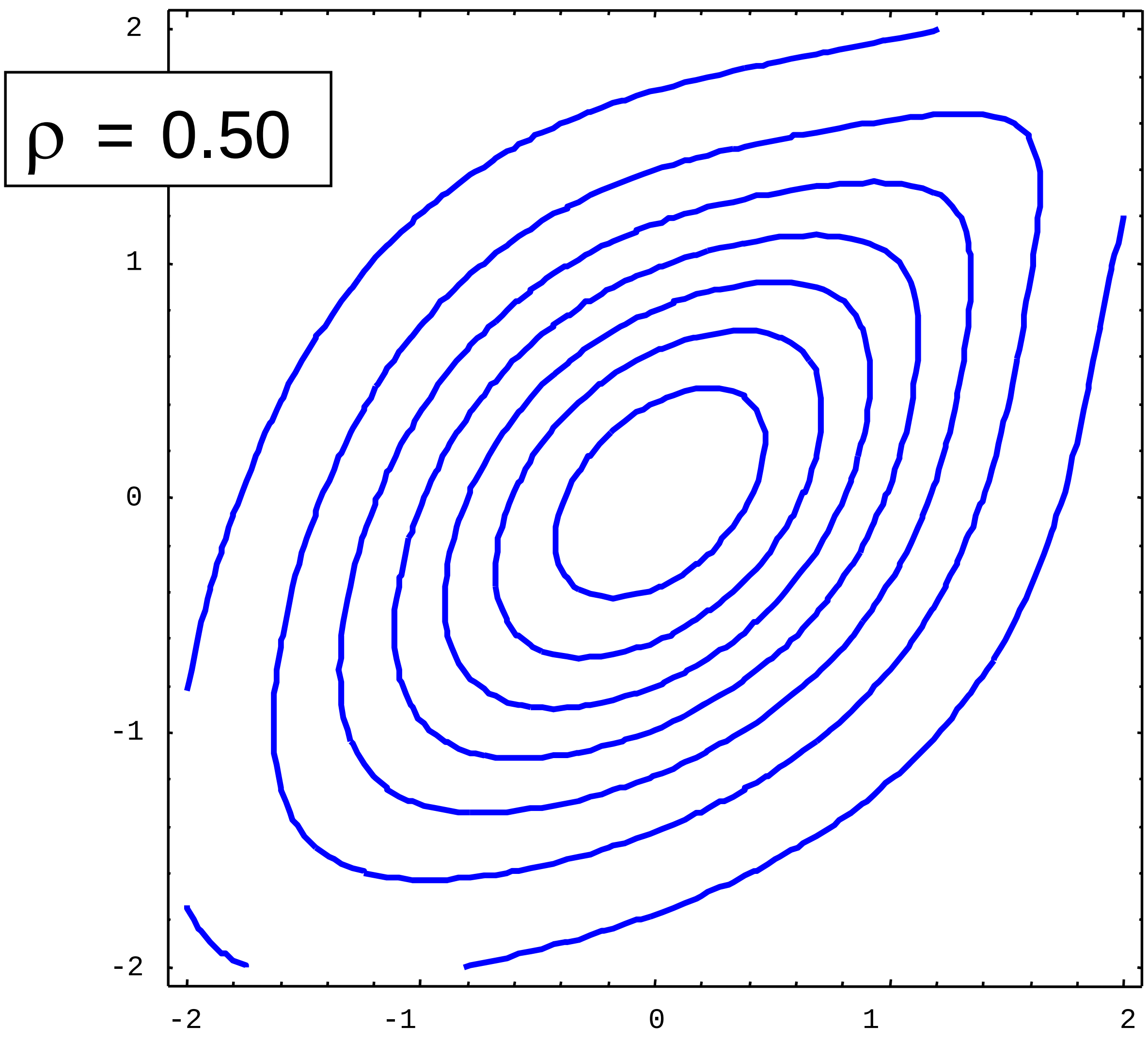
θ=1

Gumbel's (1960) copula



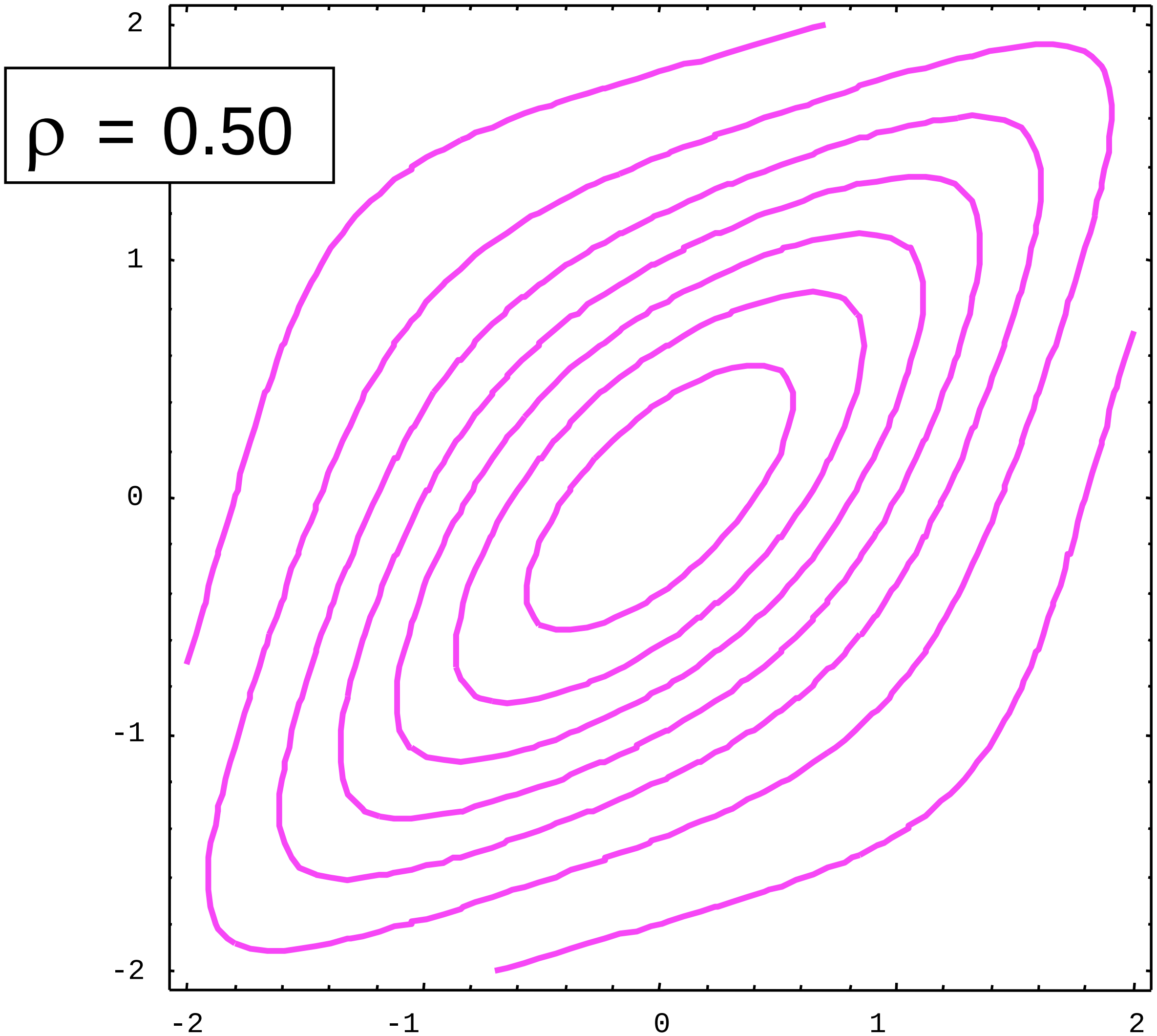
δ=1.5

Joe-Clayton (1997) copula



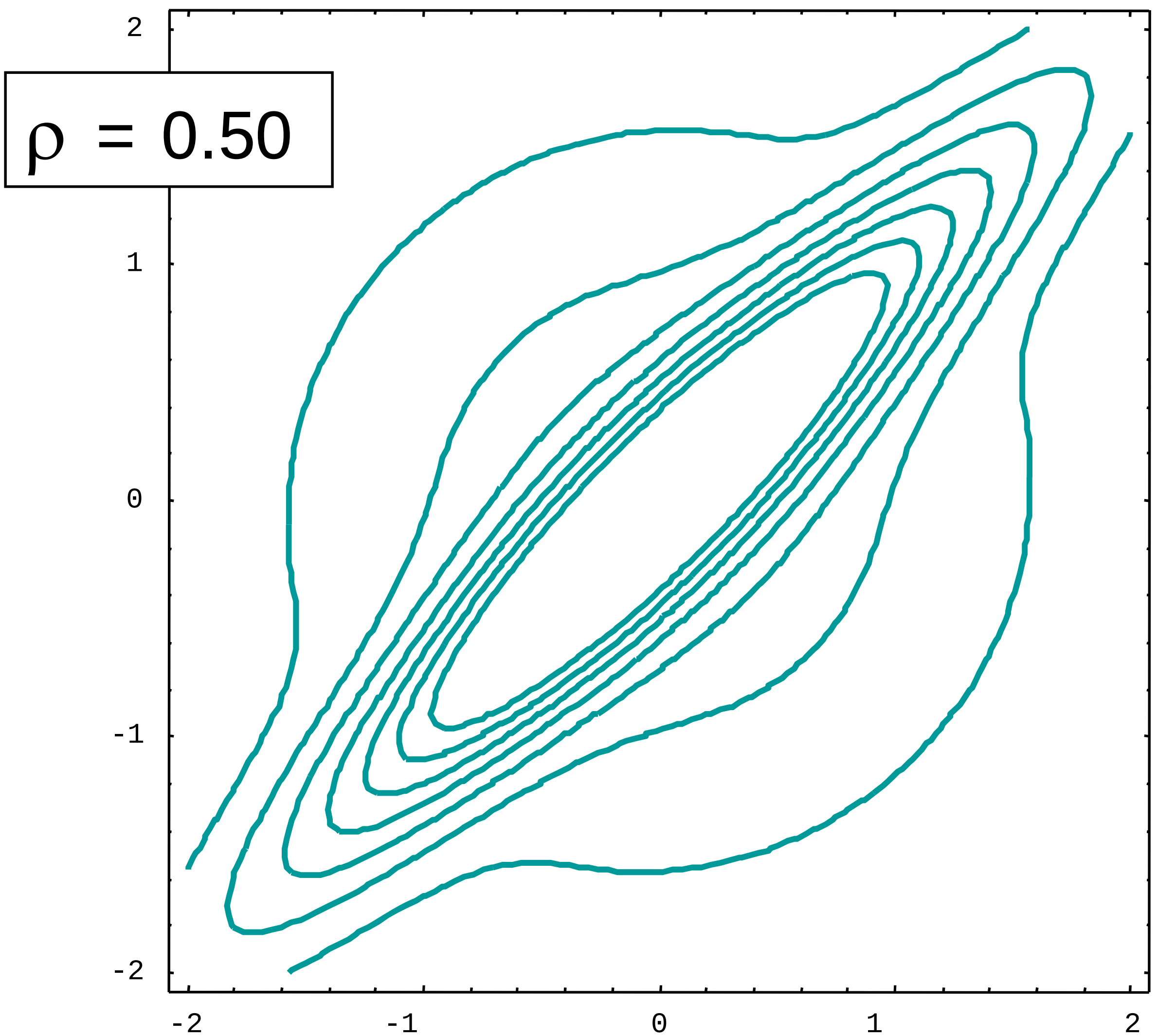
κ=1.42,
γ=0.47

Student's t_3 copula



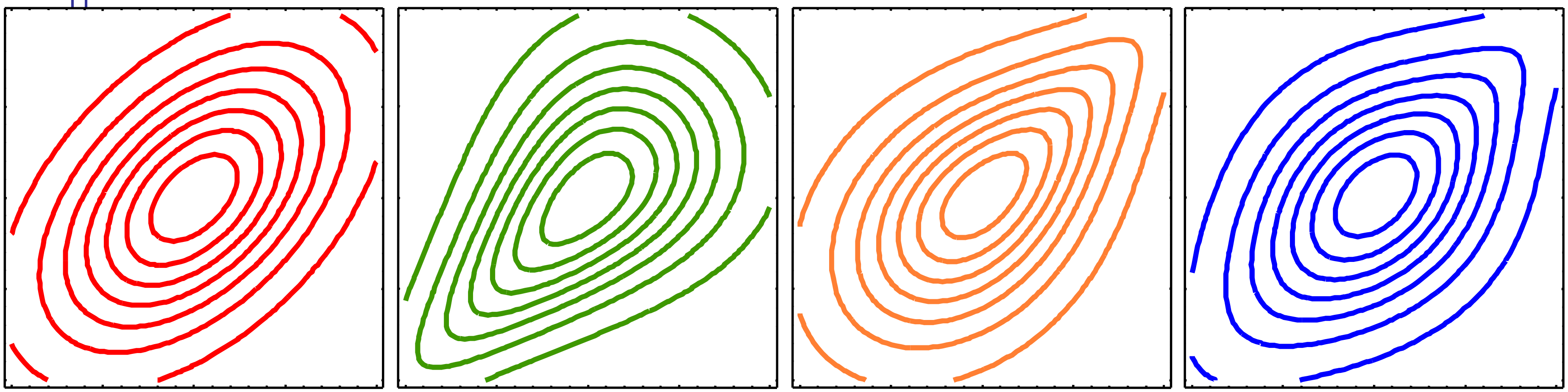
ρ=0.50,
υ=3

Mixture of Normal copulas



ρ_1=0.95,
$\rho_{2=}$0.05,
ω=0.5

All of these distributions have N(0,1) marginal distributions and ρ=0.50



	Gaussian	Clayton	Gumbel	Joe-Clayton
ρ	0.50	0.50	0.50	0.50
ρ_s	*0.4878*	0.4664	0.4755	0.4512
τ	*0.3370*	0.3273	0.3272	*0.3151*
β	0.3333	0.3333	0.3311	0.2972
τ^U	0.0000	0.0000	0.4126	0.3707
τ^L	0.0000	0.5000	0.0000	0.2288

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Goodness-of-fit testing

$$(X,Y) \sim H = C(\, F(X) \,,\, G(Y) \,)$$

- We need to ensure that the marginal distribution models are well specified before modelling the copula
  - Evaluation of univariate density models

- Recall that the copula can be defined as the joint distribution of two *i.i.d.* Unif(0,1) random variables
  - So the evaluation of copula models is a special case of the more general problem of evaluating multivariate density models

Goodness-of-fit testing

- No single best method of evaluating density models has yet emerged – we will employ two different methods:

1. Diebold, *et al.* (1998) and (1999): Test (separately) whether $U_t \equiv F_t(X_t | I_{t-1})$ and $V_t \equiv G_t(Y_t | I_{t-1})$ are *i.i.d.* and Unif(0,1)

2. Break the density model into multiple interval (or region) models and evaluate these
