Economics 201FS: The Basic Diffusive Model, Jumps, Variance Measures, and Noise Corrections

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Economics 201FS
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Main Points

1. The Continuous Time Model, Theory and Simulation
2. Observed Data, Plotting and Pitfalls
3. Empirical Variance Measures
4. Options and Implied Volatility
5. Microstructure Noise
1. The Continuous Time Model

Dynamics of the log price ($p$) process:

Continuous

$$dp(t) = \mu(t)dt + \sigma(t)d\omega(t).$$

With jumps

$$dp(t) = \mu(t)dt + \sigma(t)d\omega(t) + dJ(t),$$

where $J(t) - J(s) = \sum_{s<\tau\leq t} \kappa(\tau).$
Generate Gaussian (normal) random variables:

\[ Z_j \sim N(0, 1), \quad j = 1, 2, \ldots, J \]

\( M \) steps per day:

\[ M = 80 \Rightarrow \Delta = 1/80 \]

Standard deviation

\[ \sigma = 0.011 \Rightarrow 1.10 \text{ percent per day} \]

\( T \) days:

\[ T = 1.25 \times 252 \Rightarrow J = T \times M = 315 \times 80 \text{ (steps)} \]
Euler Scheme (continued)

Wiener increments
\[ dW_j = \sqrt{\Delta} Z_j \quad j = 1, 2, \ldots, J \]

Initialize \( p_0 = \log(75) \) (\$75 per share)

Iterate:
\[ p_j = p_{j-1} + \mu \Delta + \sigma dW_j, \quad j = 1, 2, \ldots, J \]

Convert to levels:
\[ \{ P_j \}_{j=1}^{J}, \quad P_j = e^{pj} \quad j = 1, 2, \ldots, J (= M \times T), \]

and make a nice plot.
Euler Scheme: Time Varying Variance

Wiener increments

\[ dW_j = \sqrt{\Delta Z_j} \quad j = 1, 2, \ldots, J \]

Initialize \( p_0 = \log(75) \) (\$75 per share)

Iterate:

\[ p_j = p_{j-1} + \mu \Delta + \sigma_{j-1} dW_j, \quad j = 1, 2, \ldots, J \]

Convert to levels:

\[ \{ P_j \}_{j=1}^J, \quad P_j = e^{p_j} \quad j = 1, 2, \ldots, J (= M \times T), \]

and make a nice plot.
Sample the number of jumps $N \sim \text{Poisson}$, then draw

$N$ jumps $\kappa_k \sim N(0, \sigma_{jmp})$, $k = 1, \ldots, N$, and scatter the jumps randomly.
Jump Diffusion

With jumps

\[ dp(t) = \mu(t)dt + \sigma(t)dw(t) + dJ(t), \]
2. Observed Data: XOM September-October 2008
Within-day geometric returns

\[ r_{t,j} = p(t - 1 + j\Delta) - p(t - 1 + (j - 1)\Delta), \]

for \( j = 1, 2, \ldots, M \), integer \( t = 1, 2, \ldots, T \).
XOM Returns, September-October, 2008
Default Plots from Graphics are not Acceptable

Poorly labeled axis, scales, and colors:
Default gives very poor graphic:
Care with Data: e.g., Stock Splits

Consider Apple (APPL):

AAPL Price

Clearly, something is wrong, or needs attention:
Culprit

Data recording error or maybe a stock split?

AAPL (Yahoo Finance, Charts, click max)

Need to adjust the data for any stock splits. The convention is to adjust backwards. Watch it, sometimes there can be a reverse stock split such as 1:3.
3. Variance Measures

The Realized Variance

$$RV_t = \sum_{j=1}^{M} r_{t,j}^2$$

As $\Delta \to 0$, $M \to \infty$, then from advanced probability theory:

$$RV_t \to \int_{s=t-1}^{t} \sigma^2(s) \, ds + \sum_{t-1<s\leq t} \kappa_s^2$$

$RV_t$ converges to the integrated variance plus sum of all within-day jumps squared.
Bipower Variation

