Hidden Action Case

With a fixed wage contract the manager would just exert $e_L$.

The principal must make the agent face some risk in order to give her the incentive to exert $e_H$.

Study two cases:

- Risk-Neutral agent: $v(w) = w$.

- Risk-Averse agent: $v'' < 0$. 
Risk-Neutral Agent: \( v(w) = w \).

**Proposition:** when the agent is risk neutral, there exists a contract that generates the same effort choice and same expected payoffs for the principal and agent as when effort is observable. This contract therefore must be an optimal contract.

**Proof.** First, suppose we have such a contract. It must be optimal because the principal can never do better when effort is unobservable than when it is not.

Next, consider a contract \( w(\pi) = \pi - \alpha \). The manager becomes the residual claimant and will choose effort to maximize expected profits net of effort costs:

\[
\max_{e \in \{e_L, e_H\}} \int \pi \cdot f(\pi \mid e) d\pi - g(e) - \alpha.
\]

The effort level which maximizes these profits is \( e^* \). The principal will set \( \alpha \) to extract all the rents from the manager:

\[
\alpha^* = \int \pi \cdot f(\pi \mid e^*) d\pi - g(e^*) - w.
\]

This gives the manager exactly her reservation utility, and gives the principal the rest of the surplus, as in the full information case. \( \blacksquare \)
Strictly Risk-Averse Agent: $v'' < 0$.

**Two-step solution:**

First, find $w^o(\pi \mid e)$, the best compensation scheme that will induce the agent to undertake a particular action $e$.

Second, determine the best overall compensation scheme by comparing principal’s expected profits from implementing $e_H$ using $w^o(\pi \mid e_H)$ and expected profits from implementing $e_L$ using $w^o(\pi \mid e_L)$. 
Wage Contract to Implement Low Effort Level

**Proposition:** The principal will use a *fixed wage contract* to implement $e_L$:

$$w^0(\pi | e_L) = w^*(\pi | e_L) = v^{-1}[\pi + g(e_L)] = w^*_L.$$

**Proof.**

With a fixed wage contract, the agent will choose $e_L$.

Since the agent is risk averse, the principal would have to increase expected compensation if risk were introduced into the compensation schedule.

The principal will therefore simply ensure that the agent earns her reservation value and is compensated for her effort cost. ■
Wage Contract to Implement High Effort Level

Solve for $w^o(\pi \mid e_H)$. The principal’s problem is:

$$\min_{w(\pi\mid e_H)} \int_{\Pi} w(\pi \mid e_H) \cdot f(\pi \mid e_H) d\pi$$

s.t.

$$\int_{\Pi} [v(w(\pi \mid e_H)) \cdot f(\pi \mid e_H) d\pi] - g(e_H) \geq \bar{u}$$

(Individual Rationality (IR))

$$\int_{\Pi} [v(w(\pi \mid e_H)) \cdot f(\pi \mid e_H) d\pi] - g(e_H) \geq$$

$$\int_{\Pi} [v(w(\pi \mid e_H)) \cdot f(\pi \mid e_L) d\pi] - g(e_L).$$

(Incentive Compatibility (IC))

Let $\gamma \geq 0$ be the Langrange multiplier on the IR constraint, and let $\mu \geq 0$ be the multiplier on the IC constraint.
Strictly Risk-Averse Agent (cont.)

**Proposition:** $\gamma > 0$ and $\mu > 0$. Both IR and IC are binding at the optimal contract $w^o(\pi \mid e_H)$.

**Proof.** Examine the FOC

\[ -f(\pi \mid e_H) + \gamma v'(w^o(\pi)) f(\pi \mid e_H) + \mu [f(\pi \mid e_H) - f(\pi \mid e_L)] v'(w^o(\pi)) = 0 \]

which becomes

\[ \frac{1}{v'(w^o(\pi))} = \gamma + \mu \left[ 1 - \frac{f(\pi \mid e_L)}{f(\pi \mid e_H)} \right]. \]
Strictly Risk-Averse Agent (cont.)

\[
\frac{1}{v'(w^o(\pi))} = \gamma + \mu \left[ 1 - \frac{f(\pi \mid e_L)}{f(\pi \mid e_H)} \right].
\]

Suppose first $\gamma = 0$.

We have assumed that $F(\pi \mid e_H)$ FOSD $F(\pi \mid e_L)$.

Hence, $f(\pi \mid e_L) > f(\pi \mid e_H)$ for some profits levels.

At these profit levels, the term $\left[ 1 - \frac{f(\pi \mid e_L)}{f(\pi \mid e_H)} \right]$ is negative.

Since $\mu \geq 0$, $v'(w^o(\pi))$ is negative for these profit levels.

Contradiction, since we have assumed that $v' > 0$.

Hence $\gamma > 0$.

Suppose next $\mu = 0$.

This implies that $w^o(\pi)$ is a fixed wage contract.

So the manager will exert low effort. Hence, $\mu > 0$.

Note: Since the IR is binding, the agent earns expected utility $\bar{u} + g(e_H)$. 

