Q1. \( z \) is the certain income in current period. \( y(r) \) is random second period income, \( y(r) \sim F(y \mid r) \) on \([0, z]\). An increase in \( r \) represents increase in risk of \( y \), in the sense of mean preserving spread (MPS): \( Eg(r) = w, \forall r \).

\( u'() > 0, u''() < 0, u'(0) = \infty, u'(z) = 0. \) Individual is prudent: \(-\frac{u'''(\cdot)}{u''(\cdot)} > 0, \) i.e. \( u'''(\cdot) > 0. \)

a) Show \( 0 < s^*(r) < w. \)

\[
\max_r V(r, s) = u(w - s) + \int_0^z u(y + s)f(y \mid r)dy, \quad s.t. \ 0 \leq s(r) \leq w
\]

\[
FOC : \phi(r, s) = \int_0^z u'(y + s)f(y \mid r)dy - u'(w - s)
\]

\[
\phi(r, s^*) \leq 0 \quad \text{for} \quad s^* = 0
\]

\[
\phi(r, s^*) = 0 \quad \text{for} \quad 0 < s^* < w
\]

\[
\phi(r, s^*) \geq 0 \quad \text{for} \quad s^* = w
\]

i)

\[
\phi(r, 0) = \int_0^z u'(y)f(y \mid r)dy - u'(w)
\]

\[
= Eu'(y) - u'(Ey), \quad \text{since} \ Eg(r) = w, \forall r \ \text{by MPS}
\]

By prudence \( u'''(\cdot) > 0, \) we know \( u'(\cdot) \) is convex. Jensen’s Inequality implies \( Eu'(y) > u'(Ey). \) Thus \( \phi(r, 0) > 0. \) Hence, \( s^*(r) > 0. \)

ii)

\[
\phi(r, w) = \int_0^z u'(y + w)f(y \mid r)dy - u'(0)
\]

\[
< 0, \quad \text{since} \ u'(0) = \infty
\]

Hence, \( s^*(r) < w. \) Combining i) and ii) gives \( 0 < s^*(r) < w. \) In the following, we will use \( \phi(r, s^*) = \int_0^z u'(y + s^*)f(y \mid r)dy - u'(w - s^*) = 0, \) which defines \( s^*(r) \) as a function of \( r. \)

Preliminaries

An increase in \( r \) represents increase in risk of \( y, \) in the sense of mean preserving spread. Explicitly, let \( F(y \mid r') \) be a MPS of \( F(y \mid r), \) then

\[
\int_0^z [F(x \mid r') - F(x \mid r)]dx
\]

\[
= x[F(x \mid r') - F(x \mid r)]_0^z - \int_0^z x[f(x \mid r') - f(x \mid r)]dx
\]

\[
= 0 \quad \text{by MPS}
\]
\[ A + C = B + D \text{ for MPS} \]

That is, the areas below the two distributions are the same over the interval \([0, z]\). Figure:

\[
\int_0^y [F(x \mid r') - F(x \mid r)]dx \geq 0, \forall y \in [0, z], \forall r' > r
\]

\[
\Rightarrow \int_0^y \frac{F(x \mid r') - F(x \mid r)}{r' - r}dx \geq 0
\]

taking limit \Rightarrow \int_0^y F_r(x \mid r)dx \geq 0, \forall y \in [0, z], \forall r' > r

\[
\int_0^z [F(x \mid r') - F(x \mid r)]dx = 0, \forall r > r'
\]

\[
\Rightarrow \int_0^z \frac{F(x \mid r') - F(x \mid r)}{r' - r}dx = 0
\]

taking limit \Rightarrow \int_0^z F_r(x \mid r)dx = 0, \forall r' > r

Alternatively, to see \( \int_0^z F_r(x \mid r)dx = 0 \).

