Supplemental Appendix for

Bespoke Realized Volatility:
Tailored Measures of Risk for Volatility Prediction

by Andrew J. Patton and Haozhe Zhang

This version: April 22, 2024

This appendix contains nine sections, presenting additional details and analyses relevant to the main paper.

S.1. Data cleaning details

We use trade data from the NYSE Trade and Quote (TAQ) database. We follow Barndorff-Nielsen et al. (2009) to clean this data, using the following rules:

1. Keep only entries with time step between 9:30 am to 4:00 pm (when the exchange is open).
2. Delete entries with zero transaction prices.
3. Retain entries originating from NYSE and NASDAQ only.
4. Delete entries with corrected trades (CORR $\neq 0$).
5. Delete entries with unusual sale condition (COND with letter code except for letters E and F).
6. We use the median price if multiple entries have the same time stamp.

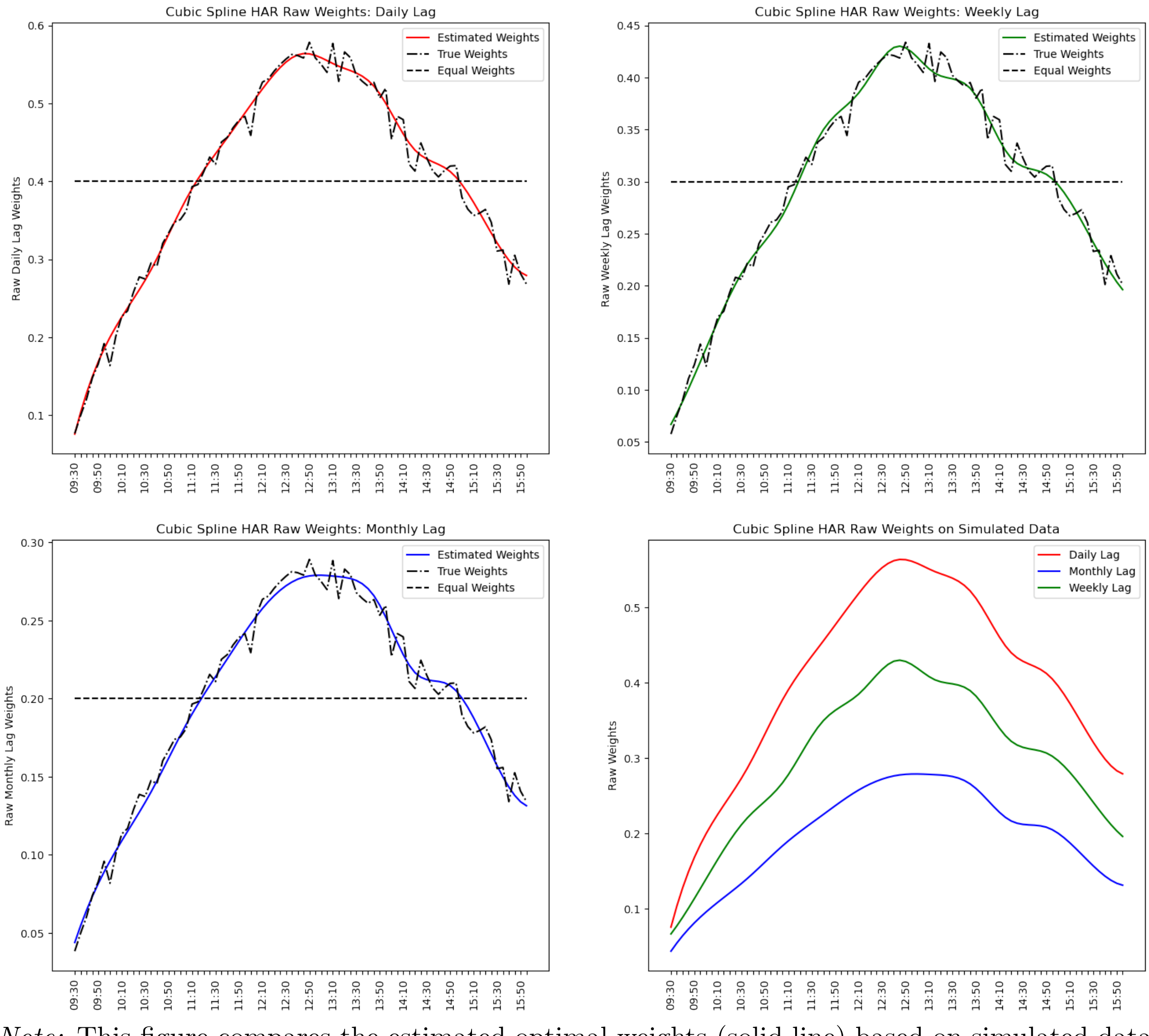
S.2. Simulation study

In this section we simulate data to illustrate the ability of the proposed Cubic HAR model to recover the true coefficients. We assume that high-frequency returns, $r_{i,t}$, are i.i.d. N(0,1/M), where M is the number of high-frequency returns each trade day. We combine these high frequency returns to obtain realized variances using time-of-day (TOD) weights, ω_i, as in equation 7 of the main paper, and we use a HAR model to simulate the time series dynamics of RV.

