A Unified Analysis of Rational Voting with Private Values and Group-Specific Costs*

Curtis R. Taylor
Department of Economics
Duke University
Box 90097
Durham, NC 27708
E-mail: crtaylor@econ.duke.edu

Huseyin Yildirim†
Department of Economics
Duke University
Box 90097
Durham, NC 27708
E-mail: yildirh@econ.duke.edu

February 5, 2010

Abstract

We provide a unified analysis of the canonical rational voting model with privately known political preferences and costs of voting. Focusing on type-symmetric equilibrium, we show that for small electorates, members of the minority group vote with a strictly higher probability than do those in the majority, but the majority is strictly more likely to win the election. As the electorate size grows without bound, equilibrium outcome is completely determined by the individuals possessing the lowest cost of voting in each political group. We relate our equilibrium characterization to Myerson's Poisson games, and examine the potential uniqueness of equilibrium.

JEL Classifications: C72, D72, D82.

Keywords: costly voting, free-riding incentive, coordination problem, underdog effect.

1 Introduction

Rational voting theory, originally proposed by Downs (1957) in decision-theoretic terms, and later formulated by Ledyard (1981, 1984), and Palfrey and Rosenthal (1983, 1985) in game-theoretic terms, lays out the most basic incentives to vote and assumes that each agent trades off the net benefit of winning discounted by the probability of casting the pivotal

*We thank the advisory editor, two referees, John Aldrich, Han Hong, Ozgur Kibris, Mike Munger, Thomas Palfrey, and especially Martin Osborne for comments and discussions. This paper is a substantially revised version of the one circulated under " A Characterization of Voter Turnout in Large Elections". All remaining errors are ours.

†Corresponding author.

vote against the cost of voting.[1] Despite severely underestimating turnout rate in large electorates [Palfrey and Rosenthal (1985)], rational voting theory is still widely believed to yield empirically reasonable comparative statics even in its purest form.[2] Perhaps this is why there is a renewed interest in the theory's applications. Most notably, Campbell (1999) applies the theory to rationalize election upsets by demonstrating that the minority group is likely to win the election if the electorate size is sufficiently large and if the minority is composed of agents with relatively low cost-benefit ratios. In a small electorate with *ex ante* symmetric agents, Börgers (2004) shows that voluntary participation may lead to too much turnout from the social viewpoint. Krasa and Polborn (2009) extend Börgers' analysis to asymmetric groups and large electorates, and point to the potential benefits of mandatory voting policies.[3] In two related papers, Goeree and Grosser (2007) and Taylor and Yildirim (2010) examine the impact of releasing information about the distribution of political preferences through pre-election polls, political stock markets, etc. on equilibrium electoral outcomes and welfare.

While providing valuable insights, these papers have also recorded some important – and at times startling – theoretical results. For instance, Goeree and Grosser (2007) for small electorates, and Krasa and Polborn (2009), and Taylor and Yildirim (2010) for large electorates have noted that even in the presence of a clear majority, each alternative is *equally likely* to win the election in a type-symmetric equilibrium, which is also the basis for Campbell's finding. In addition, there seems to be a common understanding that an agent's vote becomes less pivotal as electorate size grows and/or others vote with a greater probability. Finally, despite the inherent coordination problem among the supporters of each alternative, Börgers (2004) establishes the uniqueness of type-symmetric equilibrium.

It is, however, often difficult to discern what factors drive these results and how robust they are, given that each paper employs a costly voting model with a varying degree of generality. The present paper aims to fill in this gap by providing a unified analysis while taking the celebrated "paradox of not voting" as given. The model we analyze is a generalization of Börgers (2004) and a slight variation of Palfrey and Rosenthal (1985). There are n agents divided randomly into two groups: supporters of alternative A and supporters

[1] See Aldrich (1997), Blais (2000), Feddersen (2004), and Merlo (2006) for excellent overviews of the literature.

[2] See, Hansen, Palfrey and Rosenthal (1987) for empirical evidence, and Levine and Palfrey (2007) for experimental evidence in favor of this model.

[3] On the same topic, also see Ghosal and Lockwood (forthcoming).

of alternative B. These political preferences are distributed independently across agents. Moreover, within each group the costs of voting are also independently distributed. Each agent privately knows both his realized preference and voting cost.[4] Most significantly, this generalization allows for group-specific cost distributions with potentially different supports. An agent receives a net benefit normalized to 1 if his preferred alternative wins, and 0 otherwise. Agents decide whether to vote or abstain simultaneously, and ties are broken by a fair coin toss. As is common in the literature, we focus on type-symmetric equilibrium.