\( F(0 \mid r) = 0, F(z \mid r) = 1, F_r(0 \mid r) = 0, F_r(z \mid r) = 0, \forall r \)
Thus,

\[
E(y \mid r) = \int_0^z y f(y \mid r) dy = y F(y \mid r) \bigg|_0^z - \int_0^z F(y \mid r) dy
\]

\[
= z F(z \mid r) - 0 F(0 \mid r) - \int_0^z F(y \mid r) dy = z - \int_0^z F(y \mid r) dy
\]

\[
= \int_0^z dy - \int_0^z F(y \mid r) dy = \int_0^z [1 - F(y \mid r)] dy
\]

\[
= w \text{ by MPS}
\]

\[
\frac{\partial E(y \mid r)}{\partial r} = - \int_0^z F_r(y \mid r) dy = 0
\]

\[
\frac{d s^*(r)}{dr} > 0.
\]

\[
\frac{d s^*(r)}{dr} = - \frac{\partial \phi(r, s^*)}{\partial r} \text{ / } \partial \phi(r, s^*) / \partial s \text{ By Implicit Function Theorem,}
\]

\[
\text{sign } \frac{d s^*(r)}{dr} = \text{ sign } \frac{\partial \phi(r, s^*)}{\partial r}
\]

since \( \frac{\partial \phi(r, s^*)}{\partial s} = \int_0^z u''(y + s^*) f(y \mid r) dy + u''(w - s^*) < 0 \)

\[
\frac{\partial \phi(r, s^*)}{\partial r} = \int_0^z u'(y + s^*) F_r(y \mid r) dy
\]

\[
= u'(y + s^*) F_r(y \mid r) \bigg|_0^z - \int_0^z u''(y + s^*) F_r(y \mid r) dy
\]

\[
= u'(z + s^*) F_r(z \mid r) - u'(s^*) F_r(0 \mid r) - \int_0^z u''(y + s^*) F_r(y \mid r) dy
\]

\[
= - \int_0^z u''(y + s^*) F_r(y \mid r) dy \text{ since } F_r(z \mid r) = F_r(0 \mid r) = 0
\]

\[
= - \int_0^z u''(y + s^*) dy \int_0^y F_r(x \mid r) dx
\]

\[
= -u''(y + s^*) \int_0^y F_r(x \mid r) dx \bigg|_0^z + \int_0^z u''(y + s^*)[ \int_0^y F_r(x \mid r) dx] dy
\]

\[
= -u''(z + s^*) \int_0^z F_r(x \mid r) dx + \int_0^z u''(y + s^*)[ \int_0^y F_r(x \mid r) dx] dy
\]

\[
= \int_0^z u''(y + s^*)[ \int_0^y F_r(x \mid r) dx] dy \text{ since } \int_0^z F_r(x \mid r) dx = 0
\]

\[
> 0 \text{ since } u''(y + s^*) > 0 \text{ and } \int_0^y F_r(x \mid r) dx \geq 0
\]

Thus, \( \frac{d s^*(r)}{dr} > 0 \)
Q2. Investment of $l$ has a random gross return of \( U(l) \), where \( U'' > 0, U' < 0 \), and \( x \) is the realization of a random variable \( X(r) \) with distribution \( F(x \mid r) \) on \([0, 1]\). An increase in \( r \) represents an increase in riskiness of future income in the sense of a MPS.

Suppose individual has to choose \( i \) before observing \( U \):

\[
\max_i -i + \int_0^1 xR(i) f(x \mid r) dx, \text{ s.t. } 0 < i(r) < \infty
\]

\[
\text{FOC}_1 : \int_0^1 xR'(i^*(r)) f(x \mid r) dx - 1 = 0
\]

Note: optimal \( i^*(r) \) conditional on risk \( r \)

Suppose individual can choose \( i \) after observing \( U \):

\[
\max_i -i + xR(i), \text{ s.t. } 0 < i(x) < \infty
\]

\[
\text{FOC}_2 : xR'(i^{**}(x)) - 1 = 0
\]

Note: optimal \( i^{**}(x) \) conditional on observed \( x \)

Value of information:

\[
V(r) = \int_0^1 [xR(i^{**}(x)) - i^{**}(x) - (xR(i^*(r)) - i^*(r))] dF(x \mid r)
\]