   - → Extension of work by Christoffersen (1998) and Engle and Manganelli (1999).

1. Diebold, *et al.* (1998) and (1999)

$U_t \equiv F_t(X_t|I_{t-1})$ and $V_t \equiv G_t(Y_t|I_{t-1}) \sim$ *i.i.d.* Unif(0,1)

- Employ two tests: one for the *i.i.d.* property, and one for the Unif(0,1) property.
- ***i.i.d.*** **:** regress $(u_t\text{-ubar})^k$ and $(v_t\text{-vbar})^k$ on 20 lags of each, for k = 1,2,3,4. Use LM test that all coefficients are zero.
- **Unif(0,1):** Use Kolmogorov-Smirnov test that $\{u_t\}$ and $\{v_t\}$ are not significantly different from Unif(0,1).

2. 'Hit' tests

- Christoffersen (1998) and Engle and Manganelli (1999) both proposed looking at the sequence of 'hits' to determine the adequacy of an interval forecast (like a VaR(q) forecast)

  Let $Hit_t \equiv \mathbf{1}\{x_t < VaR_t(0.05)\}$, t = 1,2,...,T.

- If the forecast is good, the sequence of hits should be *i.i.d.* Bernoulli(0.05).

- We can thus test the accuracy of an interval forecast by testing

  H_0: $Hit_t \sim$ *i.i.d.* Bernoulli(0.05).

Multivariate Hit tests

- In the bivariate case, a hit would be defined as the observation (x_t , y_t) lying in some *region*, R, of the support of the distribution:

$$Hit_t \equiv \mathbf{1}\{(x_t , y_t) \in R\}, \ t = 1,2,\ldots,T.$$

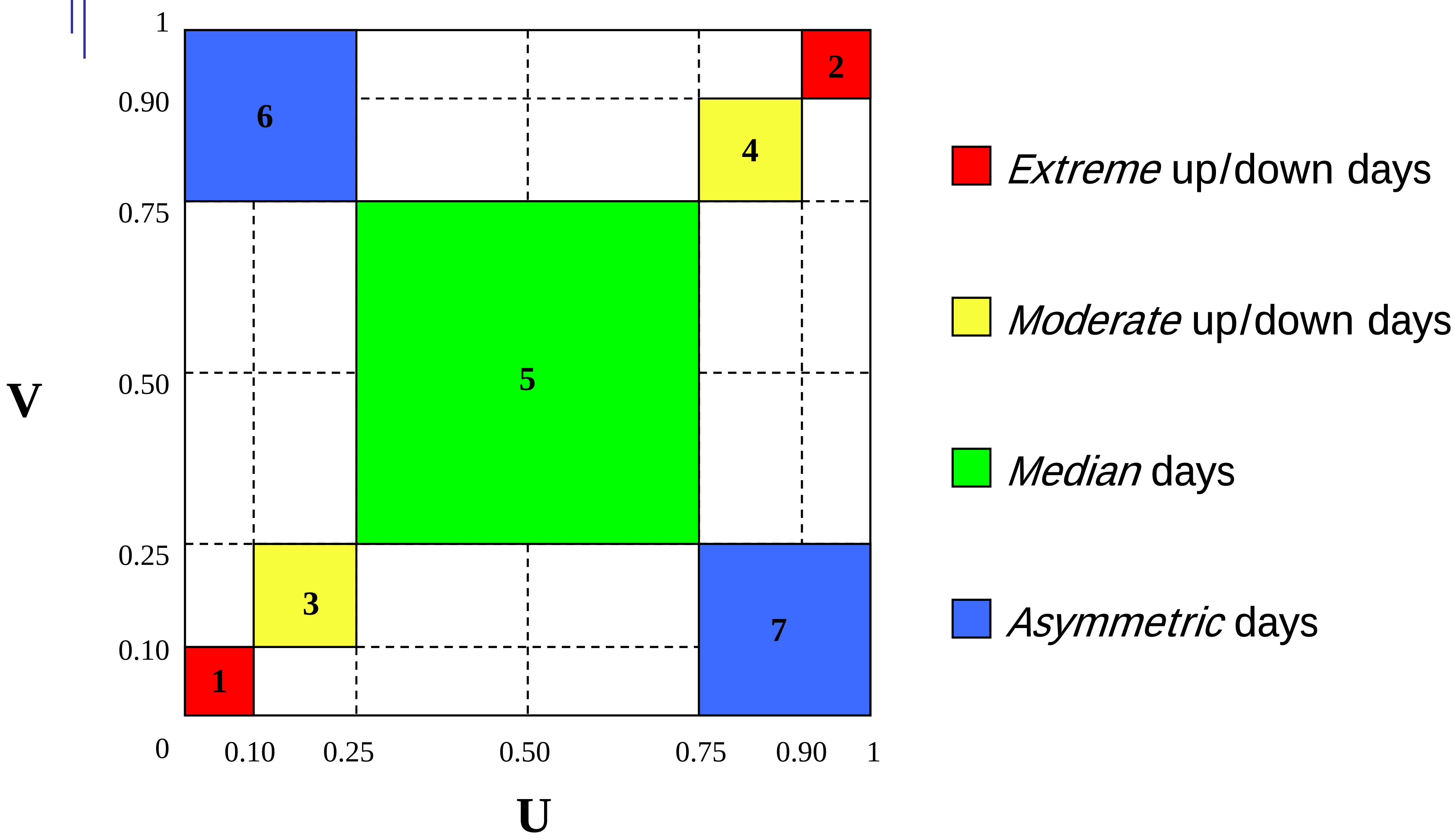
- Unlike univariate interval forecasts, the probability of a hit will *not* be constant: it depends on the amount of copula probability mass in that region.