Our main result pertaining to small electorates formalizes the "underdog effect": given the same cost distribution, the members of the minority group vote with a strictly higher probability than do those in the majority.[5] Nonetheless, the majority never completely loses its initial advantage. This contrasts with the political neutrality findings of Goeree and Grosser (2007), and Taylor and Yildirim (2010) when voting costs are assumed fixed and equal for all agents.[6] As electorate size grows without bound, consistent with Campbell (1999), and Krasa and Polborn (2009), we show that only the agents with the lowest possible costs vote, independent of the distributions of preferences and costs. Moreover, unlike Krasa and Polborn, by allowing for different cost supports across the two political groups, we discover that each alternative is equally likely to win the election if and only if the lower bounds of the supports are equal. Otherwise, the group with a cost advantage (in the sense of the lowest possible cost) is strictly more likely to win, as intuition suggests.

Our equilibrium characterization of large elections also bridges a gap between the costly voting model with a fixed population size and Myerson's Poisson games with a random population [Myerson (1998, 2000)]. We demonstrate that a large election can be considered a Poisson game – in Myerson's sense – where the population mean is the sum of equilibrium limit turnouts for each group and an appropriately defined probability of voting for each alternative in terms of these limits, which is, in general, different from the initial distribution of preferences.[7]

[4] As in the Ledyard-Palfrey-Rosenthal model, agents in our private-values setup are also differentiated by their intrinsic preferences over political alternatives. Hence, we do not study the information aggregation problem that is the focus of common-value models such as Feddersen and Pesendorfer (1997), Krishna and Morgan (2008), and Razin (2003).

[5] To be sure, the underdog effect has been articulated in several empirical and experimental studies, the most recent being Levine and Palfrey (2007); but, to our knowledge, it has not been formally shown in a framework as general as ours.

[6] Such an underdog effect is not present in Börgers (2004) due to *ex ante* symmetry.

[7] Myerson (1998, pp. 386-92) makes a similar point but within a numerical example with a fixed cost of voting.

Finally, we establish a sufficient condition for equilibrium uniqueness, which is satisfied if agents are sufficiently symmetric. In doing so, we demonstrate that Börgers' (2004) uniqueness result with symmetric agents is robust to small perturbations.

The rest of the paper is organized as follows. In the next section, we set up the formal model, followed by the equilibrium characterizations for small and large electorates in Sections 3 and 4, respectively. In Section 5, we examine the question of equilibrium uniqueness, and we gather some concluding remarks in Section 6. The proofs not appearing in the text have been relegated to the Appendix.

2 The Model

There are $n \geq 2$ agents who may cast a vote in an election between two alternatives, $r = A, B$. Each agent i privately knows his 2-dimensional type, $t_i = (r_i, c_i)$, consisting of his political preference $r_i \in \{A, B\}$ and his cost of voting c_i. Political preferences are independently drawn from a Bernoulli distribution with $\lambda_r \in (0, 1)$ representing the probability of alternative r, and conditional on these preferences, the agents who favor alternative r pick their voting costs independently from the differentiable distribution $G_r(c)$ where $G_r'(c) = g_r(c) > 0$ for all $c \in [\underline{c}_r, \overline{c}_r] \subset \mathbb{R}_+$. Note that we allow voting costs across the two political groups to differ not only in their densities but also in their supports.[8] Upon privately observing their types, agents simultaneously choose whether to vote for their preferred alternative or to abstain. The election is decided by a simple majority rule and ties are broken by a fair coin toss. Agent i receives a gross payoff normalized to 1 if r_i wins; and 0 otherwise.

Action/Outcome	r_i wins	r_i loses
Abstain	1	0
Vote	$1 - c_i$	$-c_i$

Table 1: Ex Post Payoffs of Agent i

As is clear from Table 1, abstaining strictly dominates voting for one's less preferred alternative, resulting in "sincere" voting in this setup.[9] In order to rule out trivial equilibria in which it is a dominant strategy for all agents in some political group to abstain or for

[8]If $G_A = G_B$, then supports must, of course, be the same, but the converse is not true.