$$RV_t = \beta_0 + \beta_d \sum_{i=1}^{M} \omega_i r_{i,t-1}^2 + \beta_w \frac{1}{4} \sum_{j=2}^{5} \sum_{i=1}^{M} \omega_i r_{i,t-j}^2 + \beta_m \frac{1}{16} \sum_{j=6}^{21} \sum_{i=1}^{M} \omega_i r_{i,t-j}^2 + \epsilon_t \qquad \text{(S.1)}$$

where $\epsilon_t \sim N(0,1)$. We set $\beta_0 = 0$, $\beta_d = 0.4$, $\beta_w = 0.3$, $\beta_m = 0.3$, and $M = 78$. We impose that ω_i follows the empirical TOD weights presented in Figure 4. Then we use the cubic HAR model with the exact same set-up (hyperparameter grid and estimation algorithm) to estimate the coefficients on the simulated data. Figure S.1 shows that when the true data generating process indeed is a linear combination with the TOD weights, our cubic HAR model will recover the true weights almost perfectly. (As the true weights in this simulation are not a smooth function of time, the cubic spline model clearly cannot obtain a perfect fit.) This simulation results give us confidence that the proposed Cubic HAR has enough estimation power to recover the optimal weighting scheme for realized variance forecasting.

Figure S.1: Cubic HAR vs True Weights: Simulated Data



Note: This figure compares the estimated optimal weights (solid line) based on simulated data based with the true weights (dash-dotted line).

S.3. Alternative target for shrinkage

In the analysis in Section 3.2 of the main paper, we consider shrinkage methods (Ridge, LASSO and elastic net) that shrink the estimated parameters towards zero. This is a standard shrinkage target in high-dimensional estimation, but in our application an interesting alternative target is to shrink the parameters towards the benchmark HAR parameters. To do this, we re-write the regularized regression models as:

$$RV_t = \beta_0 + \widetilde{RV}_{t-1}(\beta^d_{HAR} + \boldsymbol{\gamma}^d) + \frac{1}{4}\sum_{j=2}^{5}\widetilde{RV}_{t-j}(\beta^w_{HAR} + \boldsymbol{\gamma}^w) + \frac{1}{16}\sum_{j=6}^{21}\widetilde{RV}_{t-j}(\beta^m_{HAR} + \boldsymbol{\gamma}^m) + e_t \tag{S.1}$$

That is, we split the coefficients on high frequency returns into the HAR coefficients and a perturbation term to be estimated. Then, instead of regularizing the entire coefficient, we only penalize the perturbation terms, and shrink these towards zero. That is, the penalty term becomes:

$$\alpha(\lambda\|\boldsymbol{\gamma}\|_1 + (1-\lambda)\|\boldsymbol{\gamma}\|_2) \tag{S.2}$$

where $\boldsymbol{\gamma} = [\boldsymbol{\gamma}^d, \boldsymbol{\gamma}^w, \boldsymbol{\gamma}^m]$. The estimation procedure and hyperparameters grid are identical to those in the original regularized regression analysis. We compare the cubic HAR model with these alternative shrinkage estimators in Table S.1. (The OLS and HAR results are identical to those in Table 2 of the main paper, and are included here for ease of comparison.) With this alternative target the optimal degree of shrinkage is found to be large, and the Ridge, LASSO and elastic net models are all shrunk almost all the way to the benchmark HAR parameters. Given that, it is unsurprising that the cubic HAR continues to significantly out-perform these alternative regularized models, as shown in Table S.1, which have performance comparable to the benchmark HAR model.