- In our case if the model is accurate we will find:

$$Hit_{it} \equiv \mathbf{1}\{(x_t , y_t) \in R^i\} \sim i.n.i.d.\ \text{Bernoulli}(p_{it})$$

  - where p_{it} is given by the copula at time t : $C_t(\cdot,\cdot|I_{t-1})$

Hit regions


1
0.90
0.75
0.50
0.25
0.10
0
V
0.10
0.25
0.50
0.75
0.90
1
U
1
2
3
4
5
6
7
Extreme up/down days
Moderate up/down days
Median days
Asymmetric days

Individual region hit tests

- We model the hits in a logistic regression framework:

H_0: the model is correctly specified

$\Rightarrow$ $Hit_{it} \sim$ Bernoulli(p_{it}), for t = 1,2,...,T.

H_1: the model is not correctly specified

$\Rightarrow$ $Hit_{it} \sim$ Bernoulli(π_{it}), for t = 1,2,...,T, where

$$\pi_{it} = \pi_i(Z_{it}, \beta_i, p_{it}) = \Lambda\left(Z_{it}^{/} \cdot \beta_i - \ln\frac{1-p_{it}}{p_{it}}\right)$$

- where $\Lambda(x)=(1+e^{-x})^{-1}$ is the logistic function, Z_{it} contains anything thought to influence the probability of a hit, β_i is the parameter vector to be estimated.
- Thus the test becomes H_0: $\beta_i = 0$ vs. H_1: $\beta_i \neq 0$.

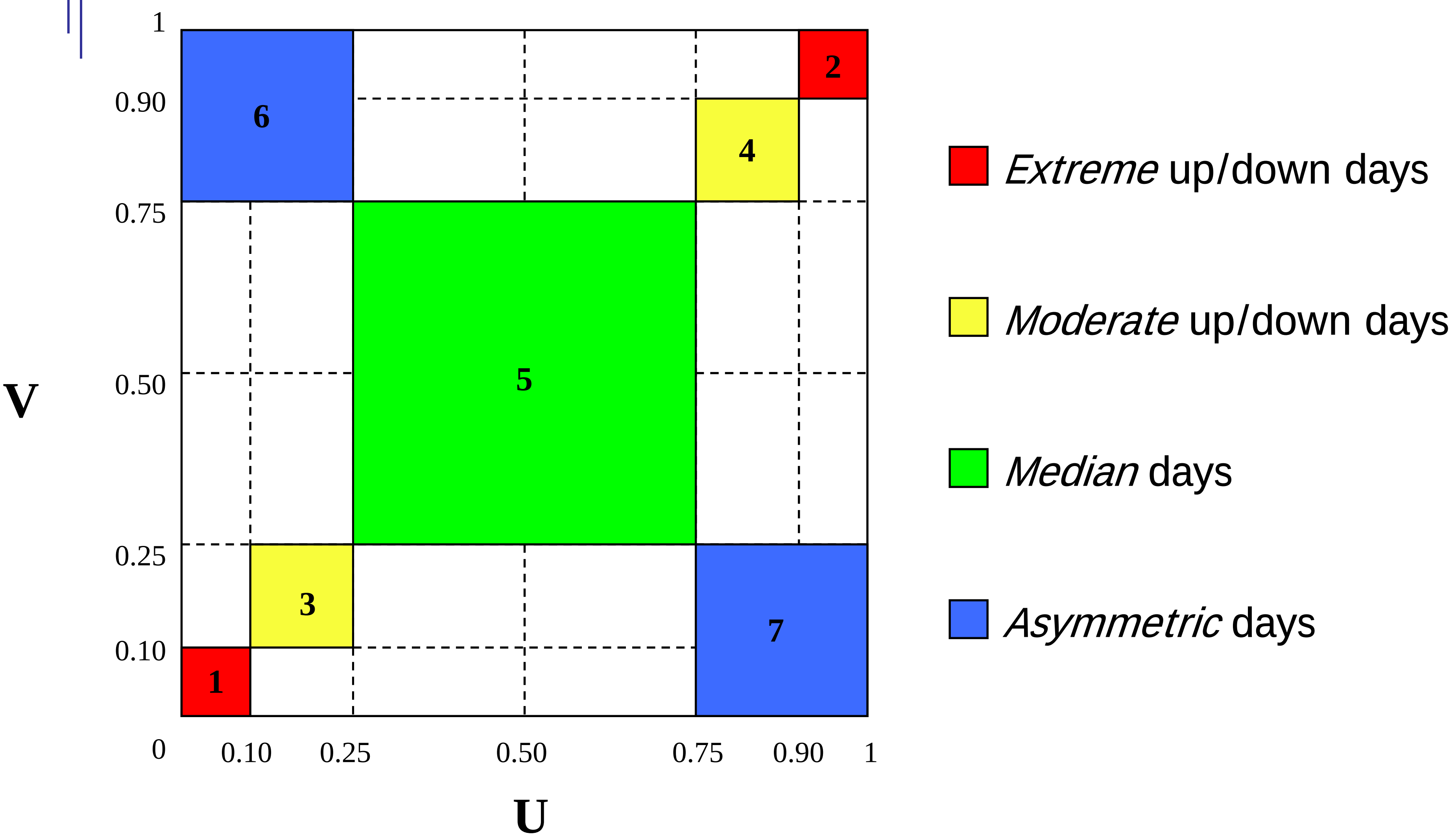
Testing multiple region forecasts at once

- Define $Hit_{it} \equiv \mathbf{1}\{(x_t , y_t) \in R^i\}$, where $R^i \cap R^j = \emptyset$ and $\cup_{i=0}^{K} R^i = S$, then define a new random variable:

$$M_t \equiv \sum_{i=0}^{K} i \cdot Hit_{it}$$

- M_t is a multinomial random variable
- Let $P_t = [p_{0t} , p_{1t} , \ldots , p_{Kt}]$, where p_{it} is $\Pr[Hit_{it}=1]$ implied by model.
- Then $M_t \sim$ Multinomial (P_t) if the model is correctly specified

Hit regions



1
0.90
0.75
0.50
0.25
0.10
0
V
6
2
4
5
3
7
1
0.10
0.25
0.50
0.75
0.90
1
U
Extreme up/down days
Moderate up/down days
Median days
Asymmetric days

Multiple region hit tests

H_0: the model is correctly specified

$\Rightarrow M_t \sim$ Multinomial(P_{it}), for t = 1,2,...,T.

H_1: the model is not correctly specified

$\Rightarrow M_{it} \sim$ Multinomial(Π_t), for t = 1,2,...,T, where

$$\pi_{1t} = \pi_1(Z_t, \beta, P) = \Lambda\left(Z_{1t}^{/} \cdot \beta_1 - \ln \frac{1 - p_{it}}{p_{it}} \right)$$

$$\pi_{jt} = \pi_j(Z_t, \beta, P) = \left(1 - \sum_{i=1}^{j-1} \pi_{it} \right) \cdot \Lambda\left(Z_{jt}^{/} \cdot \beta_j - \ln \frac{1 - \sum_{i=1}^{j} p_{jt}}{p_{jt}} \right), \quad for\ j = 2,...,K$$

$$\pi_{0t} = 1 - \sum_{j=1}^{K} \pi_{jt}$$

Again, the test reduces to H_0: $\beta = 0$ vs. H_1: $\beta \neq 0$.

Difference from previous ‘hit’ tests

- Christoffersen (1998) proposed modelling the sequence of this as a first-order Markov chain, as a test of the *i.i.d.* assumption
  - $\rightarrow$ Difficult to allow for higher order serial dependence
  - $\rightarrow$ Doesn’t include exogenous variables that may influence the probability of a hit

- Engle and Manganelli (1999) model the hits in a *linear probability model*.
  - $\rightarrow$ Easy to include more lags or exogenous regressors
  - $\rightarrow$ Simple to estimate and draw inference from
  - $\rightarrow$ BUT normal distribution may not be a good approximation to a Bernoulli random variable...

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Copulas in Economics

- The original theory of copulas was developed for the *i.i.d.* case. But we can extend it:

- $H(x_t, y_t) = C(F(x_t), G(y_t))$ - *i.i.d.*

- $H_t(x_t, y_t) = C_t(F_t(x_t), G_t(y_t))$ - *i.n.i.d.*

- $H_t(x_t, y_t | I_{t-1}) = C_t(F_t(x_t | I_{t-1}), G_t(y_t | I_{t-1}) | I_{t-1})$
  where $I_t = \sigma(\mathbf{z}_{t+1}, x_t, y_t, \mathbf{z}_t, x_{t-1}, y_{t-1} \ldots)$. - *d.n.i.d.*

