Basic Processes

The reference of record is arguably Jacod and Shiryaev (2003), which many might find tough going. Our main texts are Jacod and Protter (2012) and Ait-Sahalia and Jacod (2014).

1 Brownian Semimartingales

We consider continuous time processes $X_t$ for $0 \leq t \leq T$. The unit of time is always one day. Usually $T$ is an integer equal to 1, 22, or 252 for a day, business month, or business year.

A semimartingale is a càdlàg process that consists of a piece of finite variation plus a martingale

$$X_t = X_0 + A_t + M_t$$

(1)

The elementary diffusion gives the basic idea.

1.1 Constant Coefficients Gaussian Diffusion

The constant coefficient Gaussian diffusion is:

$$X_t = at + \sqrt{c} W_t$$

(2)

where $a$ and $c > 0$ are constants and $W$ is a Wiener process with the $W_t$ jointly Gaussian, $W_0 = 0$, conditional means $E(W_t|W_s) = W_s$ for $s \leq t$, which implies $\text{Cov}(W_s, W_t) = |t - s|$. The model (2) is a very natural representation of a financial times series because $a\tau$ represents the local predictable risk premium earned over the interval $t \rightarrow t + \tau$ for bearing the risk of the outcome $\sqrt{c}W_{t+\tau} - \sqrt{c} W_t$.

The easiest way to learn the model is to think about simulating it given values for $a, c$. Suppose the discretization interval is (a very small number) $\Delta_n$, and we want $n = \lfloor 1/\Delta_n \rfloor$ steps per day over $T$ days in total. Generate $nT$ independent and identically distributed standard $N(0, 1)$ random variables $Z_i, i = 1, 2, \ldots, nT$ and set

$$X^n_t = \sum_{i=1}^{[t/\Delta_n]} \left( a\Delta_n + \sqrt{c} \sqrt{\Delta_n} Z_i \right), \quad t \in [0, T].$$

(3)

With this scaling convention $E(X^n_1) = a$, $\text{Var}(X^n_1) = c$, and $E(X^n_T) = aT$, $\text{Var}(X^n_T) = cT$. In Jacod and Protter (2012) and many other places it is shown that $X^n \rightarrow X$ as a stochastic process as $\Delta_n \rightarrow 0$.

1 Actually a local martingale, but do not worry about that now.
1.2 Variable Coefficients Gaussian Diffusion

We can easily let the local drift and variance be random and time varying in continuous time \( \{a_s, c_s\}_{s \in [0,T]} \); for now, independent of the driving Brownian motion and left continuous. To generate the simulation again generate \( nT \) independent and identically distributed standard \( \mathcal{N}(0,1) \) random variables \( Z_i \). Now define

\[
X^n_t = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left( a_{(i-1)/n} \Delta_n + \sqrt{c_{(i-1)/n}} \sqrt{\Delta_n} Z_i \right), \quad t \in [0,T].
\]  

(4)

In Jacod and Protter (2012) and many other places it is shown that \( X^n \to X \), where

\[
X_t = \int_0^t a_s \, ds + \int_0^t \sqrt{c_s} \, dW_s,
\]  

(5)

and the first integral is the ordinary integral and the second is a stochastic integral.

2 Compound Poisson Processes

Compound Poisson processes for \( t \in [0,T] \) are fundamental to Lévy processes and financial econometrics.

Let \( \lambda > 0 \) and \( N \sim \text{Poisson}(\lambda T) \). \( N \) is the total finite number of jumps on \( [0,T] \). Let \( \{X_i\}_{i=1}^N \) be independent and identically distributed random variables independent of \( N \). Conditional on \( N \), all that needs to be done is to scatter the \( N \) jumps uniformly over \( [0,T] \). To this end, Let \( \{U_i\}_{i=1}^N \) be independent \( \text{Uniform}([0,T]) \) on \( [0,T] \) variables, and set

\[
Y_t = \sum_{i=1}^N 1_{\{U_i \leq t\}} X_i, \quad 0 \leq t \leq T. \]  

(6)

Then \( Y_t \) is a compound Poisson process with intensity parameter \( \lambda \) and jump pdf \( f(x) \). The characteristic function of \( Y_1 \) is

\[
E(e^{iuY_1}) = e^{\lambda \int (e^{ix} - 1) f(x) \, dx}
\]

and that of \( Y_t \) is

\[
E(e^{iuY_t}) = e^{t \int (e^{ix} - 1) m(x) \, dx}, \quad t \geq 0,
\]

where we write

\[
m(x) = \lambda f(x)
\]

for the intensity density. If \( a < b \), then the expected number of jumps in \( Y_t \) of size between \( a \) and \( b \) is

\[
t \int_a^b m(x) \, dx.
\]
Since $f(x)$ is a density that integrates to 1, then $m(x)$ integrates to $\lambda$:

$$\int m(x)dx = \lambda < \infty,$$

which is why we call $m(x)$ the intensity density. We can always compute the relative frequency of jumps in $[a, b]$ relative to $[c, d]$ as the ratio

$$0 \leq \frac{\int_a^b m(x)dx}{\int_c^d m(x)dx} < \infty$$

where $c < d$.

Note that jumps of size 0 make no sense so we can redefine the jump intensity at 0 as $m(0) = 0$. If $a < 0$ and $b > 0$ then strictly speaking we should write the expected number of jumps of $Y_t$ in the interval $[a, b]$ is

$$t \int_b^0 m(x) + t \int_0^b m(x)dx$$

which is tedious so we will just write this as

$$t \int_a^b m(x)dx$$

with the understanding that the above integral skips over the point $x = 0$. We will see later that the behavior of the intensity function around 0, i.e., for very small positive or negative jumps is delicate and very important for studying a Lévy process.

### 3 Jump Diffusion

We put the pieces together

$$X_t = \int_0^t a_s \, ds + \int_0^t \sqrt{c_s} \, dW_s + \tilde{Y}_t \quad 0 \leq t \leq T. \tag{7}$$

As before, the first integral is the ordinary integral, the second is a stochastic integral, and $\tilde{Y}_t = Y_t - t\lambda \mu_x$ is a centered compound Poison process with intensity density $m(x)$; centering makes the $a_t$ process be the true drift. $X_t$ is a semimartingale and is the most commonly used models for financial prices. Note that with $\Delta X_t \equiv X_t - X_{t-}$

$$X_t' = X_t - \Delta X_t \tag{8}$$

is continuous and thereby called the continuous part of $X$. We often think of the decomposition

$$X_t = X_t' + \Delta X_t \tag{9}$$
4 Sampling

Now let’s suppose the process $X_t$ in (7) is well defined and consider discrete equi-spaced observations at sampling interval $\Delta_n$ on it: $X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{nT\Delta_n}$, where $n = [1/\Delta_n]$. We always work with the increments

$$\Delta^n_i X = X_{i\Delta_n} - X_{(i-1)\Delta_n}, \quad i = 1, 2, \ldots, nT.$$  

(10)

Following Barndorff-Nielsen and Shephard (2004) (and references therein) We can define the realized variance and bipower variance via sums of $|\Delta^n_i X|^2$ and $|\Delta^n_i X| |\Delta^n_{i-1} X|$.

5 The Three Uses of $\Delta$ in high Frequency Financial Econometrics

A. $\Delta X_t = X_t - X_{t-}$ (the jump operator)

B. $\Delta_n$ (the width of the sampling interval) $\Delta_n = \frac{1}{n}$, or $n = \lfloor \frac{1}{\Delta_n} \rfloor$

C. $\Delta^n_i X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$ (first difference operator)

The meaning is deduced from the context.

References