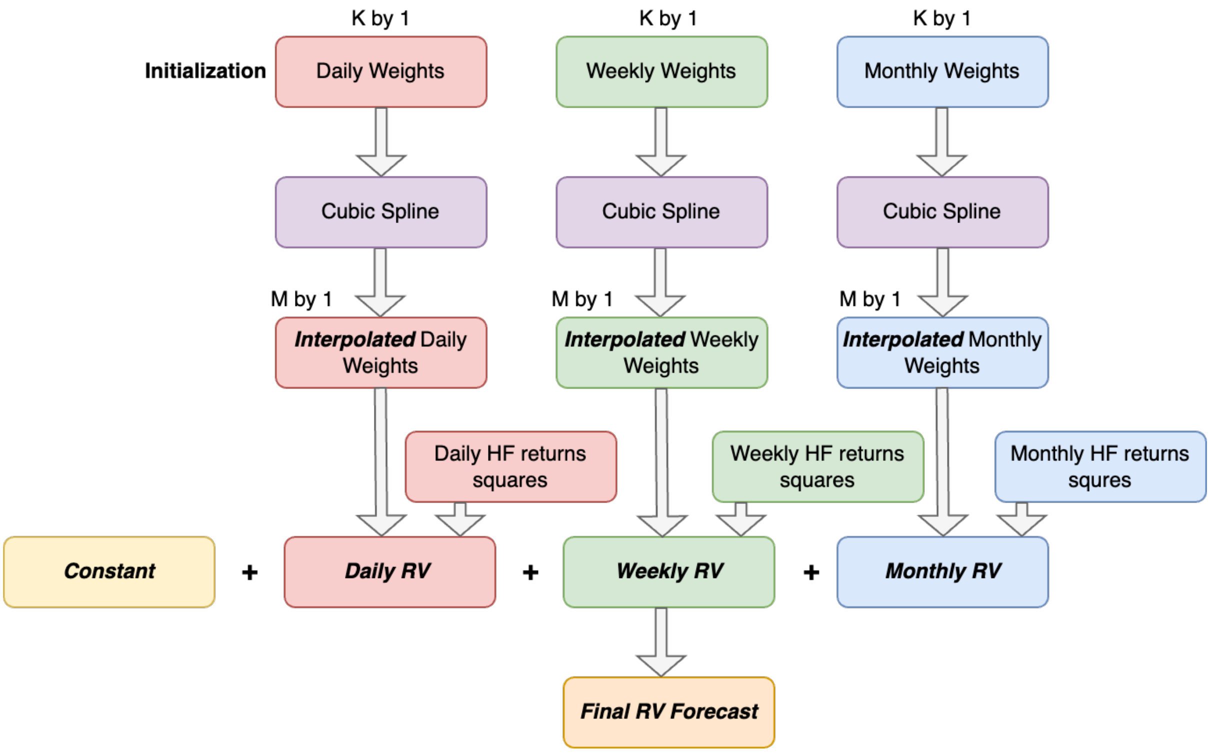
Table S.1: Alternative regularized regression models: Shrink towards HAR

	GW Losses		**GW Wins**		**GW t-stat**
CubicHAR vs:	**Total**	**Signif**	**Total**	**Signif**	**Panel**
Ridge-Alt	166	44	720	423	-17.3
LASSO-Alt	165	46	721	437	-4.2
Elastic net-Alt	170	46	716	412	-16.4
OLS	54	17	832	564	-6.4
HAR	155	41	731	470	-21.7

Note: This table reports individual and panel Giacomini-White (2006) tests comparing the cubic HAR model against competing models across 886 S&P 500 stocks. A positive panel GW t-statistic indicates that the competing model out-performs the cubic HAR model, while a negative t-statstic indicates the opposite. The models labeled "-Alt" shrink the estimated coefficients towards the HAR model coefficients, rather than towards zero as in Table 2 of the main paper.

S.4. Cubic HAR Model Architecture

Figure S.2: Cubic HAR Model Architecture

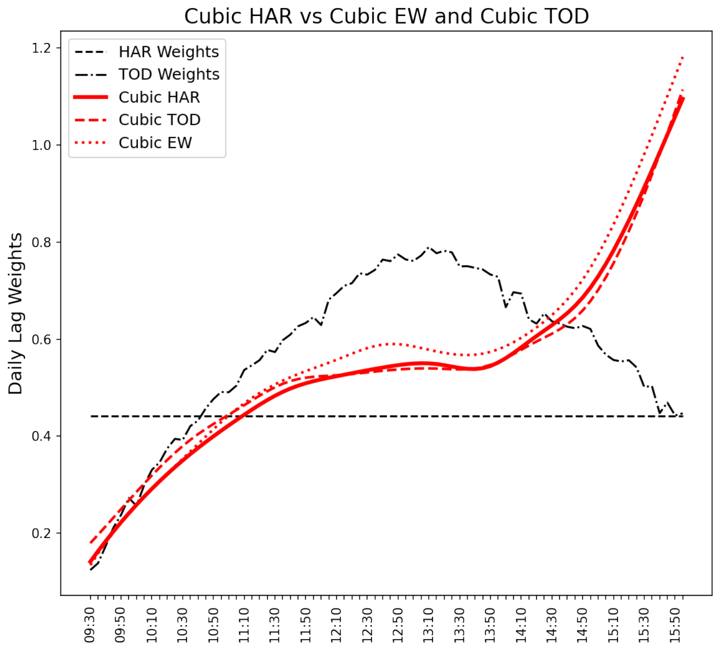


Note: This figure illustrates the cubic HAR model architecture. In particular, it shows how we use miniBatch gradient descent to iteratively solve the optimal parameters $(3 \times K + 1)$ and then use them to get the cubic spline interpolated weights for constructing final RV forecast. Here K represents the number of basis points for cubic spline interpolation or the true number of parameters, and M represents the desired number of points as cubic spline interpolation output.