[9]Unlike a private values election, sincere voting, in general, does not obtain in equilibrium with common-values. However, Krishna and Morgan (2008) have shown that if voting is costly, then there always exists an equilibrium with sincere voting.

all to vote with certainty, we assume $0 < \underline{c}_r < \frac{1}{2} < \overline{c}_r$. All aspects of the environment are common knowledge. In the analysis below, we frequently refer to political group r as the majority and r' as the minority if $\lambda_r > \lambda_{r'}$, because, with $\lambda_r > \lambda_{r'}$, the expected size of group r is strictly greater than that of group r'.[10]

3 Equilibrium in Small Electorates

As is standard in the costly-voting literature, we concentrate on type-symmetric Bayesian Nash Equilibrium (BNE) in which all agents preferring alternative r follow the same equilibrium strategy. It is straightforward to verify that in a type-symmetric BNE, agents adopt a cutoff strategy in which a player favoring r votes if and only if his cost is less than some critical level, c_r^*. In order to characterize such a BNE, denote the *ex ante* probability that a type r agent[11] votes by $\phi_r \equiv G_r(c_r^*)$, and the *ex ante* probability that an agent votes for alternative r by $\alpha_r \equiv \lambda_r \phi_r$. Hence, the *ex ante* probability that an agent abstains is $(1 - \alpha_r - \alpha_{r'})$. Now, recall that the number of ways k other agents can vote for r, k' can vote for r', and $n - 1 - k - k'$ can abstain is given by the trinomial coefficient

$$\binom{n-1}{k, k', n-1-k-k'} \equiv \frac{(n-1)!}{k!k'!(n-1-k-k')!}.$$

Given this, the net expected utility from voting to an agent with voting cost, c, and political preference r may be written (see the proof of Lemma 1 in the Appendix)

$$\Delta_r \equiv \frac{1}{2} P(\alpha_r, \alpha_{r'}, n) - c, \tag{1}$$

where

$$P(\alpha_r, \alpha_{r'}, n) \equiv \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{k, k, n-1-2k} \alpha_r^k \alpha_{r'}^k (1 - \alpha_r - \alpha_{r'})^{n-1-2k} + \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1}{k, k+1, n-2-2k} \alpha_r^k \alpha_{r'}^{k+1} (1 - \alpha_r - \alpha_{r'})^{n-2-2k}, \tag{2}$$

for $r = A, B$, $r \neq r'$, and $\lfloor \cdot \rfloor$ is the usual operator that rounds a number to the lower integer when necessary.

[10] This terminology is commonly used in the literature, e.g., Campbell (1999), Goree and Grosser (2007), and Krasa and Polborn (2009).

[11] To avoid repetition, we sometimes abuse terminology and say "type" r to refer to one's political type only, keeping in mind type also includes his cost.

To understand expression (1), observe that $P(\alpha_r, \alpha_{r'}, n)$ is the probability that a type r agent casts a decisive vote; i.e., that his vote is pivotal in determining the outcome when each of the other $n-1$ agents: votes for r with probability α_r, votes for r' with probability $\alpha_{r'}$, and abstains with probability $1 - \alpha_r - \alpha_{r'}$. In particular, his vote may be pivotal for one of two reasons corresponding to the two summations in (2). First, if k of the other agents vote for r, k vote for r', and $n-1-2k$ abstain, then the agent in question will *break* a tie by voting. The first summation in (2) is, therefore, the probability that the agent breaks a tie that would otherwise occur. Second, if k agents vote for alternative r, $k+1$ vote for r', and $n-2-2k$ abstain, then the agent in question will *create* a tie by voting. The second summation in (2) is, therefore, the probability that the agent in question creates a tie when alternative r' would otherwise have won. When his vote breaks a tie, the probability that alternative r is implemented rises from $1/2$ to 1, and when his vote creates a tie, the probability that r is implemented rises from 0 to $1/2$. This accounts for the factor $1/2$ in (1). Of course, when an agent votes, his net expected benefit must also account for his voting cost, c.

An important step in understanding equilibrium voting behavior now, and the possibility of equilibrium uniqueness later, is to derive some basic properties of the pivot probability. To our knowledge, these properties have not been recorded elsewhere, except for the special case of $\alpha_r = \alpha_{r'}$.

Lemma 1. *For* $(\alpha_r, \alpha_{r'}) \in (0, \lambda_r) \times (0, \lambda_{r'})$ *where* $r, r' = A, B$ *and* $r \neq r'$,

(i) $P(\alpha_r, \alpha_{r'}, n) - P(\alpha_{r'}, \alpha_r, n) =^{sign} \alpha_{r'} - \alpha_r$,

(ii) $\frac{\partial}{\partial \alpha_{r'}} P(\alpha_r, \alpha_{r'}, n) =^{sign} \begin{cases} 0 & if \quad n = 2 \\ \alpha_r - \alpha_{r'} & if \quad n > 2 \end{cases}$,

(iii) $\frac{\partial}{\partial \alpha_r} P(\alpha_r, \alpha_{r'}, n) < 0$, *if* $\alpha_r \geq \left(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor}\right) \alpha_{r'}$,

(iv) $P(\alpha_r, \alpha_{r'}, n) > P(\alpha_r, \alpha_{r'}, n+2)$, *but* $P(\alpha_r, \alpha_r, n) > P(\alpha_r, \alpha_r, n+1)$.