Applying copulas to FX modelling

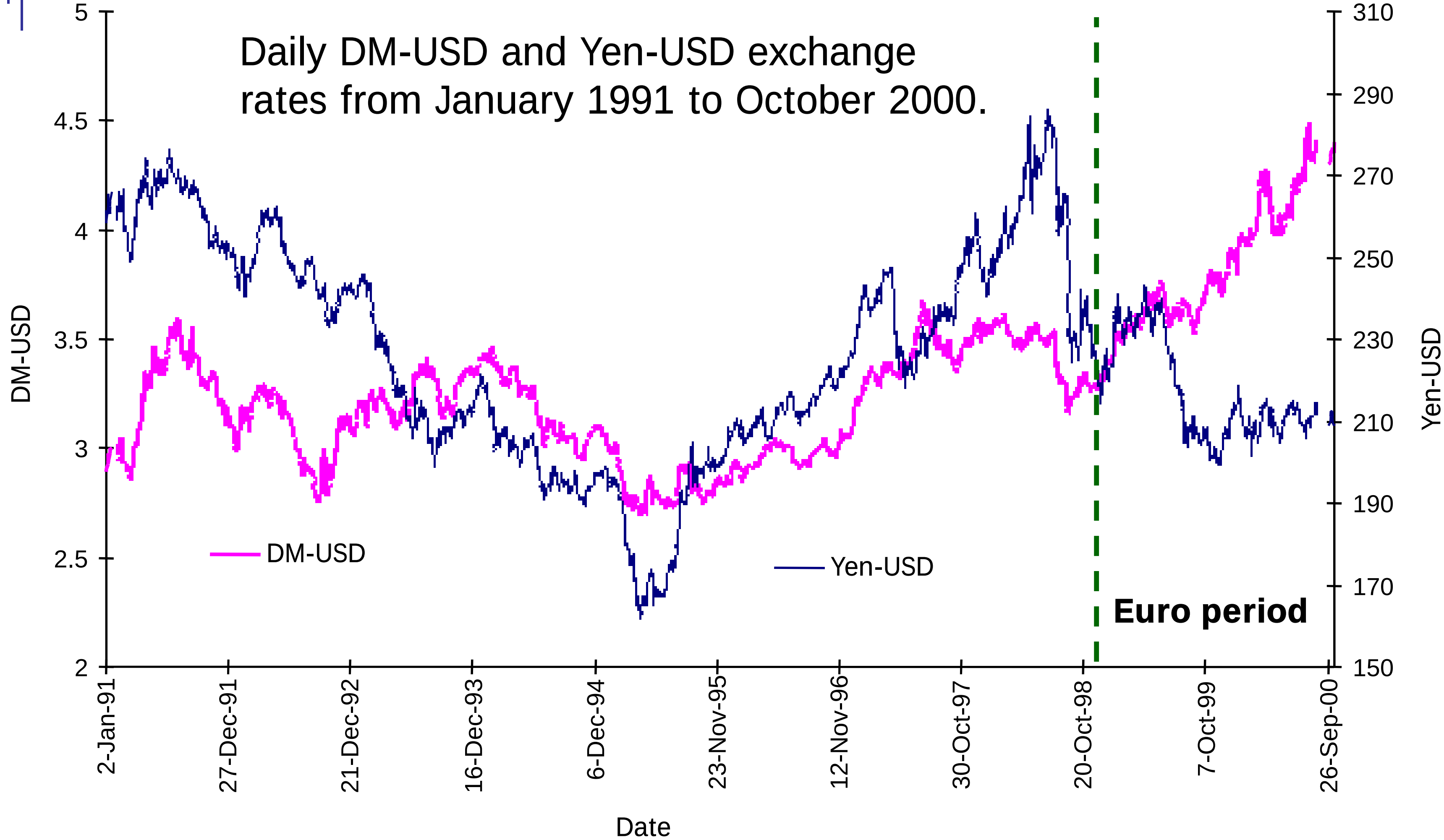
- As an application we will model the joint distribution of the DM-USD and Yen-USD exchange rates, allowing for time-variation in the conditional distribution.

  → The two most heavily traded exchange rates

    → The DM-USD and Yen-USD exchange rates made up 29% and 18% of total foreign exchange trading in 1996 (Melvin, 2000). Next largest were DM-FFr and GBP-USD with 6% and 5% respectively. (AUD-USD was 8^{th} largest at 2%).

  → Provide an application where time variation in dependence is likely to be significant

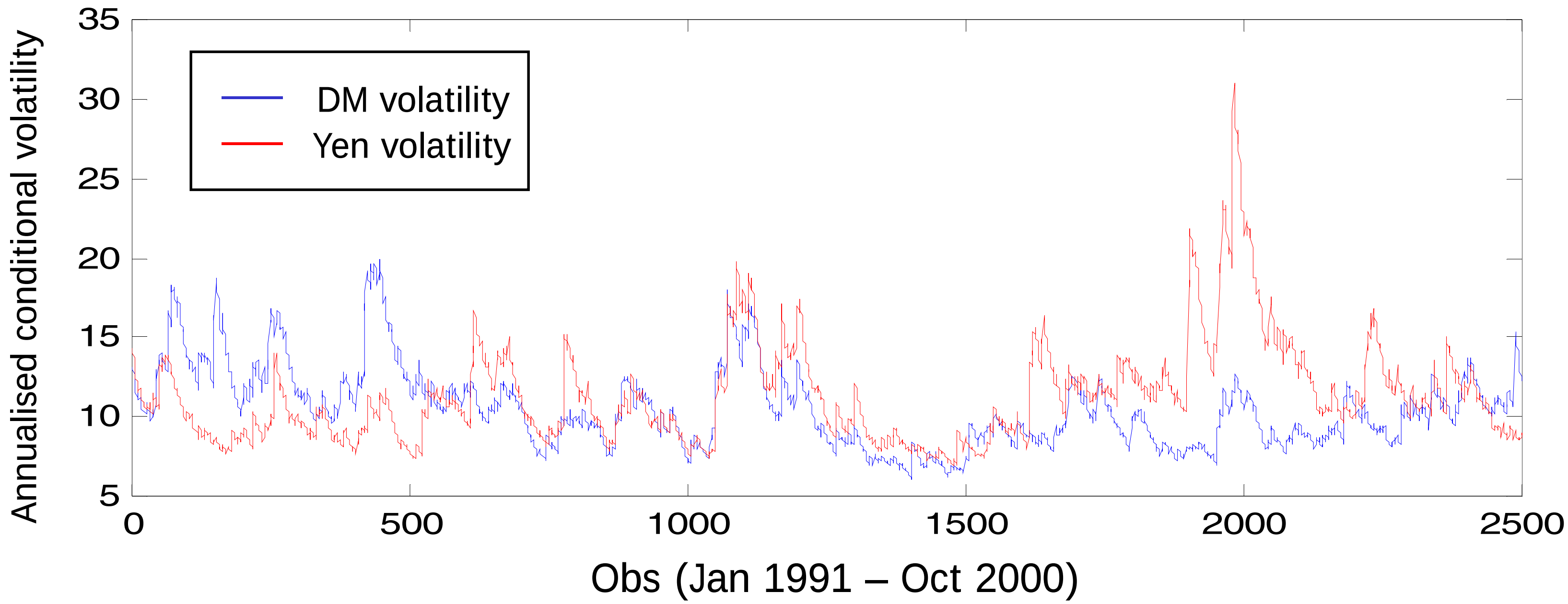
The data



Daily DM-USD and Yen-USD exchange rates from January 1991 to October 2000.
DM-USD
Yen-USD
Euro period
Date
5
4.5
4
3.5
3
2.5
2
310
290
270
250
230
210
190
170
150
2-Jan-91
27-Dec-91
21-Dec-92
16-Dec-93
6-Dec-94
23-Nov-95
12-Nov-96
30-Oct-97
20-Oct-98
7-Oct-99
26-Sep-00

Why a time-varying copula?

- We have substantial evidence of time variation in the conditional volatility of exchange rates (Bollerslev, 1987, Engle *et al. 1990,* Andersen *et al.* 2000, amongst others)



Why a time-varying copula?

- It then seems natural to allow for time variation in the conditional dependence also...

The model

- We will exploit the knowledge we have on modelling individual exchange rates, and then work on modelling the copula...

- Let $I_t = \sigma(x_t, y_t, x_{t-1}, y_{t-1}, \ldots)$

- $\hat{C}_t(\ \hat{F}_t(x_t|I_{t-1})\ ,\ \hat{G}_t(y_t|I_{t-1})\ |I_{t-1}\ ) \Rightarrow \hat{H}_t(\ x_t\ ,\ y_t\ |I_{t-1})$

Normal copula
vs.
Joe-Clayton copula

AR-*t* GARCH

Model for the
joint
distribution

The model (cont'd)

- The marginal distributions:

$$\left.\begin{aligned}
&X_t = \mu_x + \phi_x X_{t-1} + \varepsilon_t \\
&h_t^x = \omega_x + \alpha_x \varepsilon_{t-1}^2 + \beta_x h_{t-1}^x \\
&\sqrt{\upsilon_x / [h_t^x(\upsilon_x - 2)]} \cdot \varepsilon_t \sim t_{\upsilon_x}
\end{aligned}\right\} \quad F_t(\,\cdot\,|I_{t-1})$$

$$\left.\begin{aligned}
&Y_t = \mu_y + \phi_{1y} Y_{t-1} + \phi_{10y} Y_{t-10} + \eta_t \\
&h_t^y = \omega_y + \alpha_y \eta_{t-1}^2 + \beta_y h_{t-1}^y \\
&\sqrt{\upsilon_y / [h_t^y(\upsilon_y - 2)]} \cdot \eta_t \sim t_{\upsilon_y}
\end{aligned}\right\} \quad G_t(\,\cdot\,|I_{t-1})$$

Two models for the copula

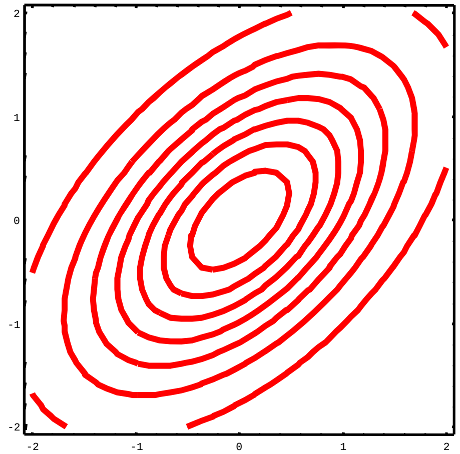
- We consider two possible models for the copula:
  - the Normal copula, and the Joe-Clayton copula.