The Bipower variation of Barndorff-Nielsen and Shephard (2003)

\[ BV_t = \mu_1^{-1} \frac{M}{M-1} \sum_{j=2}^{M} |r_{t,j}| r_{t,j-1} \]

As \( \Delta \to 0 \), \( M \to \infty \), and

\[ BV_t \to \int_{s=t-1}^{t} \sigma^2(s) \, ds \]

\( BV_t \) converges to the integrated variance. It is jump robust. In the absence of jumps

\[ RV_t - BV_t \to 0. \]
Continuous and Jump Variation

Model:

\[ dp(t) = \mu(t)dt + \sigma(t)dw(t) + dJ(t), \]

Split the quadratic variation into two pieces:

\[
RV_t \rightarrow QV_t = \int_{s=t-1}^{t} \sigma^2(s) \, ds + \sum_{t-1 < s \leq t} \kappa_s^2
\]

\[ QV_t = IV_t + TJV_t \]

\[
IV_t = \int_{s=t-1}^{t} \sigma^2(s) \, ds, \quad TJV_t = \sum_{t-1 < s \leq t} \kappa_s^2
\]
Decompose the Realized Variance

\[ RV_t = CV_t + JV_t \]

\[ CV_t = RV_t I(z_t \leq c) + BV_t I(z_t > c) \]

\[ JV_t = (RV_t - BV_t) I(z_t > c) \]
Realized Variance (Annualized)

XOM 2007-2010
Bipower Variation (Annualized)

XOM 2007-2010
Relative Contribution of Jumps

XOM 2007-2010
A poorly scaled graphic:

![XOM: Relative Contribution of Jumps](image_url)
Relative Contribution of Jumps

XOM 2007-2010
A reasonable well-scaled graphic:
Truncated Variance (TV) as Estimator of $IV_t$

Threshold Estimator (Mancini, 2009)

Truncated Variance:

$$TV_t = \sum_{j=1}^{M} |r_{t,j}|^2 \times I[|r_{t,j}| \leq \text{cutoff}_t]$$

where $I[\ ]$ is the 0-1 indicator function (1 = true, 0 = false).
Value of cutoff for Threshold Variation

What is the cutoff? Something like 4 (or 2, 3) standard deviations would be about right, but what standard deviation? Papers by Ait-Sahalia, Todorov, and others provide some guidance. A reasonable choice would be

\[
\text{cutoff}_t = 4 \times \sqrt{\frac{1}{M} BV_{t-1}}
\]

where \( BV_{t-1} \) is the bipower variation of the preceding day.
4. Options and Implied Volatility

XOM Call Options Expire at close Friday, April 20, 2012

Close of Trading, Friday, January 13, 2012 \( S_0 = 84.88 \)

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<tr>
<th>Strike</th>
<th>Call Price at Close</th>
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<tr>
<td>65.00</td>
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<td>70.00</td>
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<td>80.00</td>
<td>6.45</td>
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<td>85.00</td>
<td>3.20</td>
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</table>
\[ S_0 = \$84.88 \]

\[ T = \frac{98}{365} = 0.2685 \]

\[ r = 1 \text{ percent} \]

Historical Volatility (Ameritrade) \( \sigma_H = 25 \text{ percent} \)
<table>
<thead>
<tr>
<th>Option</th>
<th>Market Price</th>
<th>( \sigma )</th>
<th>Black-Scholes Value</th>
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<td>Call-80</td>
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<td>Historical 25</td>
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<td>5.34</td>
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<td>8.05</td>
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<td><strong>Put</strong></td>
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<td></td>
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<td>Historical 25</td>
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<tr>
<td>Option</td>
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<td>$\sigma$</td>
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<tr>
<td>Call-80</td>
<td>$6.45$</td>
<td>Implied 19.38</td>
<td>6.45</td>
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<td>Put-80</td>
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<td>Implied 24.67</td>
<td>2.13</td>
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<td>$\sigma$</td>
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</tr>
<tr>
<td><strong>Call</strong></td>
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<td>Call-90</td>
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<td>Call-90</td>
<td><strong>Implied 16.28</strong></td>
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5. Market Microstructure Noise

A better term would be trading frictions, or small deviations from the appropriate price.