Show \( V'(r) > 0 \). Write \( V(r) = \int_0^1 W(x, r) dF(x \mid r) \). To show \( V'(r) > 0 \), follow two steps:

a) Show that \( W(x, r) = [xR(i^{**}(x)) - i^{**}(x) - (xR(i^*(r)) - i^*(r))] \) is convex in \( x \).

\[
\frac{\partial W(x, r)}{\partial x} = R(i^{**}(x)) + xR'(i^{**}(x)) \frac{\partial i^{**}(x)}{\partial x} - \frac{\partial i^{**}(x)}{\partial x} - R(i^*(r))
\]

\[
= R(i^{**}(x)) - R(i^*(r)) + [xR'(i^{**}(x)) - 1] \frac{\partial i^{**}(x)}{\partial x}
\]

\[
= R(i^{**}(x)) - R(i^*(r)) \text{ by FOC}_2
\]

\[
\frac{\partial^2 W(x, r)}{\partial x^2} = R'(i^{**}(x)) \frac{\partial i^{**}(x)}{\partial x} > 0
\]

\[\implies W(x, r) \text{ is convex in } x \]

since \( R'(i^{**}(x)) > 0 \), and \( \frac{\partial i^{**}(x)}{\partial x} = -\frac{R'(i^{**}(x))}{R'(i^{**}(x))} > 0 \) by FOC$_2$

b) Show that \( \int_0^1 v(x) f(x \mid r) dx \) is increasing in \( r \) for any convex function \( v(x) \). Let \( S(r) = \int_0^1 v(x) f(x \mid r) dx \)
\[ (*) \quad S'(r) = \int_0^1 v(x) f_r(x \mid r) dx \]

\[ = v(x) F_r(x \mid r) \bigg|_0^1 - \int_0^1 v'(x) F_r(x \mid r) dx \]

\[ = v(1) F_r(1 \mid r) - v(0) F_r(0 \mid r) - \int_0^1 v'(x) F_r(x \mid r) dx \]

\[ = - \int_0^1 v'(x) F_r(x \mid r) dx, \text{ since } F_r(1 \mid r) = F_r(0 \mid r) = 0 \]

\[ = - \int_0^1 v'(x) d\int_0^x F_r(y \mid r) dy \]

\[ = -v'(x) \int_0^x F_r(y \mid r) dy \bigg|_0^1 + \int_0^1 v''(x) \int_0^x F_r(y \mid r) dy dx \]

\[ = -v'(1) \int_0^1 F_r(y \mid r) dy + \int_0^1 v''(x) \int_0^x F_r(y \mid r) dy dx \]

\[ = \int_0^1 v''(x) \int_0^x F_r(y \mid r) dy dx \text{ since } \int_0^1 F_r(y \mid r) dy = 0 \text{ by MPS} \]

\[ > 0 \text{ since } \int_0^1 F_r(y \mid r) dy \geq 0 \text{ for } x \in (0, 1) \]

In our particular case, \( v(x) = W(x, r) \)

\[ V(r) = \int_0^1 [xR(i^*(x)) - i^*(x) - (xR(i^*(r)) - i^*(r))] f(x \mid r) dx \]

\[ V'(r) = \int_0^1 [xR(i^*(x)) - i^*(x) - (xR(i^*(r)) - i^*(r))] f_r(x \mid r) dx \]

\[ = \int_0^1 \frac{\partial}{\partial r}(i^*(r)) \left( \frac{\partial}{\partial r} f(x \mid r) \right) dx \]

\[ = \int_0^1 W(x, r) f_r(x \mid r) dx - \frac{\partial}{\partial r} \int_0^1 [xR'(i^*(r)) - 1] f(x \mid r) dx \]

\[ = \int_0^1 W(x, r) f_r(x \mid r) dx \text{ since } FOC_1 : \int_0^1 xR'(i^*(r)) f(x \mid r) dx - 1 = 0 \]

Since \( W(x, r) \) is convex in \( x \), we are back in \( (*) \) in part b). Therefore \( V'(r) > 0 \).