S.5. Visualizing the optimal Cubic TOD and Cubic EW weights

This section presents the optimal daily weights for the cubic HAR, cubic TOD and cubic EW models, which differ in what shapes are estimated or imposed on the weekly and monthly weights. We observe the cubic TOD and cubic EW daily weights are almost identical to the cubic HAR weights for daily lags, which shows that restricting the shapes of weekly and monthly weights has almost zero affect on the estimated daily weights.

Figure S.3: Cubic TOD and Cubic EW weights



Note: This figure compares estimated cubic TOD and cubic EW weights against the cubic HAR model weights, averaged across the 886 S&P 500 stocks in our sample.

S.6. Forecast comparisons using mean-squared error loss

Table S.2: HAR vs other models performance on S&P500. MSE loss.

HAR vs:	GW Losses Total	GW Losses Signif	GW Wins Total	GW Wins Signif	GW t-stat Panel
Fully Flexible	421	169	465	106	0.8
Flexible HAR	726	408	160	11	3.3
Cubic HAR	714	406	172	7	1.9
TOD HAR	653	348	233	62	2.2
Ridge	462	186	424	120	-0.1
LASSO	203	46	683	338	-4.1
Elastic net	434	166	452	136	-2.0
OLS	64	7	822	507	-3.3

Note: This table reports individual and panel Giacomini-White (2006) tests comparing the baseline HAR model against competing models across 886 S&P 500 stocks, using MSE loss. A positive panel GW t-statistic indicates that the competing model out-performs the HAR model, while a negative t-statstic indicates the opposite. This table is related to Table 1 of the main paper, which uses QLIKE loss.

Table S.3: Cubic HAR vs other models performance on S&P 500. MSE loss.

Cubic HAR vs:	GW Losses Total	GW Losses Signif	GW Wins Total	GW Wins Signif	GW t-stat Panel
Fully Flexible	151	16	735	383	-0.7
Flexible HAR	411	55	475	90	0.0
TOD HAR	405	69	481	196	-1.0
HAR	172	7	714	406	-1.9
Ridge	171	12	715	349	-1.7
LASSO	66	2	820	542	-5.2
Elastic net	159	8	727	372	-3.4
OLS	23	3	863	664	-3.3

Note: This table reports individual and panel Giacomini-White (2006) tests comparing the cubic HAR model against competing models across 886 S&P 500 stocks, using MSE loss. A positive panel GW t-statistic indicates that the competing model out-performs the cubic HAR model, while a negative t-statstic indicates the opposite. This table is related to Table 2 of the main paper, which uses QLIKE loss.

S.7. Multi-days ahead forecasting comparisons: TOD HAR vs HAR

Table S.4: Multidays Ahead Volatility Forecasting: TOD HAR vs HAR

TOD vs HAR	**GW Losses**		**GW Wins**		**GW t-stat**
Horizon	**Total**	**Signif**	**Total**	**Signif**	**Panel**
1-Day	206	63	680	458	-18.6
2-Day	253	72	633	349	-12.5
3-Day	257	74	629	314	-10.1
4-Day	264	74	622	293	-13.6
5-Day	270	71	616	298	-11.5
20-Day	253	68	633	345	-6.8
60-Day	345	126	541	255	-0.2

Note: In this table, we report the individual and panel Diebold Mariano tests results of TOD HAR model against the HAR model in the S&P500 cross section for longer horizon forecasting. Note that negative test statistics favors the TOD HAR model.

S.8. GARCH-X with RV as the target variable

The GARCH-X model in Section 4.2 of the main paper uses, effectively, the daily squared return as the volatility proxy when evaluating forecast accuracy. Realized variance is known to be a more accurate volatility proxy (see, e.g., Andersen and Bollerslev (1998) and Andersen et al. (2003)), and more accurate proxies lead to more powerful forecast comparisons, (see, e.g., Patton (2011)). Here we consider the estimating and evaluating the GARCH-X model replacing the squared daily return with 5-minute realized variance. Table S.5 presents results corresponding to Table 8 in the main paper. We see that GARCH-X based on bespoke RV continues to significantly outperform both the standard GARCH-X model, and the model use time-of-day (TOD) RV, with panel GW t-statistics less than -4 for both comparisons. The main difference between Table S.5 and Table 8 is that the number of individual GW tests that reject the null (listed in the "Signif" columns of the table) is greater, ranging from 68 to 186 in Table 8 and from 75 to 293 here. This increase in significance is consistent with 5-minute RV being a more accurate volatility proxy.