To aid with discussion, it is worth repeating that $P(\alpha_r, \alpha_{r'}, n)$ is the probability that the vote of an isolated type r agent is pivotal given the voting probabilities of the other $n-1$ agents. Now, to understand part (i) of the lemma, suppose $\alpha_r > \alpha_{r'}$. In this case, it is likely that alternative r has more votes than r', and therefore a vote for r (which widens the expected lead) is less apt to be pivotal than a vote for r' (which narrows it). Part (ii) says that an increase in the probability of voting for alternative r' makes a vote for r more likely pivotal because it closes the gap in voting probabilities between the two alternatives. Hence, the pivot probability of a vote for r is *nonmonotonic* in the probability of voting for r', peaking at the point where $\alpha_{r'} = \alpha_r$.[12]

Part (iii) reveals that the vote of an isolated type r agent is less apt to be pivotal when the probability that all other type r agents vote increases, provided they vote with higher probability than type r' agents (i.e., when the gap in voting probabilities increases). The converse is, however, not necessarily true. In other words, if $\alpha_r < \alpha_{r'}$, it is not necessarily true that the vote of an isolated type r agent is more apt to be pivotal when α_r increases (i.e., the gap in voting probabilities decreases). Part (iii) implies that an agent views his vote as a *substitute* to the voting probability of others' who share his political preference, so long as this probability is not too far behind the probability for the competing alternative, and as a *complement* otherwise.[13]

Finally, part (iv) reveals that a vote for r becomes less apt to be pivotal when the electorate size increases by two. Intuition suggests that as the electorate grows, the pivot probability should decrease for all n. This turns out not to be true in general. For some fixed pair $(\alpha_r, \alpha_{r'})$, a type r agent's vote actually may be more likely to be pivotal as n increases by one.[14] This nonmonotonicity is a consequence of the different ways ties can occur when n is odd or even, and seems to be especially relevant in small electorates. Nonetheless, the monotonicity of the pivot probability is restored, if one takes increments by two rather than one, or compute it on a particular path such as $\alpha_r = \alpha_{r'}$, as in Börgers (2004), Goeree and Grosser (2007), and Taylor and Yildirim (2010). The latter plays a crucial role in establishing equilibrium uniqueness for these papers – an issue we address in Section 5.

In a type-symmetric equilibrium, the net expected payoff of a type r agent with the cutoff cost, c_r^*, must satisfy

$$\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^* \leq 0 \text{ and } \left[\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^*\right](c_r^* - \underline{c}_r) = 0 \; . \tag{3}$$

To understand why, note that if $\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^* > 0$, then a type r agent would not be indifferent but would prefer to vote with certainty, violating the definition of c_r^* as a cost

[12]This makes sense, because if $\alpha_{r'} = \alpha_r$, then each alternative is equally likely to win, making a vote for r decisive with the highest probability.

[13]Note that since $\frac{\partial}{\partial \alpha_r}P(\alpha_r, \alpha_r, n) < 0$ and $\frac{\partial}{\partial \alpha_{r'}}P(\alpha_r, \alpha_r, n) = 0$, it follows that $\frac{d}{d\alpha_r}P(\alpha_r, \alpha_r, n) < 0$, as found in Börgers (2004), Goeree and Grosser (2007), and Taylor and Yildirim (2010).

[14]As an example, let n be even and $\alpha_r + \alpha_{r'} = 1$. Then, $P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) = \binom{n-1}{\frac{n}{2}}\alpha_r^{\frac{n}{2}-1}\alpha_{r'}^{\frac{n}{2}}(1 - 2\alpha_r)$. Hence, $P(\alpha_r, \alpha_{r'}, n) < P(\alpha_r, \alpha_{r'}, n+1)$ whenever $\alpha_r > \frac{1}{2}$.

cutoff. Conversely, if $\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^* < 0$, then the agent would prefer to abstain with certainty or would have $c_r^* = \underline{c}_r$. Finally, if $c_r^* > \underline{c}_r$, then the agent would, by the definition of c_r^*, vote for some cost realizations, but because $\frac{1}{2} < \overline{c}_r$, not for all. Thus, in equilibrium, he must be indifferent at the cutoff cost.

Given $\phi_r \equiv G_r(c_r^*)$, $\alpha_r \equiv \lambda_r \phi_r$, and defining

$$\Phi_r(\alpha_r, \alpha_{r'}) \equiv G_r\left(\frac{1}{2}P(\alpha_r, \alpha_{r'}, n)\right) - \frac{\alpha_r}{\lambda_r},$$

we can rewrite (3):

$$\Phi_r(\alpha_r^*, \alpha_{r'}^*) \leq 0 \text{ and } \alpha_r^* \Phi_r(\alpha_r^*, \alpha_{r'}^*) = 0. \tag{4}$$

Finding an equilibrium, therefore, amounts to finding a pair $(\alpha_A^*, \alpha_B^*) \in [0, \lambda_A] \times [0, \lambda_B]$ that satisfies (4).

