Realized Beta GARCH
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ECON201
Resources:

- **Main:**

- **Others Referenced:**
  - Barndoff-Neilson, Hansen, Lunde, Shephard. *Multivariate realised kernels: Consistent positive semi-definite estimators of the covariation of equity prices with noise and non-synchronous trading*
Realized Beta: Persistence and Predictability

- Assesses dynamics and predictability in realized betas,
  - relative to dynamics in underlying market variance and covariances with market
- Does not require an assumed volatility model
- Does not require an assumed model of time-varying-beta
- Conclusion:
  - Although realized variances and covariances fluctuate widely and are highly persistent and predictable
  - Realized betas display much less persistence and predictability
- Areas for Extension:
  - Data only went through 1999
  - Allow variance to change with time: so that beta innovations become more volatile as the constituent parts, the market variance and stock/market covariance, increase
Introduction

• Multivariate GARCH model that utilizes and models realized measures of volatility and covolatility
  ▫ Utilizes multivariate kernel approach developed in Barndorff-Neilson, Hansen, Lunde and Shephard (2010)
• Research investigates how realized measures can be used to specify better models of volatility dynamics and provide more accurate forecasts.
  ▫ Reduced-form: based on time-series
  ▫ Model-Based: based on parametric
• Realized GARCH Model
  ▫ Involves different approach to the joint modeling of returns and realized volatility
  ▫ Key difference: measurement equation
Realized GARCH Framework:

- Model Strategy Combines:
  - Marginal model for market returns and corresponding realized measures of volatility
  - Conditional model for the asset-specific return, its realized volatility and covolatility between the asset and the factor (i.e. the market)
- Marginal model: Used for market specific time series is variant of Realized GARCH *Hansen et al. (2010)*
  - Called Realized EGARCH: shares certain features with EGARCH by Nelson (1991)
- Bivariate Set-Up
Notation & Modeling Strategy

- Market Return: $r_{o,t}$ with realized volatility $x_{o,t}$
- Time Series for Individual Asset: $r_{1,t}$ and $x_{1,t}$
- Realized Correlation Measure: $y_{1,t}$
- Realized Volatility and Correlation obtained using multivariate kernel methodology
- Natural Filtration:
  $$\mathcal{F}_t = \sigma(\chi_t, \chi_{t-1}, \ldots) \quad \text{with} \quad \chi_t = (r_{0,t}, r_{1,t}, x_{0,t}, x_{1,t}, y_{1,t})$$
- Decomposition of Conditional Density
  $$f(r_{0,t}, x_{0,t}, r_{1,t}, x_{1,t}, y_{1,t}|\mathcal{F}_1, t) = f(r_{0,t}, x_{0,t}|\mathcal{F}_1, t)f(r_{1,t}, x_{1,t}, y_{1,t}|r_{0,t}, x_{0,t}, \mathcal{F}_1, t)$$
Realized EGARCH Model for Market Returns

- EGARCH Model is given by the following three equations
  - (1) \( r_{0,t} = \mu_0 + \sqrt{h_{0,t}} z_{0,t} \)
  - (2) \( \log h_{0,t} = a_0 + b_0 \log h_{0,t-1} + c_0 \log x_{0,t-1} + \tau_0 (z_{0,t-1}) \)
  - (3) \( \log x_{0,t} = \xi_0 + \varphi_0 \log h_{0,t} + \delta_0 (z_{0,t}) + u_{0,t} \)
- Where we model the following as:
  - \( z_{0,t} \sim iidN(0,1) \)
  - \( u_{0,t} \sim iidN(0,\sigma_u^2) \)
  - \( h_{0,t} \) denotes conditional variance, \( h_{0,t} = \text{var}(r_{0,t} \mid F_{t-1}) \)
Measurement Equation

- \( \log x_{0,t} = \hat{\xi}_0 + \varphi_0 \log h_{0,t} + \delta_0(z_{0,t}) + u_{0,t} \)

- Tying the realized measure, \( x_t \), to the conditional variance, \( h_t \), is motivated by the fact that conventional GARCH models imply:
  - \( \log(r_t - \mu)^2 = \log h_t + \log z_t^2 \)

- Since \( x_t \) is similar to \( r_t^2 \), the assumption follows that:
  - \( \log x_t \approx \log h_t + f(z_t) + \text{error}_t \)

- Intercept and slope allow some flexibility in the specification that may be needed due to the fact that realized measures of volatility are computed over a shorter period of time than the return.
Conditional Model for Individual Asset Returns, Volatility, and Covolatility