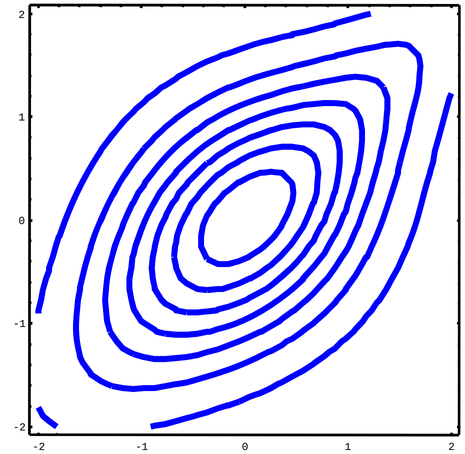
$$C(u,v|\rho) \quad = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-(s^2 - 2\rho st + t^2)}{2(1-\rho^2)} \right\} ds\, dt$$

$$C(u,v|\kappa,\gamma) = 1-(1-((1-(1-u)^{\kappa})^{-\gamma}+(1-(1-v)^{\kappa})^{-\gamma}-1)^{-1/\gamma})^{1/\kappa}$$

where $U_t \equiv F_t(X_t | I_{t-1})$ and $V_t \equiv G_t(Y_t | I_{t-1})$

Comparing the two copula models





Normal copula	*Joe-Clayton* copula
One parameter: ρ	Two parameters: κ , γ
Symmetric tail behaviour	Asymmetric tail behaviour
Parameter maps to linear correlation	Parameters map to upper and lower tail dependence
Positive and negative dependence	Positive dependence only
Zero tail dependence	Non-zero tail dependence

**

Upper and lower tail dependence

- As the name suggests, it measures dependence in the *tails*, or the extreme values, of the dist'n. [Recall $U_t \equiv F_t(X_t|I_{t-1})$ and $V_t \equiv G_t(Y_t|I_{t-1})$.]

$$\tau^L = \lim_{q \to 0^+} \Pr[U \le q \mid V \le q] = \lim_{q \to 0^+} \frac{\Pr[U \le q \cap V \le q]}{\Pr[V \le q]} = \lim_{q \to 0^+} \frac{C(q,q)}{q}$$

$$\tau^U = \lim_{q \to 1^-} \Pr[U > q \mid V > q] = \lim_{q \to 1^-} \frac{\Pr[U > q \cap V > q]}{\Pr[V > q]} = \lim_{q \to 1^-} \frac{1 - 2q + C(q,q)}{1 - q}$$

Time-varying copulas and dependence

- We attempt to capture time-varying dependence by allowing the parameters of the copulas to evolve over time, in a similar fashion to a GARCH model for heteroscedasticity.

Time-varying Joe-Clayton copula

- The parameters of the Joe-Clayton copula are strictly increasing functions of the upper and lower tail dependence measures (Joe, 1997):

$$\tau^L(\gamma) = 2^{-1/\gamma} \quad \Leftrightarrow \quad \gamma(\tau^L) = -[\log_2(\tau^L)]^{-1}$$

$$\tau^U(\kappa) = 2\text{-}2^{1/\kappa} \quad \Leftrightarrow \quad \kappa(\tau^U) = [\log_2(2-\tau^U)]^{-1}$$

- As κ and γ are not easily interpretable, we will instead model τ_t^U and τ_t^L, and then obtain κ_t and γ_t by the above expressions.

Time-varying Joe-Clayton copula

- Joe-Clayton copula:

$$\tau_t^U = \Lambda\left(\omega_U + \beta_U \tau_{t-1}^U + \alpha_U \frac{1}{10}\sum_{j=1}^{10}\left|u_{t-j} - v_{t-j}\right|\right)$$

Upper tail dependence

$$\tau_t^L = \Lambda\left(\omega_L + \beta_L \tau_{t-1}^L + \alpha_L \frac{1}{10}\sum_{j=1}^{10}\left|u_{t-j} - v_{t-j}\right|\right)$$

Lower tail dependence

$$\Lambda(\mathrm{x}) = \frac{1}{1+\mathrm{e}^{-\mathrm{x}}}$$

The logistic function

**

Time-varying Joe-Clayton copula

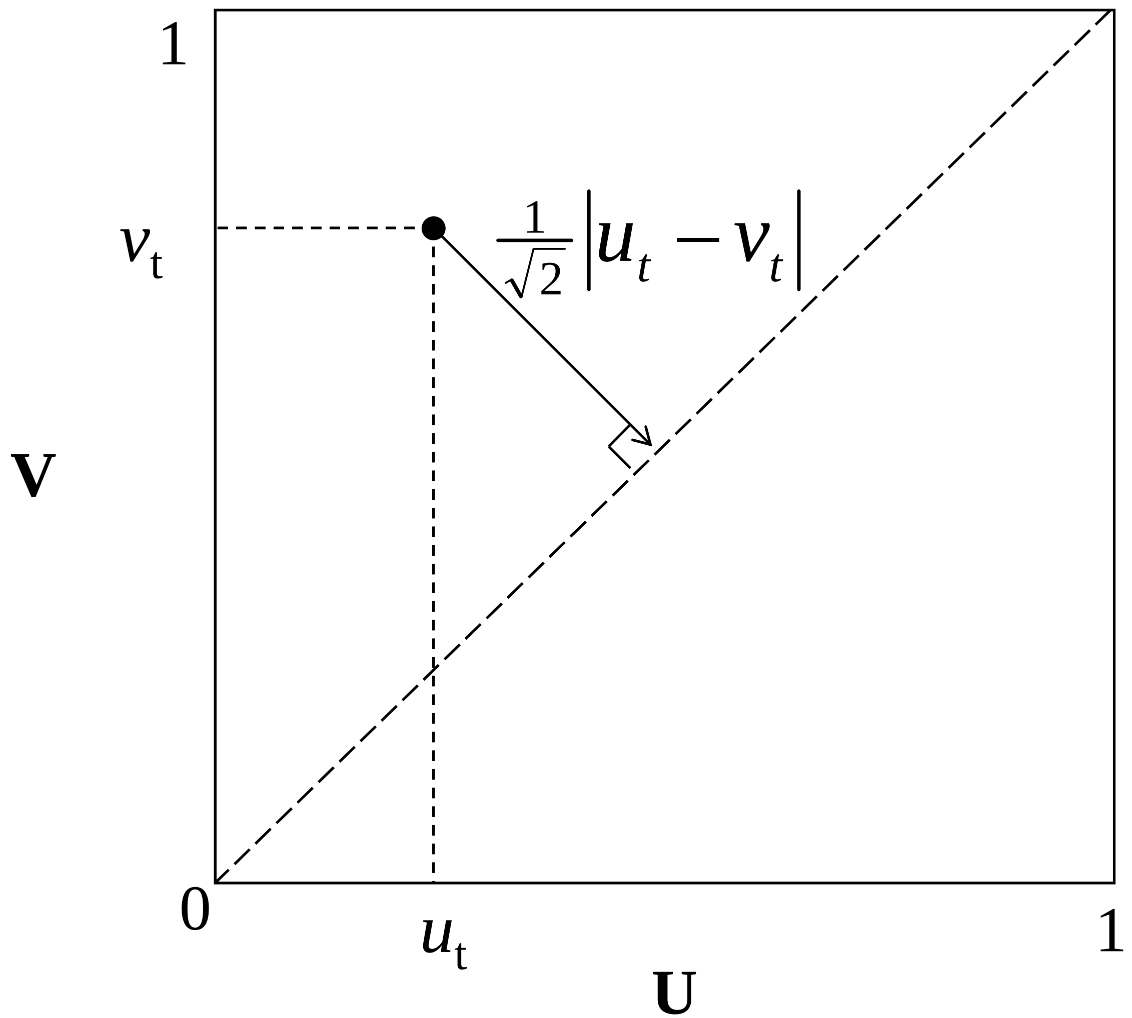
- On the choice of forcing variable in the Joe-Clayton copula: $\frac{1}{10}\sum_{j=1}^{10}\left|u_{t-j}-v_{t-j}\right|$

  - Under perfect positive dependence, all points (u_t , v_t) lie on the main diagonal of the copula support, while under independence they are scattered all over the support

  - Thus, average distance from the point to the main diagonal is an approximate measure of how 'close' the last ten observations were to being perfectly dependent.

**

Time-varying Joe-Clayton copula



Time-varying normal copula

- Normal copula:

$$\rho_t = \tilde{\Lambda}\left(\omega_\rho + \beta_\rho \rho_{t-1} + \alpha_\rho \frac{1}{10}\sum_{j=1}^{10} \Phi^{-1}(u_{t-j}) \cdot \Phi^{-1}(v_{t-j})\right)$$

where Φ^{-1} is the inverse of the standard normal c.d.f.

$$\tilde{\Lambda}(x) = \frac{1-e^{-x}}{1+e^{-x}}$$

The modified logistic function

Overview

1. Intro to / Review of Copulas
2. Motivation
3. Evaluating copula models
4. Applying copulas to exchange rate modelling
   a) The data
   b) *The models*
   c) Results
5. Conclusions and potential future work

Overview

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The best existing alternatives

- An alternative to the model just presented may be an AR-BEKK model for the mean and variance, with some residual distribution assumption:

1. AR-BEKK with bivariate Normal residuals:
   1. Passes LM tests of dynamic specification
   2. Fails (miserably!) K-S and hit tests of density specification of margins and copula
2. AR-BEKK with bivariate Student's *t* residuals:
   1. Passes LM tests
   2. Passes K-S tests
   3. Fails Yen marginal density hit test, and fails copula hit test

Estimation

- Two-stage maximum likelihood was employed:
- $(X,Y) \sim h(\theta) = f(x;\varphi) \cdot g(y;\gamma) \cdot c(\, F(x;\varphi), G(y;\gamma);\, \kappa)$
- $LL_H(\theta) = LL_F(\varphi) + LL_G(\gamma) + LL_C(\varphi\, ,\, \gamma\, ,\, \kappa)$

$$\hat{\varphi}_n = \arg\max_{\varphi\in\Phi} n^{-1} \sum_{t=1}^{n} \log f(x_t;\varphi)$$

$$\hat{\gamma}_n = \arg\max_{\gamma\in\Gamma} n^{-1} \sum_{t=1}^{n} \log g(y_t;\gamma)$$

$$\hat{\kappa}_n = \arg\max_{\kappa\in\mathrm{K}} n^{-1} \sum_{t=1}^{n} \log c(F(x_t;\hat{\varphi}_n), G(y_t;\hat{\gamma}_n);\kappa)$$

Results for the marginal distributions

	DM Margin		Yen Margin	
	Coeff	Std Error	Coeff	Std Error
μ	0.0276*	0.0111	0.0144	0.0111
ϕ_1	0.0142	0.0200	-0.0043	0.0195
ϕ_{10}			0.0664*	0.0183
ω	0.0039	0.0030	0.0059	0.0034
β	0.9485*	0.0161	0.9453*	0.0161
α	0.0448*	0.0126	0.0458*	0.0125
ν	5.8073*	0.6383	4.3817*	0.3800

Specification tests of marginal distributions

- Both margins pass the LM and K-S tests for correct dynamic specification and correct density specification

- Hit tests:
  - Break unit interval into five regions:

    0 , 10% , 25% , 75% , 90% and 1.
  - Include as regressors: a constant (to capture bias in probability of a hit) and three variables counting number of hits in past day, week and month (to capture serial dependence in hits)
  - Results: DM margin passes all regions and the joint test, Yen margin fails lower tail region but passes the joint test