Proposition 1. *There exists a type-symmetric equilibrium, and every type-symmetric equilibrium has the following properties:*

(i) $\phi_r^* < 1$ *for all* r*; and* $\phi_r^* > 0$ *for some* r*.*

(ii) *If* $\phi_r^* = 0$*, then* $\underline{c}_r > \underline{c}_{r'}$*.*

(iii) *If* $G_A = G_B$ *and* $\lambda_A > \lambda_B$*, then* $0 < \phi_A^* < \phi_B^*$*;* $\alpha_A^* > \alpha_B^* > 0$*; and* $\frac{1}{2} < \Pr\{A$ *wins*$\} < 1$*.*

(iv) *If* $\lambda_A = \lambda_B$*, and* G_A *first-order stochastically dominates* G_B*, then* $\phi_A^* \leq \phi_B^*$*;* $\alpha_A^* \leq \alpha_B^*$*; and* $0 < \Pr\{A$ *wins*$\} \leq \frac{1}{2}$*.*

Proof. Let $\Psi(\alpha_A, \alpha_B) \equiv (\lambda_A G_A(\frac{1}{2}P(\alpha_A, \alpha_B, n), \lambda_B G_B(\frac{1}{2}P(\alpha_B, \alpha_A, n))$. From (4), it is clear that an equilibrium pair (α_A^*, α_B^*) is a fixed point of Ψ. Since Ψ maps the compact and convex set $[0, \lambda_A] \times [0, \lambda_B]$ into itself, and it is continuous in this region, by Brouwer's fixed theorem, there exists a type-symmetric equilibrium. Next, we prove each part in turn.

(i) Suppose, to the contrary, $\phi_r^* = 1$, or equivalently $\alpha_r^* = \lambda_r (\neq 0)$ for some r. Then, since $\frac{1}{2} < \overline{c}_r$, we have $\Phi_r(\lambda_r, \alpha_{r'}^*) < 0$, which, from (4) implies $\alpha_r^* = 0$, yielding a contradiction. Hence, $\phi_r^* < 1$ for all r. Next, suppose $\phi_r^* = 0$, or $\alpha_r^* = 0$ for all r. Then, $\Phi_r(0, 0) = G_r\left(\frac{1}{2}\right) > 0$, contradicting (4). Thus, $\phi_r^* > 0$ for some r.

(ii) Let $\phi_r^* = 0$ for some r. Then, $\phi_{r'}^* > 0$ by part (i). By (3), this means $\frac{1}{2}P(0, \alpha_{r'}^*, n) - \underline{c}_r \leq 0$ and $\frac{1}{2}P(\alpha_{r'}^*, 0, n) - c_{r'}^* = 0$ where $c_{r'}^* > \underline{c}_{r'}$. From (2), note that $P(0, \alpha_{r'}^*, n) = (1-\alpha_{r'}^*)^{n-1} + (n-1)\alpha_{r'}^*(1-\alpha_{r'}^*)^{n-2}$ and $P(\alpha_{r'}^*, 0, n) = (1-\alpha_{r'}^*)^{n-1}$, which together require $c_{r'}^* + \frac{n-1}{2}\alpha_{r'}^*(1-\alpha_{r'}^*)^{n-2} - \underline{c}_r \leq 0$, and imply $c_{r'}^* < \underline{c}_r$ because $\alpha_{r'}^* \in (0,1)$.

(iii) Suppose $G_A = G_B = G$ and $\lambda_A > \lambda_B$, but, to the contrary, $\alpha_A^* \leq \alpha_B^*$. We make two observations. First, $G_A = G_B$ implies $\underline{c}_A = \underline{c}_B$, and thus $\alpha_A^*, \alpha_B^* > 0$ by part (ii). Second, $\frac{\alpha_A^*}{\lambda_A} < \frac{\alpha_B^*}{\lambda_B}$. Together with (4), the latter requires $G(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n)) < G(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n))$, which, given $G' > 0$, implies $P(\alpha_A^*, \alpha_B^*, n) < P(\alpha_B^*, \alpha_A^*, n)$, and $\alpha_A^* > \alpha_B^*$, by part (i) of Lemma 1, yielding a contradiction. Hence, $\alpha_A^* > \alpha_B^*$.