- Specification for: \( f(r_{1,t}, x_{1,t}, y_{1,t} \mid r_{0,t}, x_{0,t}, F_{t-1}) \)
- Further decompose to the following:
  - \( f(r_{1,t}, x_{1,t}, y_{1,t} \mid r_{0,t}, x_{0,t}, F_{t-1}) = f(r_{1,t} \mid r_{0,t}, x_{0,t}, F_{t-1}) f(x_{1,t}, y_{1,t} \mid r_{1,t}, r_{0,t}, x_{0,t}, F_{t-1}) \)
  - \( f(r_{1,t} \mid r_{0,t}, x_{0,t}, F_{t-1}) \) modeled with three equations:

\[
\begin{align*}
\text{(4)} & \quad r_{1,t} &= \mu_1 + \sqrt{h_{1,t}} z_{1,t} \\
\text{(5)} & \quad \log h_{1,t} &= a_1 + b_1 \log h_{1,t-1} + c_1 \log x_{1,t-1} + d_1 \log h_{0,t} + \tau_1 (z_{1,t-1}) \\
\text{(6)} & \quad \log x_{1,t} &= \xi_1 + \varphi_1 \log h_{1,t} + \delta_1 (z_{1,t}) + u_{1,t}
\end{align*}
\]
Conditional Covariance

- Due to condition on contemporaneous market data it allows for $z_{1,t}$ to be correlated with $z_{0,t}$
  - $z_{1,t}$ can not be modeled as an iid sequence
- Conditional Covariance:
  - $\rho_t = \text{cov}(z_{0,t}, z_{1,t} \mid F_{t-1})$
- Fisher Transformation of Conditional Covariance
  - (7) $F(\rho_t) = a_{01} + b_{01} F(\rho_{t-1}) + c_{01} F(y_{1,t-1})$
- Conditional on contemporary market variables, the covariance structure for the error terms in equations (3), (6), and (7) is as follows:
  - $\Sigma = \begin{pmatrix} \sigma_{u_0}^2 & \sigma_{u_0,u_1} & \sigma_{u_0,v_1} \\ \sigma_{u_0,u_1} & \sigma_{u_1}^2 & \sigma_{u_1,v_1} \\ \sigma_{u_0,v_1} & \sigma_{u_1,v_1} & \sigma_{v_1}^2 \end{pmatrix}$
Empirical Analysis

- Used high-frequency asset prices for thirty assets.
  - SPY, exchange-traded fund, was used as a proxy for the market
- Sample Period runs from January 3, 2002 to the end of 2009
- Results:
  - Consistent across stocks: lends plausibility to modeling framework
  - Realized measure loading (c coefficients) are large
  - GARCH effects (b coefficients) are of smaller magnitude w.r.t. traditional GARCH
  - Parameter $d_1$ significantly positive
  - Results for leverage functions are similar to the ones reported in Hansen et al (2010).
  - Intercepts of measurement equations are negative and the slopes near zero as predicted, given the realized measures are based on data spanning the trading session only
Realized Variance vs. Model-Implied Conditional Variance

Figure 1: Realized kernel (RK) variance and conditional variance of CVX (upper panel) and SPY (lower panel) over the period 2007 – 2009.
Model Implied Beta vs. Realized Beta

- Given as (Model vs. Realized):

  \[
  \hat{\beta}_t = \frac{\hat{\rho}_t \sqrt{\hat{h}_{0,t}\hat{h}_{1,t}}}{\sqrt{\hat{h}_{0,t}}} \quad \text{vs.} \quad \tilde{\beta}_t = \frac{y_{1,t} \sqrt{x_{1,t}x_{0,t}}}{x_{0,t}} = \frac{y_{1,t} \sqrt{x_{1,t}}}{\sqrt{x_{0,t}}}
  \]

  - Model-implied betas take into account the presence of measurement error in the realized quantities as well as the dynamic linkages between realized measures and conditional moments.

- The Difference between the model-implied beta and the realized beta is the core economic motivation
Model Implied Beta vs. Realized Beta

<table>
<thead>
<tr>
<th>Model-implied beta</th>
<th>Realized beta</th>
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<tbody>
<tr>
<td>Mean</td>
<td>Median</td>
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<td>AA</td>
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<td>Overall</td>
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Cross Sectional Variation in the Conditional Beta

Conditional GARCH Betas

Conditional GARCH Correl.
Time Variation of Conditional Beta

[Graph showing realized and model correlation and realized and model beta over time.]
Conclusion

- Model for the joint dynamics of conditional and realized measures of volatility and correlation.
- Self-contained system of equations linking realized to conditional measures.
- Allows the construction of precise conditional betas often used as measures of systematic risk in finance.
- Allows for leverage effects and spillover effects between the assets’ and the market volatility.
- Combines the flexibility of the GARCH modeling framework with the statistical precision in volatility measurement resulting from the use of high-frequency data.
- Find that betas are far more variable than typically obtained with rolling-window OLS regressions.
- Conditional betas are less variable than raw realized betas.
Areas for Extension:

• Data only goes from 2002 to 2009 apply the model across the data we have
• Large time variation in betas can help explain asset pricing “anomalies”
• Investigate use of model with other volatility measure than *kernel*
• Extend the model to multi-factor framework to extend other price factors beyond the market