Table S.5: Bespoke RV for GARCH-X models, with RV as the target variable

	GW Losses		**GW Wins**		**GW t-stat**
Model	**Total**	**Signif**	**Total**	**Signif**	**Panel**
GARCH-X: Basic vs Bespoke	281	75	605	293	-2.6
GARCH-X: TOD vs Bespoke	528	151	358	84	-7.4

In Figure S.4 we plot the cross sectional average optimal weights for the bespoke GARCH-X model, estimated with 5-minute RV as the volatility proxy. This figure is similar to Figure 8 in the main paper.

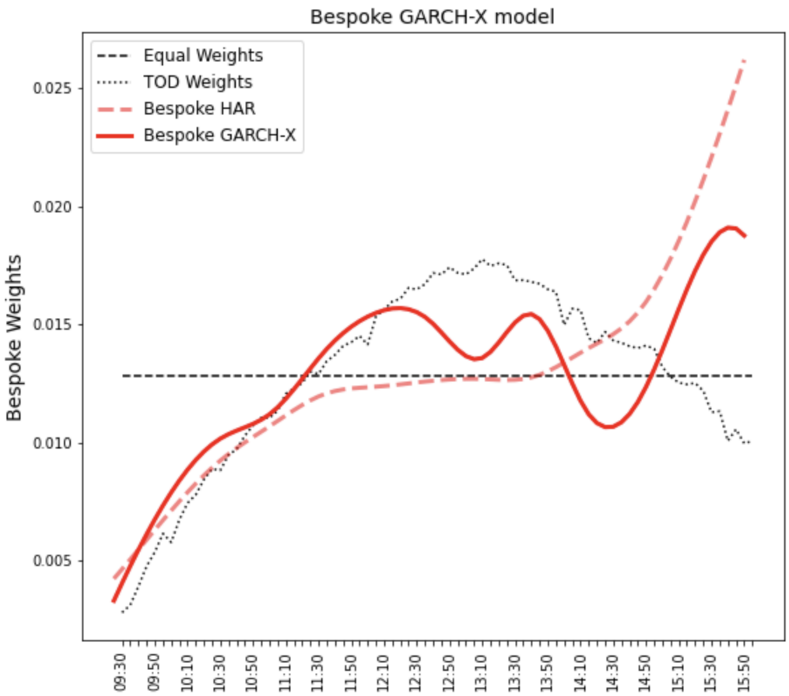
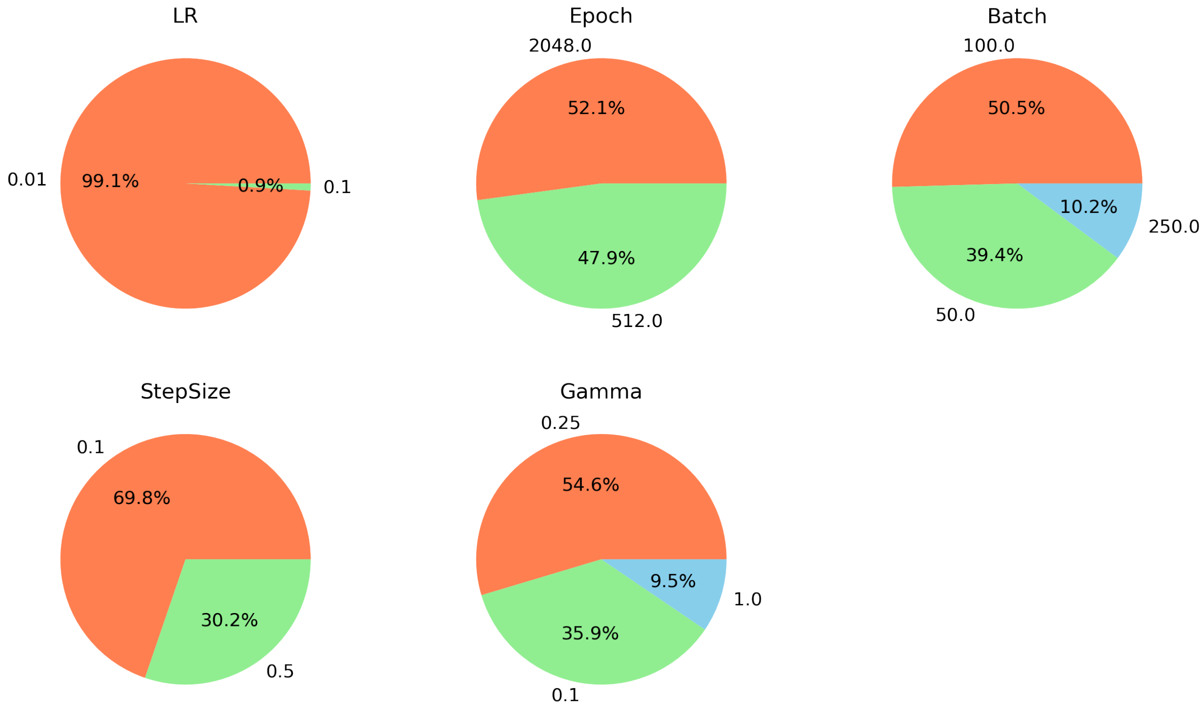