From the present value model, i.e. the Gordon Growth model, the correct stock price is:

\[
\tilde{p}_t = \frac{\Pi_t^e}{r_{a,t} - g_t}
\]
Market Price cannot Possibly Track the Correct Price Perfectly

New information continuously flows to the market about the three fundamentals, $\Pi_t^c$, $r_{a,t}$, $g_t$, and no market can keep the actual price equal to $\tilde{p}_t$. Thus, we write

$$p_t = \tilde{p}_t + \epsilon_t$$
Implications

Note the observed geometric return over the interval $t - \Delta \to t$ is:

\[ r_{\Delta,t} = \tilde{p}_t - \tilde{p}_{t-\Delta} + \epsilon_t - \epsilon_{t-\Delta} \]

\[ r_{\Delta,t} = \tilde{r}_{\Delta,t} + \epsilon_t - \epsilon_{t-\Delta} \]

\[ \text{Var}(r_{\Delta,t}) = \Delta \sigma^2_{\tilde{r}} + 2\sigma^2_\epsilon \]

As $\Delta \to 0$ the noise dominates:

\[ \lim_{\Delta \downarrow 0} \text{Var}(r_{\Delta,t}) = 2\sigma^2_\epsilon \]
Coarse Sampling

5-Minute returns, or k-Minute returns

\[ r_{t,j,k} = 100 \left[ p(t - 1 + jk/m) - p(t - 1 + (j - 1)k/m) \right], \quad j = 1, 2, \ldots, \text{end of day} \]

Volatility signature plots are helpful.
ALL 1997-2002 (Allstate Corporation)
XOM 2007-2010:

Annualized Volatility Signature

Minutes

15 20 25 30 35
Sub-Sampling

\[ RV^{(0)} = r_{9:35,9:40}^2 + r_{9:40,9:45}^2 + \cdots + r_{3:54,3:59}^2 \]
\[ RV^{(1)} = r_{9:36,9:41}^2 + r_{9:41,9:46}^2 + \cdots + r_{3:53,3:58}^2 \]
\[ \vdots \]
\[ RV^{(4)} = r_{9:39,9:44}^2 + r_{9:44,9:49}^2 + \cdots + r_{3:50,3:55}^2 \]

\[ RV_{SS} = \frac{1}{5} \times \left( RV^{(0)} + RV^{(1)} + RV^{(2)} + RV^{(3)} + RV^{(4)} \right) \]
Pre-Averaging

See papers by Mark Podolskij and many co-authors. (Tentative) Form local averages of the prices to wash out the noise:

\[
\bar{r}_1 = \frac{1}{5} \times (p_{9:40} + p_{9:41} + p_{9:42} + p_{9:43} + p_{9:44}) - \\
= \frac{1}{5} \times (p_{9:35} + p_{9:36} + p_{9:37} + p_{9:38} + p_{9:39}) \\
\bar{r}_2 = \frac{1}{5} \times (p_{9:45} + p_{9:46} + p_{9:47} + p_{9:48} + p_{9:49}) - \\
= \frac{1}{5} \times (p_{9:40} + p_{9:41} + p_{9:42} + p_{9:43} + p_{9:44})
\]

Equivalently, form local average of the returns

\[
\bar{r}_1 = \frac{1}{5} \times (r_{9:40} + r_{9:41} + r_{9:42} + r_{9:43} + r_{9:44}) \\
\bar{r}_2 = \frac{1}{5} \times (r_{9:45} + r_{9:46} + r_{9:47} + r_{9:48} + r_{9:49}) \\
\vdots
\]
Bipower after Pre-Averaging

\[ \overline{BV} = \text{scale factor} \times \sum_{j=2}^{M} |\overline{r}_j||\overline{r}_{j-1}| \]

There needs to be a second adjustment to make

\[ \text{adj}(\overline{BV}_t) \rightarrow \int_{t-1}^{t} \sigma_s^2 ds \]