Given $\alpha_A^* > \alpha_B^* > 0$, we have $P(\alpha_A^*, \alpha_B^*, n) < P(\alpha_B^*, \alpha_A^*, n)$ by Lemma 1, and $\Phi_A(\alpha_A^*, \alpha_B^*) = \Phi_B(\alpha_B^*, \alpha_A^*) = 0$ by (4). Since $G(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n)) < G(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n))$ and $\phi_r^* \equiv \frac{\alpha_r^*}{\lambda_r}$, (4) further reveals $\phi_A^* < \phi_B^*$. To complete the proof of part (iii), note that

$$\begin{aligned}\Pr\{r \text{ wins}\} &= \frac{1}{2}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k, k, n-2k} (\alpha_r^*)^k (\alpha_{r'}^*)^k (1-\alpha_r^* - \alpha_{r'}^*)^{n-2k} \\ &\quad + \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k'=0}^{k-1} \binom{n}{k, k', n-k-k'} (\alpha_r^*)^k (\alpha_{r'}^*)^{k'} (1-\alpha_r^* - \alpha_{r'}^*)^{n-k-k'}.\end{aligned} \tag{5}$$

Given $\alpha_A^* > \alpha_B^*$, it is clear that $\Pr\{A \text{ wins}\} > \Pr\{B \text{ wins}\}$, and hence $\Pr\{A \text{ wins}\} > \frac{1}{2}$. Moreover, given $\alpha_A^* < 1$, $\Pr\{A \text{ wins}\} < 1$.

(iv) Suppose $\lambda_A = \lambda_B$, and G_A first-order stochastically dominates G_B, but, to the contrary, $\alpha_A^* > \alpha_B^*$. This means $\alpha_A^* > 0$. By (4), we thus have

$$G_B(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n)) - \frac{\alpha_B^*}{\lambda_B} \leq 0 = G_A(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n)) - \frac{\alpha_A^*}{\lambda_A}. \tag{6}$$

Given $\lambda_A = \lambda_B$, (6) reveals that $G_B(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n)) < G_A(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n))$, which, because G_A first-order stochastically dominates G_B, requires that $P(\alpha_B^*, \alpha_A^*, n) \leq P(\alpha_A^*, \alpha_B^*, n)$. Then, by Lemma 1, we have $\alpha_A^* \leq \alpha_B^*$, yielding a contradiction. Hence, $\alpha_A^* \leq \alpha_B^*$. Since $\lambda_A = \lambda_B$, this implies $\phi_A^* \leq \phi_B^*$. Finally, note from (5) that $0 < \Pr\{A \text{ wins}\} \leq \frac{1}{2}$. ■

Proposition 1 highlights some basic properties of a type-symmetric equilibrium.[15] Part (i) indicates that in equilibrium, no individual votes with certainty. This is because the

[15] The existence of a type-symmetric equilibrium is well-established in the literature, e.g., Ledyard (1984) and Palfrey and Rosenthal (1985). Nevertheless, this is – to the best of our knowledge – the first formal derivation of the equilibrium properties for a small electorate.

maximum benefit from voting is $\frac{1}{2}$ and $\frac{1}{2} < \overline{c}_r$. Part (i) also indicates that at least some individuals are expected to vote. However, even though we have ruled out abstentions due to high costs, i.e, $\underline{c}_r < \frac{1}{2}$, it is possible that members of *some* political group abstain altogether for strategic reasons. Part (ii) reveals that if such full abstention occurs, the main reason must be the individuals with low costs of voting in the rival group and not necessarily the distribution of political preferences. Another important implication of part (ii) is that if $\underline{c}_r = \underline{c}_{r'}$, then the expected probability of voting is strictly positive for all individuals irrespective of the cost and political preference distributions. Hence, the knife-edge case of equal cost lower bounds, often assumed in the literature, seems to rule out the interesting case of complete abstention by one group. In our analysis of large elections, this knife-edge case will also be the source of a strong "neutrality" result.

Part (iii) formalizes the "underdog effect" alluded to in the introduction: given identical cost distributions, an agent in the minority group is strictly more likely to vote. This is due to the well-known tension between one's incentives for winning the election and free-riding on his fellow group members. Not surprisingly, the latter incentive is less pronounced in a smaller group. Nonetheless, part (iii) shows that the underdog effect never outweighs the initial majority advantage, and hence the majority is strictly more likely to win in a small electorate. Part (iv) examines the counterpart of (iii). When each agent is equally likely to support either alternative, the group whose members are more likely to have higher voting costs is less likely to win the election.