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Properties of Contract that Implements High Effort

**Proposition:** Consider two profit levels $\pi_1 > \pi_2$.

Then $w(\pi_1) > w(\pi_2)$ if and only if

$$w(\pi_1) > w(\pi_2) \text{ iff } \frac{f(\pi_1 | e_L)}{f(\pi_1 | e_H)} < \frac{f(\pi_2 | e_L)}{f(\pi_2 | e_H)}$$

i.e., iff the “likelihood ratio” $\frac{f(\pi | e_L)}{f(\pi | e_H)}$ is decreasing in $\pi$.

In this case we say that the distribution satisfies the **monotone likelihood ratio property (MLRP)**.
High Effort Contract (cont.)

**Proof.** Recall the FOC:

\[
\frac{1}{v'(w^o(\pi))} = \gamma + \mu \left[ 1 - \frac{f(\pi | e_L)}{f(\pi | e_H)} \right].
\]

Consider first the case \( \frac{f(\pi_1 | e_L)}{f(\pi_1 | e_H)} < 1 \).

When \( \frac{f(\pi_1 | e_L)}{f(\pi_1 | e_H)} < \frac{f(\pi_2 | e_L)}{f(\pi_2 | e_H)} \) then the second term on the RHS is larger for \( \pi_1 \).

So \( \frac{1}{v'(w^o(\pi_1))} \) must be equal to a larger number. This implies \( v'(w^o(\pi_1)) \) must be smaller for \( \pi_1 \).

Which implies \( w^o(\pi_1) \) must be larger, since \( v'' < 0 \).

A similar argument holds for \( \frac{f(\pi_1 | e_L)}{f(\pi_1 | e_H)} > 1 \). ■
High Effort Contract (cont.)

The likelihood ratio $\frac{f(\pi|e_L)}{f(\pi|e_H)}$ reflects the relative likelihood of a particular profit level $\pi$ being drawn from distribution $f(\pi_1 | e_L)$ rather than $f(\pi_1 | e_H)$.

Note: that this is a further restriction on $F(\pi | e_L)$ and $F(\pi | e_H)$. FOSD does not imply MLRP.
High Effort Contract (cont.)

**Proposition:** the principal pays higher expected wages to a strictly risk averse agent when effort is unobservable than when effort is observable.

We want to show that, for $v'' < 0$,

$$\int_{\pi} \left[ w^o(\pi | e_H) \cdot f(\pi | e_H) d\pi \right] > w^*_{e_H}.$$ 

**Proof.** By Jensen’s inequality:

$$v \left( \int_{\pi} \left[ w^o(\pi | e_H) \cdot f(\pi | e_H) d\pi \right] \right) > \int_{\pi} \left[ v (w^o(\pi | e_H)) \cdot f(\pi | e_H) d\pi \right]$$

The RHS is equal to $u + g(e_H) = v \left[ w^*_e \right]$, because IR is binding. Substituting, we have:

$$v \left( \int_{\pi} \left[ w^o(\pi | e_H) \cdot f(\pi | e_H) d\pi \right] \right) > v \left[ w^*_e \right].$$

Then $\int_{\pi} \left[ w^o(\pi | e_H) \cdot f(\pi | e_H) d\pi \right] > w^*_e$ since $v' > 0$. ■
Solution to Principal’s Problem

The principal will choose $w^o(\pi \mid e_H)$ over $w^*_e L$ iff

$$\int_\pi [\pi - w^o(\pi \mid e_H)] \cdot f(\pi \mid e_H) d\pi > \int_\pi [\pi \cdot f(\pi \mid e_L) d\pi] - w^*_e L.$$
Summary of Results for Risk-Averse Agent

- The principal will use a fixed wage contract to implement $e_L$.

- Both IR and IC are binding at the optimal contract $w^o(\pi \mid e_H)$.

- $w(\pi_1) > w(\pi_2)$ if and only if the “likelihood ratio” $\frac{f(\pi|e_L)}{f(\pi|e_H)}$ is decreasing in $\pi$.

- The principal pays higher expected wages to a strictly risk averse agent when effort is unobservable than when effort is observable.
Continuous Effort Case

Suppose $e \in E = [\underline{e}, \overline{e}]$.

The optimal contract $w^o(\pi \mid e)$ for implementing effort level $e$ solves the following problem $M(e)$:

$$\min_{w(\pi \mid e)_{\pi}} \int_{\pi} w(\pi \mid e) \cdot f(\pi \mid e) d\pi$$

s.t.

$$\int_{\pi} [v(w(\pi \mid e)) \cdot f(\pi \mid e) d\pi] - g(e) \geq u$$

(Individual Rationality (IR))

$$\int_{\pi} [v(w(\pi \mid e)) \cdot f(\pi \mid e) d\pi] - g(e) \geq$$

$$\int_{\pi} [v(w(\pi \mid \bar{e})) \cdot f(\pi \mid \bar{e}) d\pi] - g(\bar{e}) \quad \forall \bar{e} \in \{E \setminus e\}.$$  

(Incentive Compatibility (IC))

There are infinitely many IC constraints. So there may not exist any wage schedule $w(\pi \mid e)$ for which all constraints are simultaneously satisfied.