Figure S.4: Bespoke GARCH-X with RV target variable

S.9. Visualization of the hyper-parameters for deep learning based models

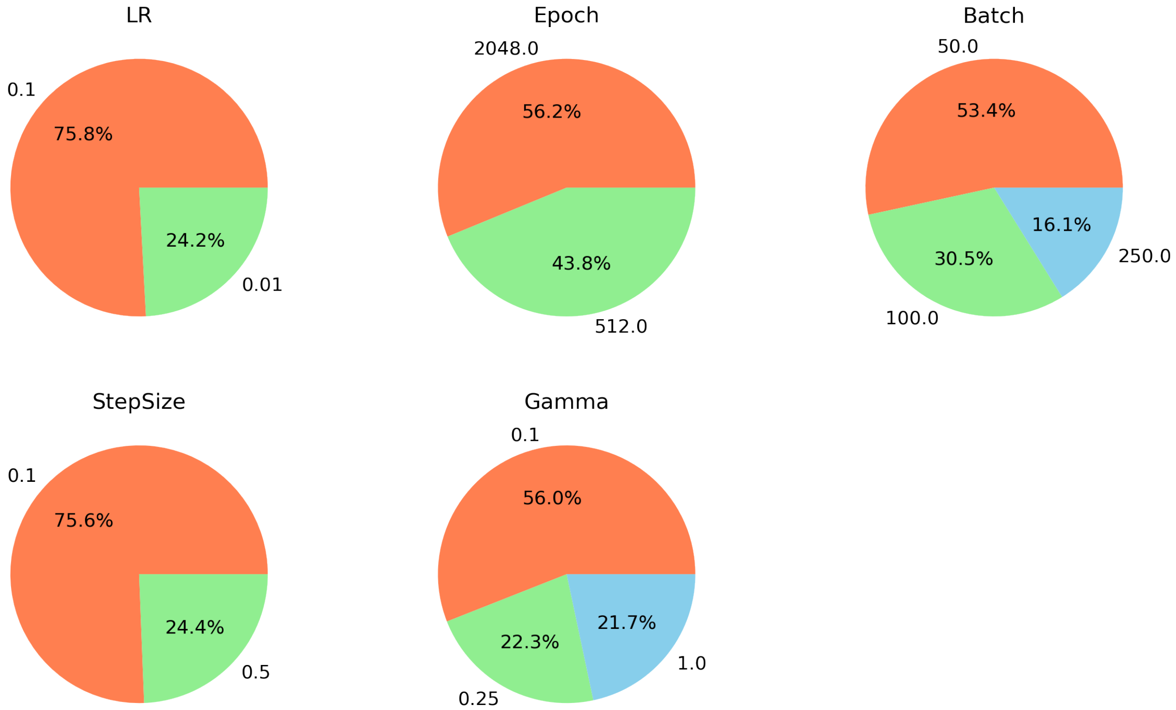
In this section, we also report visualization of the cross sectional variation in the deep learning based models.