Proposition 1 puts a perspective on recent studies of the costly-voting model with a small electorate. As mentioned in the Introduction, Börgers (2004) examines the symmetric setup in which $G_A = G_B$ and $\lambda_A = \lambda_B$ so that the underdog effect does not emerge. Goeree and Grosser (2007), and Taylor and Yildirim (2010) allow for $\lambda_A \neq \lambda_B$, and show that each group is equally likely to win the election. Part (iii) of Proposition 1 indicates that their assumption of a fixed and equal voting cost for all agents plays a crucial role in this "neutrality" result, because when there is cost uncertainty, the majority is strictly more likely to win even if the cost distributions are identical.

The underdog effect identified in Proposition 1 raises an important question: Does an increase in population size *necessarily* improve the majority's chances of winning? To answer this question, suppose $G_A = G_B$ and $\lambda_A > \lambda_B$. Let $\Pr\{A \text{ wins}|n\} \equiv \pi(\alpha^*_A(n), \alpha^*_B(n), n)$ for a pair of equilibrium strategies $(\alpha^*_A(n), \alpha^*_B(n))$. Then, by adding and subtracting the term

$\pi(\alpha_A^*(n), \alpha_B^*(n), n+1)$, the change in the majority's probability of winning can be written,

$$\Pr\{A\ wins|n+1\} - \Pr\{A\ wins|n\} = \underbrace{\pi(\alpha_A^*(n), \alpha_B^*(n), n+1) - \pi(\alpha_A^*(n), \alpha_B^*(n), n)}_{D(n)} + \underbrace{\pi(\alpha_A^*(n+1), \alpha_B^*(n+1), n+1) - \pi(\alpha_A^*(n), \alpha_B^*(n), n+1)}_{S(n)}.$$

Hence, an increase in population size has two effects on group A's probability of winning. $D(n)$ represents the direct (scale) effect because strategies are kept equal, and $S(n)$ represents the strategic effect because population size is kept equal. The following lemma shows that while the direct effect is always positive, the strategic effect can counteract and even overwhelm this positive effect.

LEMMA 2. *Suppose $G_A = G_B$ and $\lambda_A > \lambda_B$. Fix a pair of equilibrium voting strategies $(\alpha_A^*(n), \alpha_B^*(n))$. Then, $D(n) > 0$ for all n. Moreover, for an infinite subsequence of n, $S(n) < 0$ and $D(n) + S(n) < 0$.*

PROOF. Suppose $G_A = G_B$ and $\lambda_A > \lambda_B$. Fix a pair of equilibrium voting strategies $(\alpha_A^*(n), \alpha_B^*(n))$. Define $y(n) \equiv \pi(\alpha_A^*(n), \alpha_B^*(n), n)$. By Proposition 1, $\alpha_A^*(n) > \alpha_B^*(n) > 0$. Thus, $D(n) = \frac{1}{2}(\alpha_A^*(n) - \alpha_B^*(n)) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(k!)^2(n-2k)!}[\alpha_A^*(n)]^k[\alpha_B^*(n)]^k(1 - \alpha_A^*(n) - \alpha_B^*(n))^{n-2k} > 0$ and $y(n) > \frac{1}{2}$ for all n. Moreover, given $G_A = G_B$, we have $y(n) \to \frac{1}{2}$ as $n \to \infty$ by Proposition 2 below. This means $y(n+1) - y(n) < 0$ for some n; otherwise if $y(n+1) - y(n) \geq 0$ for all n, then $y(n)$ would not converge to $\frac{1}{2}$. This also means that $y(n+1) - y(n) < 0$ for an infinite subsequence of n; otherwise there would exist some $\underline{n} < \infty$ such that $y(n+1) - y(n) \geq 0$ for all $n \geq \underline{n}$, contradicting again the fact that $y(n) \to \frac{1}{2}$ as $n \to \infty$. By definition, $D(n) + S(n) < 0$ and thus $S(n) < 0$ whenever $y(n+1) - y(n) < 0$. ■

Lemma 2 answers the question we posed: an increase in population size does not necessarily improve the majority's chances of winning. It says that there is an infinite subsequence of population size under which the strategic effect due to strategic voting is sufficiently negative to diminish the probability of winning for the majority. Note that if there were no strategic voting so that $\alpha_A > \alpha_B > 0$ were fixed, then clearly $S(n) = 0$, and $\pi(\alpha_A, \alpha_B, n)$ would monotonically increase and converge to 1 as n grows.[16] To gain further insight and

[16] To be sure, Lemma 2 provides only a partial characterization of $\Pr\{A \text{ wins}|n\}$ in n. A full characterization requires a full characterization of $\alpha_A^*(n)$ and $\alpha_B^*(n)$, which, in light of Lemma 1, does not appear to be feasible unless one is willing to make strong assumptions on preference and cost distributions.

motivate our analysis of large elections, we briefly report a numerical example in which $S(n) < 0$ for all n and it dominates $D(n) > 0$ after n becomes sufficiently large, yielding a hump-shaped winning probability for the majority.