Constant dependence copulas

Constant dependence copulas

1. Normal copula: $\hat{\rho} = 0.4560$ (0.0167)
   a) Implied residual correlation: 0.4141

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   $\hat{\kappa} = 1.3356$ (0.0348) $\Rightarrow \tau^{U}(\hat{\kappa}) = 0.3197$

   $\hat{\gamma} = 0.4202$ (0.0384) $\Rightarrow \tau^{L}(\hat{\gamma}) = 0.1921$

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   a) Implied residual correlation: 0.4267

   b) Test for significance of asymmetry:

      H_0: $\tau^U(\hat{\kappa}) = \tau^L(\hat{\gamma})$ vs. H_1: $\tau^U(\hat{\kappa}) \neq \tau^L(\hat{\gamma})$

      Test stat (*p-value*) = 2.9197 (*0.0035*)

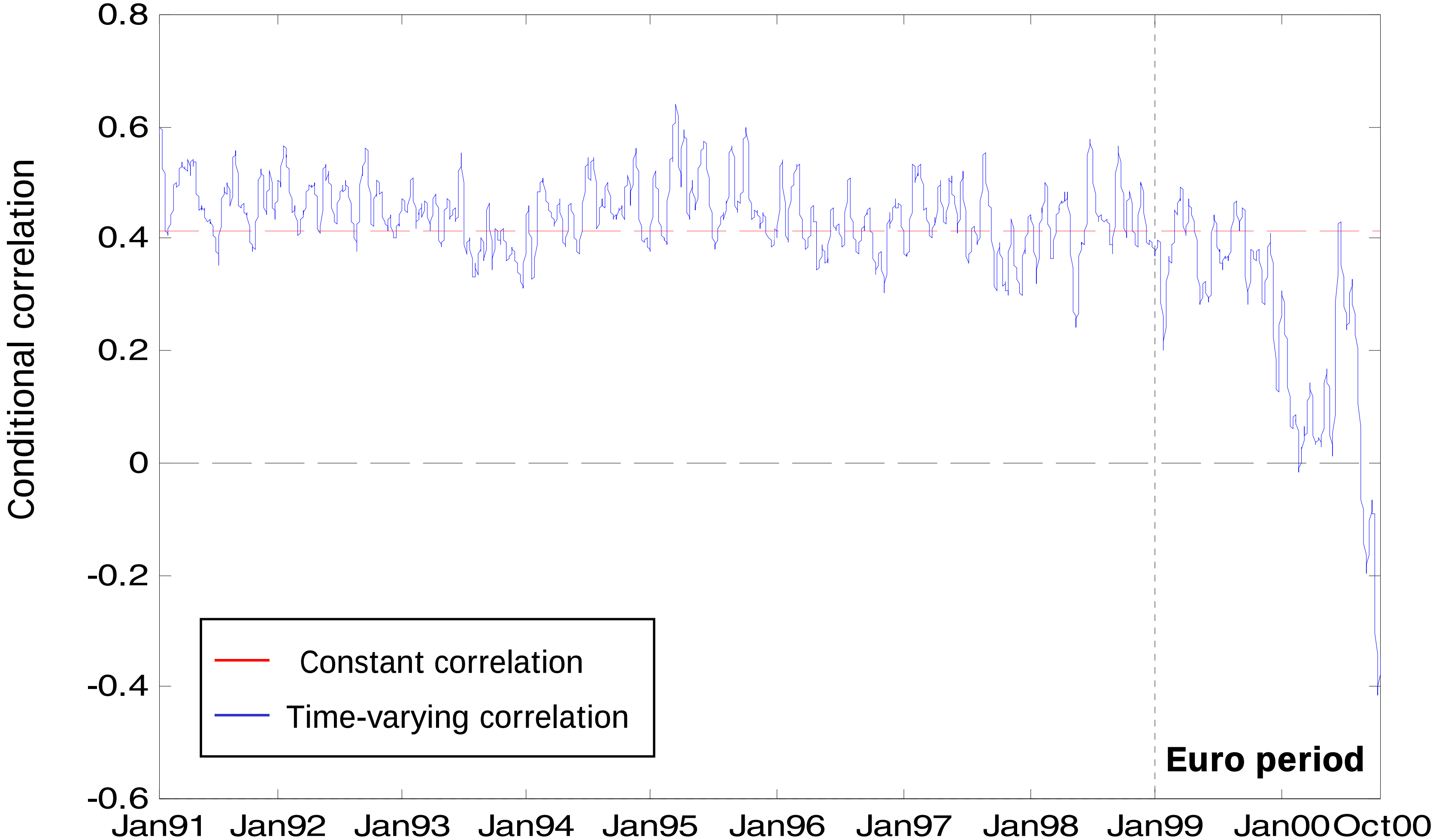
$\Rightarrow$ Reject H_0, i.e. *asymmetry is significant*.

Time-varying normal copula

	ω_ρ	β_ρ	α_ρ
Coeff	0.0015	2.0684*	0.1212
Std Error	0.0052	0.0160	0.0250

- These parameters are difficult to interpret on their own, as they enter into the logistic transformation.
- Can more easily see the results in a graph...