Figure S.5: Fully flexible model hyperparameters



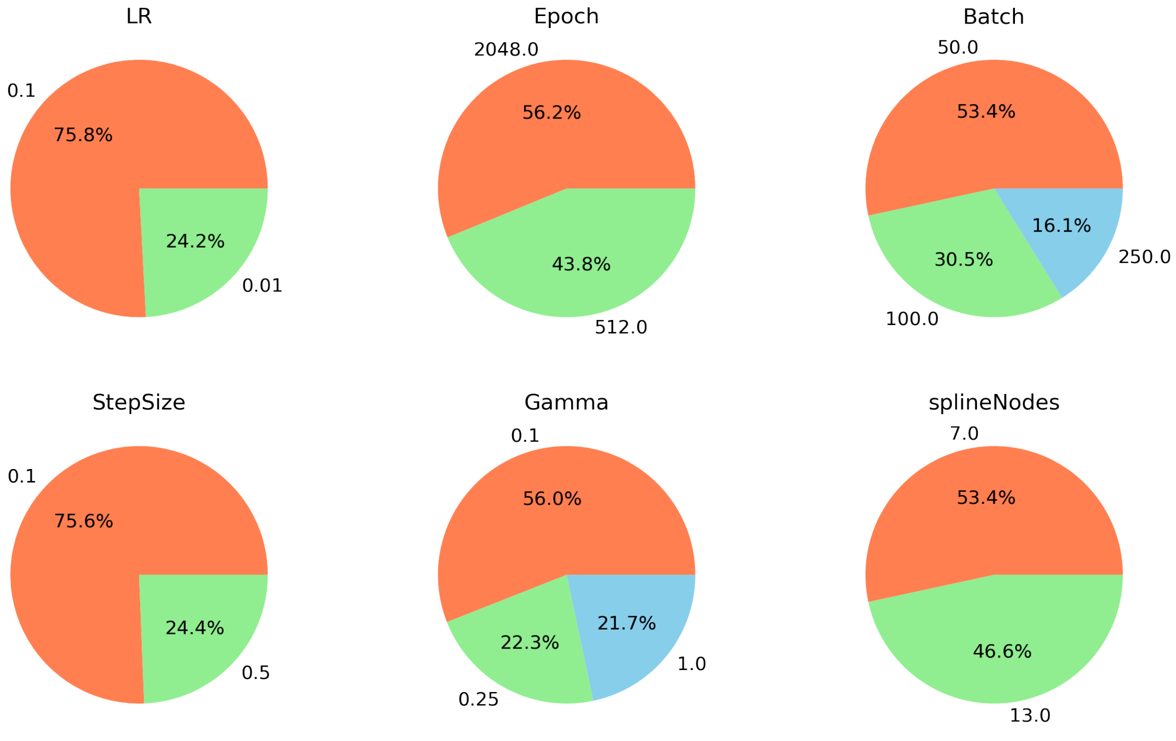
Note: This figure displays the cross-sectional variation of the optimal hyperparameters for the fully flexible model.

Figure S.6: Flexible HAR model hyperparameters



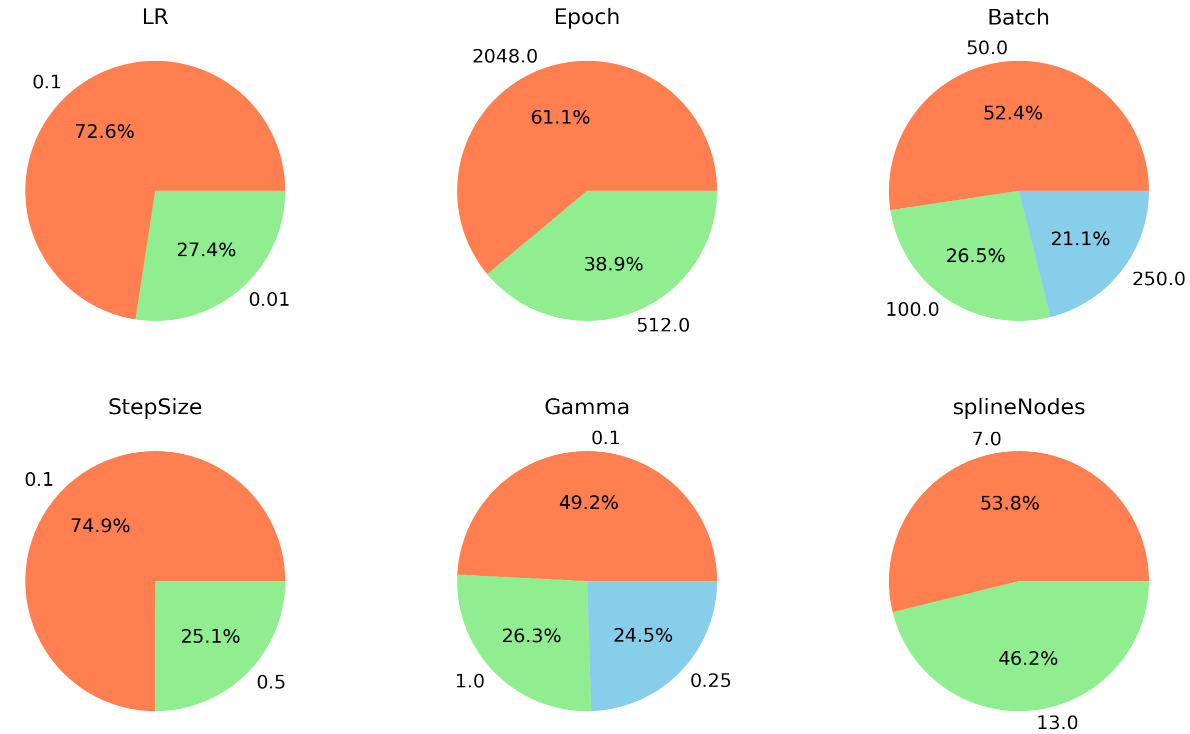
Note: This figure displays the cross-sectional variation of the optimal hyperparameters for the flexible HAR model.

