Problem Set 2

1. Suppose there is one safe asset that yields a return of \( r \geq 1 \) per unit with probability one and a risky asset. The return of the risky asset is denoted by \( x \) and is normally distributed with mean \( \bar{x} > 1 \) and variance \( \sigma^2 \). An individual has an initial wealth endowment \( w \). Let \( \alpha \) be the proportion of this wealth invested in the risky asset, with \((1 - \alpha)\) invested in the safe asset. An individual’s terminal wealth \( \tilde{w} \) is then

\[
\tilde{w} = w(1 - \alpha)r + w\alpha x.
\]

Note that since \( x \) is normally distributed, \( \tilde{w} \) is normally distributed for any \( \alpha > 0 \).

Suppose this individual is risk averse with utility function \( u(w) = -e^{-\gamma w} \).

(a) Show that an individual will maximize utility at some positive \( \alpha^* > 0 \) for \( r \) sufficiently close to 1.

(b) Assuming \( \alpha^* > 0 \) (that is, the condition you found in part (b) is satisfied) show further that the optimal portfolio share in the risky asset satisfies

\[
\alpha^* w = \frac{[\bar{x} - r]}{\lambda \sigma^2}.
\]

Interpret this result in terms of the mean and variance of the distribution as well as the Arrow-Pratt measure of absolute risk aversion (which is what in this case?).

2. Suppose an individual divides her wealth between a safe and risky asset. Let \( y^* \) denote the optimal level of investment in the risky asset, and assume \( y^* > 0 \). Let \( r_A(w) \) denote the Arrow-Pratt measure of absolute risk aversion. Show that if \( r_A(w) \) is decreasing in \( w \), then \( \frac{dy^*}{dw} > 0 \), if \( r_A(w) \) is constant, then \( \frac{dy^*}{dw} = 0 \), if \( r_A(w) \) is increasing, then \( \frac{dy^*}{dw} < 0 \). Make sure you can do this without looking at your class notes!

4. Suppose an individual has a Bernoulli utility function \( u(y) = y^2 \).

   (a) Calculate the Arrow-Pratt coefficients of absolute and relative risk aversion at the level of wealth \( w = 5 \).

   (b) Consider three possible monetary outcomes \( C = (36, 16, 4) \). Calculate the certainty equivalent and the risk premium for the lottery \( L = (0, .5, .5) \).

   (c) Calculate the certainty equivalent and the risk premium for the lottery \( L' = (.5, .5, 0) \). Compare this with the result in (b) and interpret.

5. Consider five possible monetary outcomes in \( C \). \( C = \{0, 200, 450, 470, 1000\} \) Consider two lotteries on \( C \) : \( L_A = (0, 1, 0, 0, 0) \); that is, \$200 is won with probability one, and \( L_B = (.5, .3, .1, 0, .1) \). An individual’s preferences over lotteries satisfy the von-Neumann-Morgenstern axioms. The individual is also risk-averse and prefers more money to less. Her certainty equivalent for a third lottery \( L_C = (.5, 0, 0, 0, .5) \) is \$470. Show that the individual must then prefer lottery \( L_A \) to \( L_B \).