Time-varying normal copula

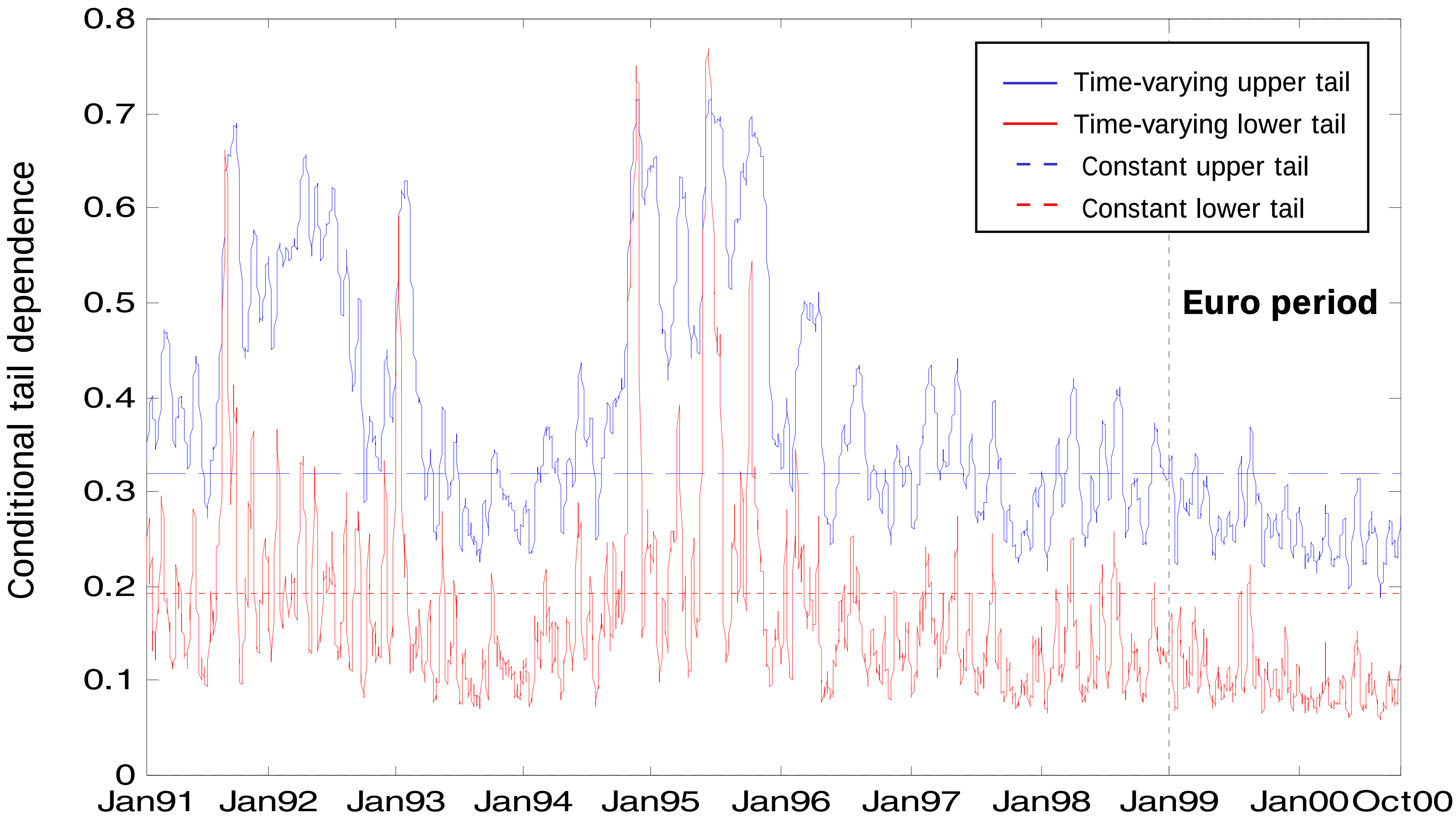


Time-varying Joe-Clayton copula

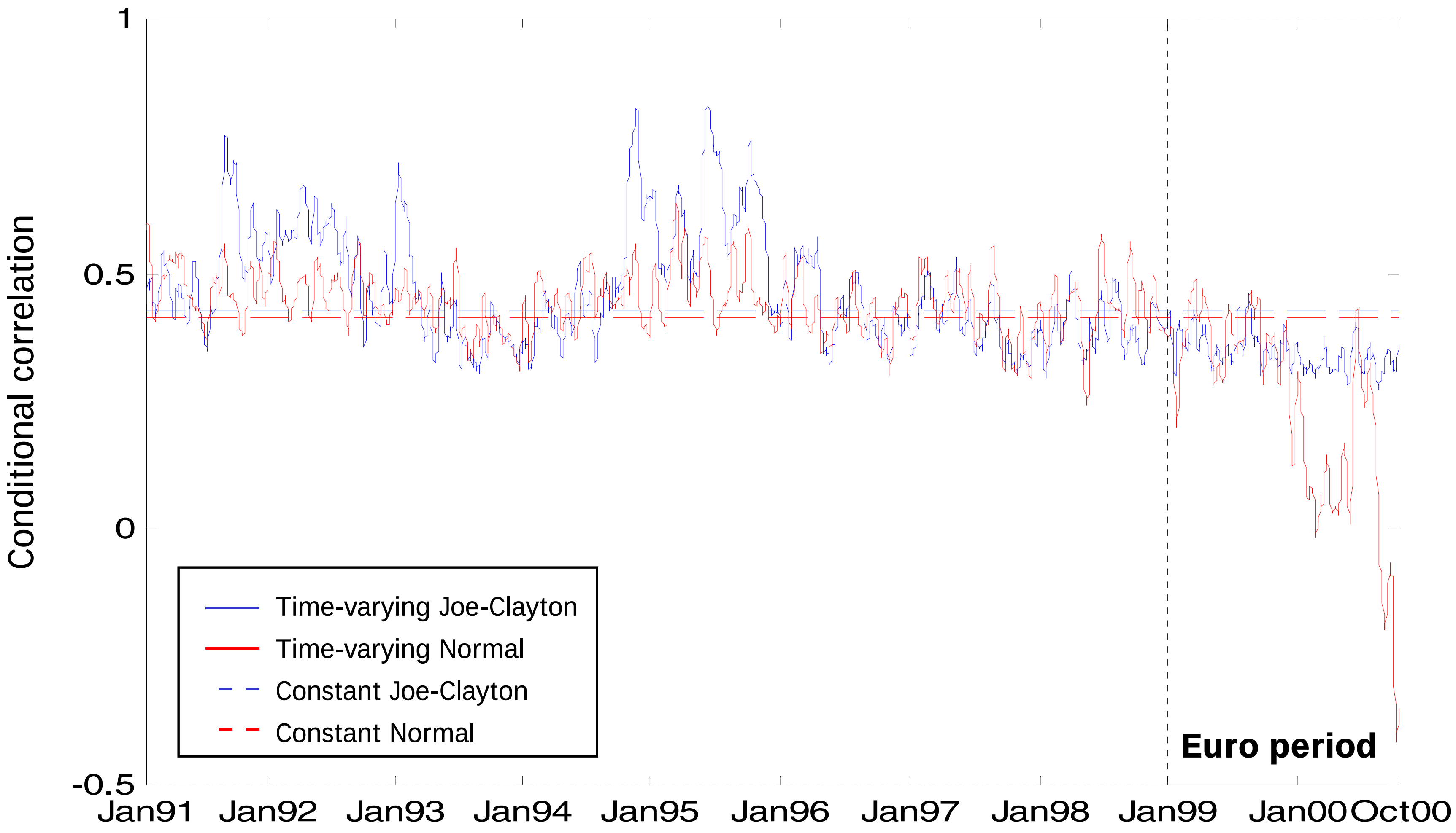
	ω_U	β_U	α_U	ω_L	β_L	α_L
Coeff	-2.0621	4.4548	-.9192	-1.3444	4.1406	-6.5119
s.e.	0.2056	0.2938	0.8966	0.5876	0.5880	3.1394

- Again interpretation of the parameter estimates directly is difficult – look at graph of time path of parameters

Tail dependence in the Joe-Clayton copula



Conditional correlation estimates



Evaluation and comparison of models

- The estimation results clearly imply quite different dependence structures - we need some way of comparing the different models' fit.

- Unfortunately, many standard means of comparing fit are not applicable/relevant here:
  a) R^2 has little or no meaning
  b) LR-type tests difficult as the models are non-nested

- We will use the tests introduced earlier (LM, K-S and Hit tests)

LM and K-S tests

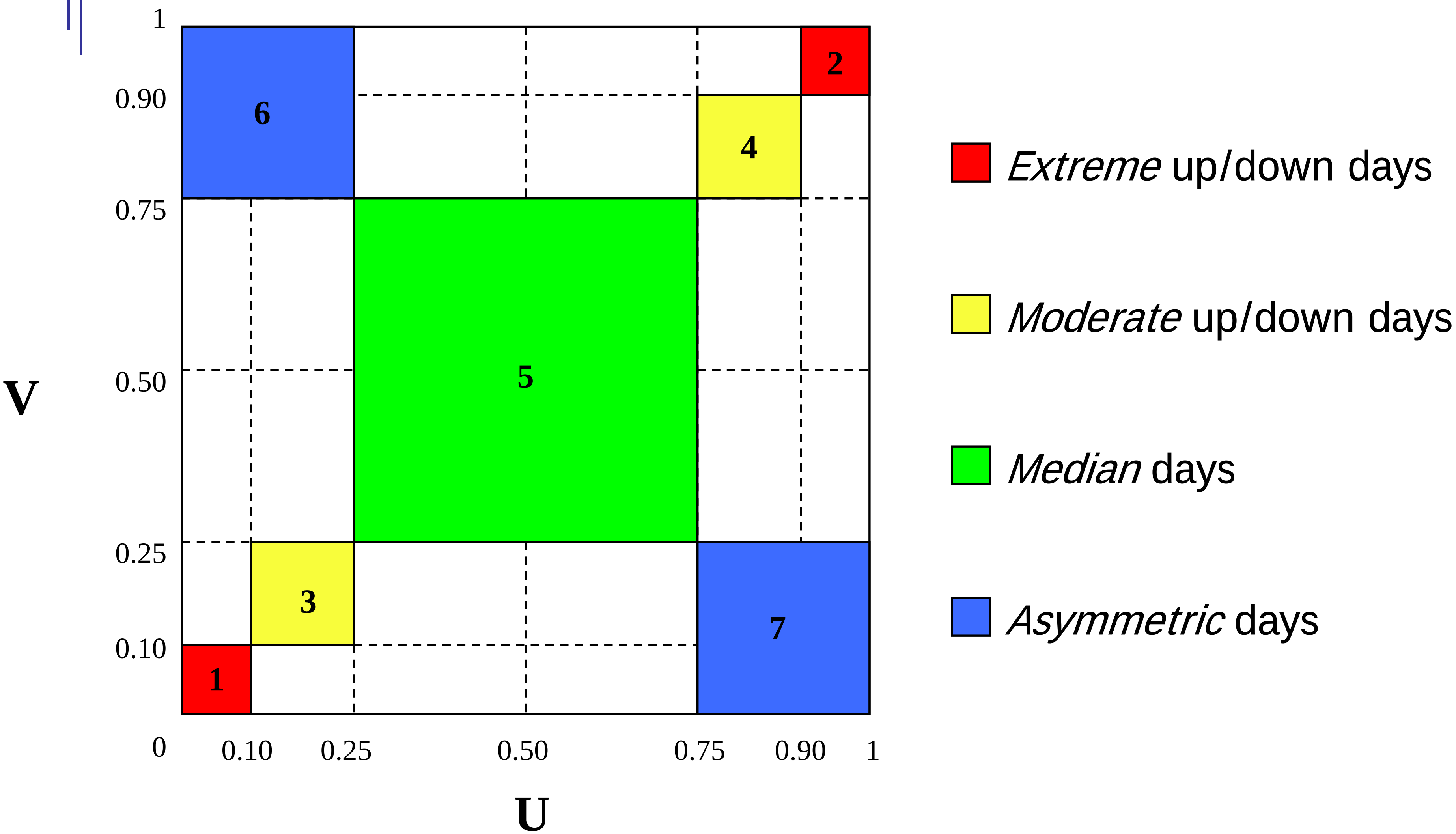
- All 4 copula models pass the LM and K-S tests for goodness-of-fit.
- We turn now to the ‘hit’ tests for hopefully a more powerful test.

Hit test specification

- Z_t contains a constant, and a count of the number of hits in the last day, one week and one month.

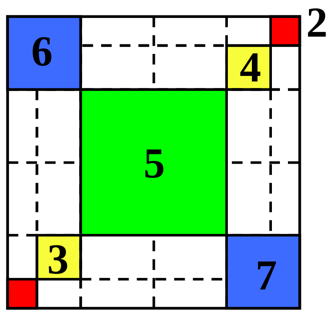
- We use a standard LR test to test the null hypothesis that $\beta = 0$.

Hit regions



1
0.90
0.75
0.50
0.25
0.10
0
0.10
0.25
0.50
0.75
0.90
1
V
U
1
2
3
4
5
6
7
Extreme up/down days
Moderate up/down days
Median days
Asymmetric days

Hit test results



Region	Constant Normal	Constant Joe-Clayton	Time-varying Normal	Time-varying Joe-Clayton
1	9.4482	7.3304	7.4209	6.6815
2	4.1361	9.8002*	2.2563	16.0750*
3	6.4374	6.1412	6.4108	6.5137
4	1.9063	1.8627	1.4813	1.4595
5	8.2495	6.7740	7.9588	3.3560
6	2.8800	6.9902	0.9284	3.1363
7	6.5945	8.4697	0.6086	3.2932
ALL	37.8395	46.1668*	26.9924	41.4195*

Individual 5% critical value: 9.49, Joint 5% critical value: 41.3371.