Figure S.7: Cubic HAR model hyperparameters



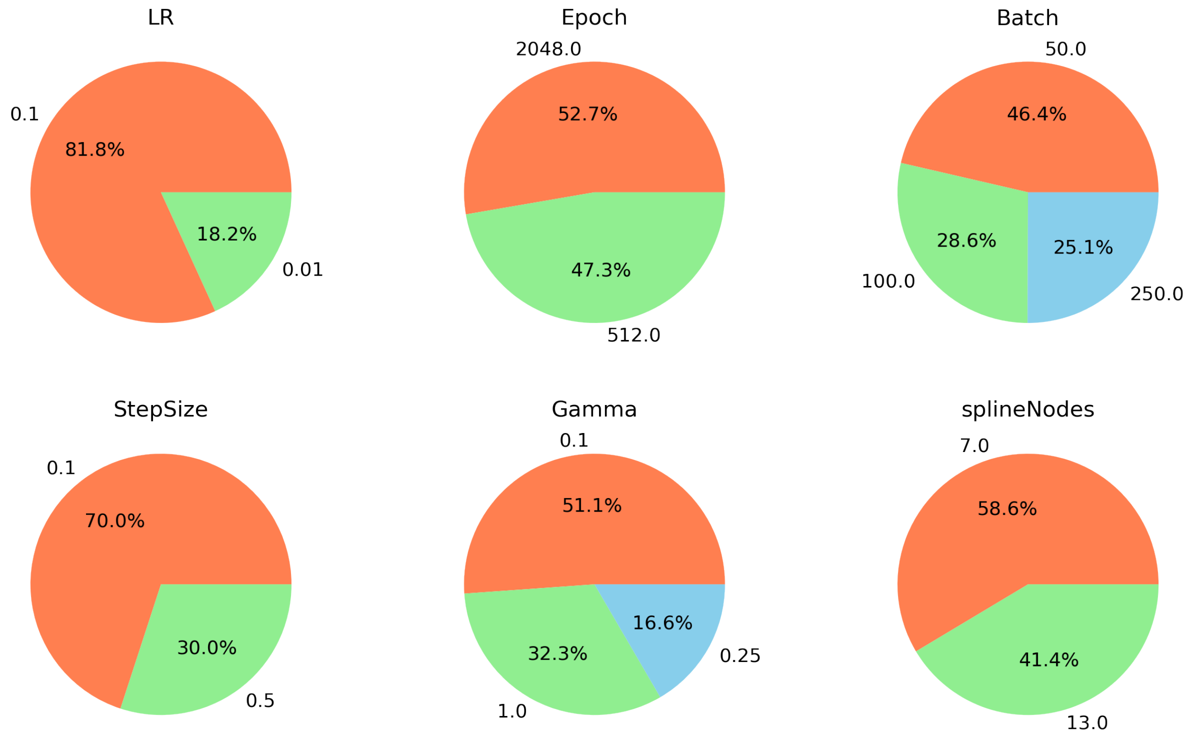
Note: This figure displays the cross-sectional variation of the optimal hyperparameters for the cubic HAR model.

Figure S.8: Cubic-TOD model hyperparameters



Note: This figure displays the cross-sectional variation of the optimal hyperparameters for the cubic-TOD model.

Figure S.9: Cubic-EW model hyperparameters



Note: This figure displays the cross-sectional variation of the optimal hyperparameters for the cubic-EW model.

Additional References

[SuppApp1] Andersen, T.G., Bollerslev, T., 1998. Answering the skeptics: Yes, standard volatility models do provide accurate forecasts. International economic review , 885–905.

[SuppApp7] Andersen, T.G., Bollerslev, T., Diebold, F.X., Labys, P., 2003. Modeling and forecasting realized volatility. Econometrica 71, 579–625.

[SuppApp12] Barndorff-Nielsen, O.E., Hansen, P.R., Lunde, A., Shephard, N., 2009. Realized kernels in practice: Trades and quotes. Econometrics Journal 12, 1–32.

[SuppApp4] Patton, A.J., 2011. Volatility forecast comparison using imperfect volatility proxies. Journal of Econometrics 160, 246–256.