Hit test results

- The two normal copulas pass all tests
- The two Joe-Clayton copulas both fail in the extreme upper tail region
  - An unexpected result, as the Joe-Clayton copula is more flexible in the tails than the Normal
  - Casts some doubt on the previous finding of asymmetric dependence

A structural break? – The euro

- Did the introduction of the euro on Jan 1, 1999 significantly change the conditional joint distribution of these exchange rates?
- We expand the information set now to include an indicator variable as to whether the data came from the pre-euro or post-euro period
- The model parameters are allowed to change over the two periods, though the functional forms are assumed to be the same

Results for the marginal distributions, with structural break

	DM Margin		Yen Margin	
	Coeff	Std Error	Coeff	Std Error
μ^1	**0.0128**	0.0120	0.0144	0.0111
μ^2	**0.0982***	0.0268	——	——
ϕ_1	0.0111	0.0200	-0.0043	0.0195
ϕ_{10}			0.0664*	0.0183
ω^1	**0.0037**	0.0029	0.0059	0.0034
ω^2	**0.0053**	0.0038	——	——
β	0.9485*	0.0160	0.9453*	0.0161
α	0.0446*	0.0125	0.0458*	0.0125
ν	5.6860*	0.6180	4.3817*	0.3800

**

Specification tests of marginal distributions, with structural break

- Multinomial test of marginal models with break:

	Full Sample		Pre-Euro		Post-Euro	
	DM	Yen	DM	Yen	DM	Yen
Test stat	14.29	35.05	16.26	25.89	13.10	21.43
p-value	0.58	0.17	0.43	0.06	0.67	0.17

Constant copulas with structural break

1. Normal copula: $\rho^1 = 0.5435$ (0.0146)

   $\rho^2 = 0.0855$ (0.0508)

2. Joe-Clayton copula:

   $\tau_1^U(\kappa) = 0.3964$ $\tau_1^L(\gamma) = 0.2542$

   $\tau_2^U(\kappa) = 0.0001$ $\tau_2^L(\gamma) = 0.0018$

   a) Implied residual correlation: 0.4983 down to 0.0781

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   a) Implied residual correlation: 0.4983 down to 0.0781
   b) Test for significance of asymmetry:
      H_0: $\tau_i^U(\kappa) = \tau_i^L(\gamma)$ vs. H_1: $\tau_i^U(\kappa) \neq \tau_i^L(\gamma)$ for $i=1,2$

   Test stats (*p-values*) = 3.13 (*0.0017*) and –0.03 (*0.9785*)

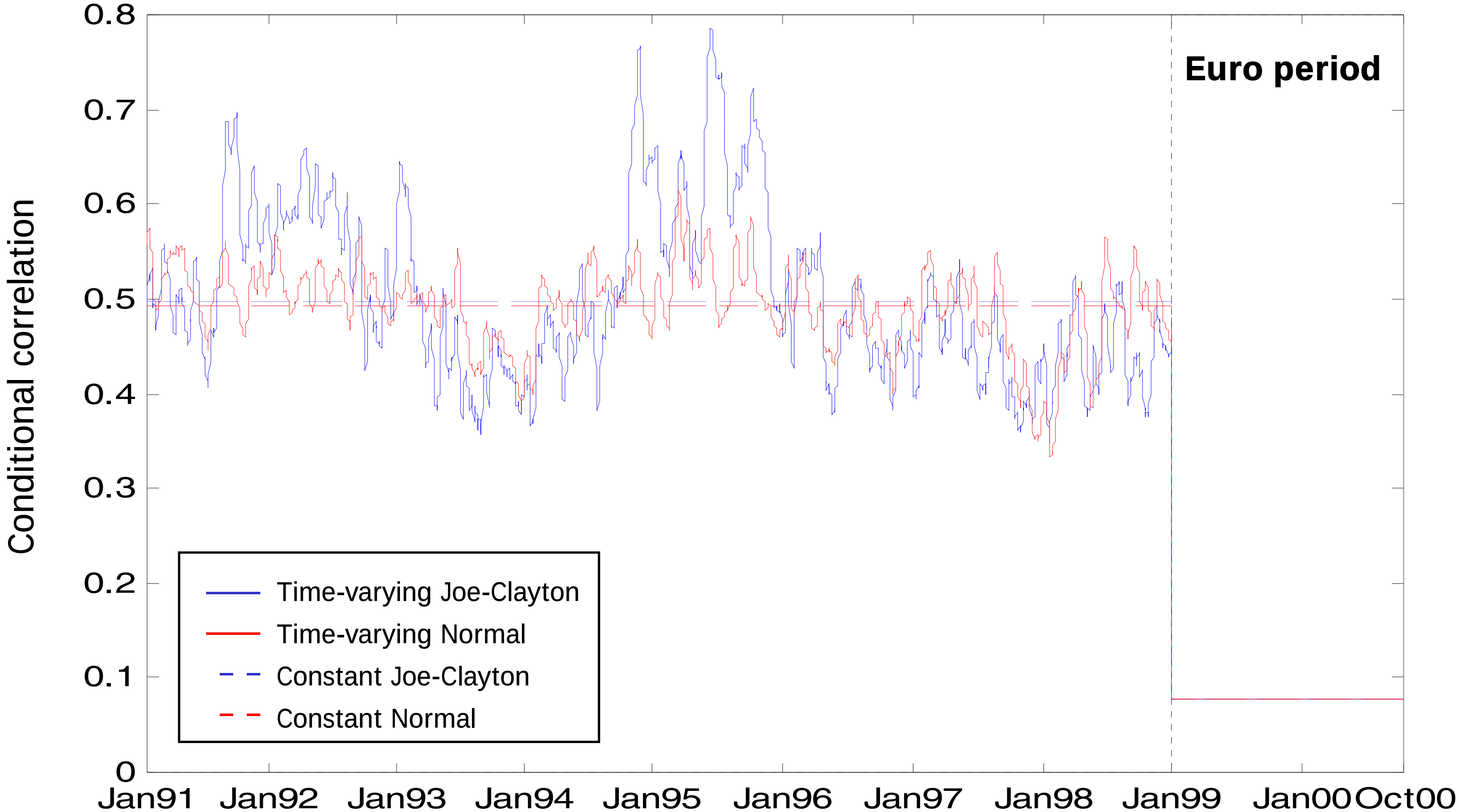
$\Rightarrow$ *Asymmetry is significant for DM/Yen, but not for Euro/Yen*

Time-varying copulas with structural break

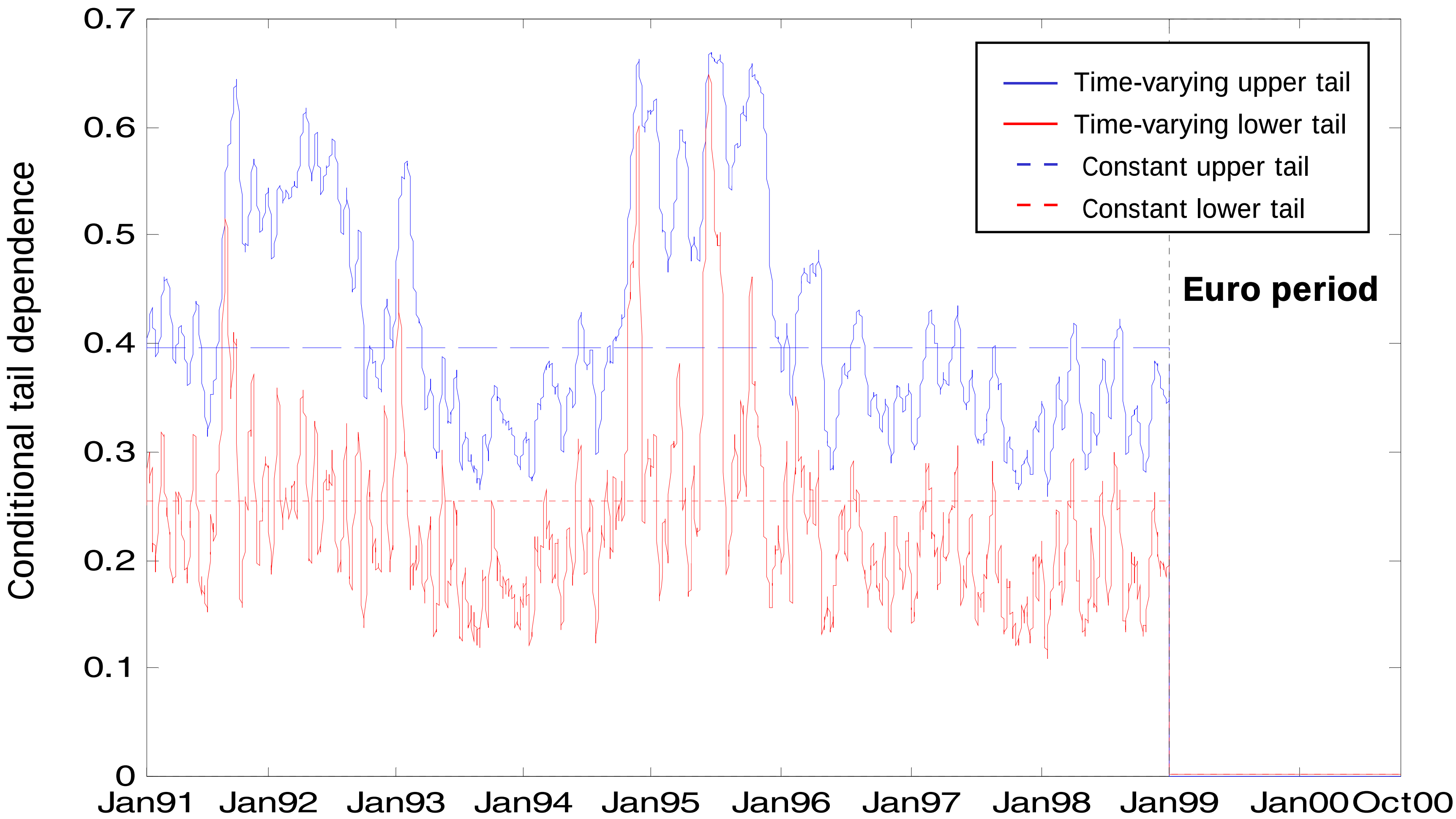
- Allowed all parameters of copula to change and then inspected resulting parameter time paths.

- Dependence appeared very near constant in the post-euro sample, and so constancy was imposed to reduce number of parameters estimated

Time-varying conditional correlation



Time-varying conditional tail dependence



Conclusions

METHODS:

- We’ve shown how copula theory may be used to develop flexible models for time-varying conditional joint distributions

- We’ve also discussed a means of evaluating multivariate density models – the logistic ‘hit’ test and multinomial test

Conclusions

EMPIRICAL RESULTS:

- Time variation in the dependence structure between the DM-USD and Yen-USD seems significant
- Asymmetry in the dependence structure was found to be important – dependence was greater during appreciations of the USD (depreciations of both the Yen and DM) than during depreciations of the USD (appreciations of both the Yen and DM)

Conclusions

EMPIRICAL RESULTS (cont'd):

- Substantial evidence of a structural break in the joint distribution was found:
  i. DM-USD unconditional drift and variance increased
  ii. Yen-USD margin did not change
  iii. Large decrease in conditional dependence following the introduction of the euro

- Conditional dependence measures (like correlation) implied by the models differ depending on the copula used

Potential future work

- For me:
  1. Two-stage estimation
  2. The case of unequal amounts of data (Yen-USD and Euro-USD exchange rates)
  3. Applications to economic models (portfolio allocation decisions, and possibly contagion)

- For someone else:
  1. Different forms of time variation in dependence – eg Markov switching
  2. Extensions to higher dimensions
  3. The search for the ‘best fitting’ FX copula, stock market copula, etc...