Example 1. *Let each agent draw his cost of voting independently from a uniform distribution in* $(.02, 1)$*, and* $\lambda_A = .55$*. Solving numerically for equilibrium (unique in this case for each* n*), we find:*

n	5	50	100	$1,000$	$10,000$	$100,000$	$1,000,000$
α_A^*	.15661	.07063	.05454	.02104	.00641	.00141	.00019
α_B^*	.13578	.06207	.04805	.01876	.00588	.00136	.00018
$\Pr\{A\ wins\}$	.53204	.56492	.57945	.64064	.68357	.62688	.52780

Table 2. Nonmonotonicity of winning probability in electorate size

Although many voting situations such as boards of directors and congressional committees involve small electorates, many others such as referendums are about large electorates, which we investigate next.

4 Equilibrium in Large Electorates

We have three main objectives in this section. First, we want to determine if the limit turnout depends on the initial distribution of political preferences. Second, we wish to identify conditions (if any) under which the advantage from being the majority group or the group with stochastically lower cost vanishes as the population becomes large. Third, we would like to know if large elections with fixed population size can be interpreted as Myerson's Poisson games with an appropriately assigned distribution of political preferences. We begin the analysis with the following well-known result:

Lemma 3. *In equilibrium,* $\lim_{n\to\infty} \alpha_r^*(n) = 0$ *and* $\lim_{n\to\infty}[n\alpha_r^*(n)] = m_r^* < \infty$ *for* $r = A, B$.

As first shown by Palfrey and Rosenthal (1985), Lemma 3 establishes that the individual probability of voting, and thus the turnout rate, becomes negligible in large elections. Moreover, the expected limit turnout for each alternative is finite. If it were infinite for some alternative, then each vote would be negligible, and no individual would vote given a strictly positive cost. But then, each vote would become pivotal with probability 1, yielding a contradiction.

Lemma 3 implies that in large elections, the equilibrium cutoff for each alternative must be close to the lower bound of the cost distribution, which, together with (3), leads to:

Lemma 4. $\lim_{n\to\infty}\left[\frac{1}{2}P(\alpha_r^*(n),\alpha_{r'}^*(n),n)\right] \le \underline{c}_r$ *(*$= \underline{c}_r$ *whenever* $c_r^*(n) > \underline{c}_r$*) for* $r, r' = A, B$ *and* $r \neq r'$.

In order to determine expected voter turnout in the limit, consider the situation facing a representative agent favoring alternative r and suppose that the other $n-1$ agents vote if and only if their costs are less than the equilibrium cutoff $c_r^*(n)$. Let $X_{A,n-1}$ and $X_{B,n-1}$ be the number of votes for alternatives A and B, respectively. Furthermore, let $X_{0,n-1} = n-1-X_{A,n-1}-X_{B,n-1}$ be the number of abstentions. Using this notation, a type r agent's vote will be pivotal if and only if $X_{r',n-1} = X_{r,n-1}$ (he breaks a tie) or $X_{r',n-1} = X_{r,n-1}+1$ (he creates a tie). Hence, the equilibrium probability that his vote is pivotal can be written

$$P(\alpha_r^*(n),\alpha_{r'}^*(n),n) = \Pr\{X_{r',n-1}^* = X_{r,n-1}^*\} + \Pr\{X_{r',n-1}^* = X_{r,n-1}^*+1\}. \tag{7}$$

Next, observe that $(X_{r,n-1}^*, X_{r',n-1}^*, X_{0,n-1}^*) \sim$Multinomial$(\alpha_r^*(n), \alpha_{r'}^*(n), 1-\alpha_r^*(n)-\alpha_{r'}^*(n)|n-1)$. Note that $X_{A,n-1}^*$ and $X_{B,n-1}^*$ are not independent for $n<\infty$, but the following result establishes independence in the limit.

Lemma 5. *The limiting marginal distributions,* $X_{A,\infty}^*$ *and* $X_{B,\infty}^*$ *are independent Poisson distributions with means* m_A^* *and* m_B^**, respectively. Hence, the limiting distribution of* $X_{A,\infty}^* + X_{B,\infty}^*$ *is Poisson with mean* $m_A^* + m_B^*$.

In light of Lemma 5, let $f(k|\mu)$ be the *p.d.f.* for a Poisson distribution with mean μ. Recall that $f(k|\mu) = \frac{\mu^k e^{-\mu}}{k!}$ for $k = 0, 1, \ldots$ Combining (7) and Lemma 5, it follows that

$$\begin{aligned}\lim_{n\to\infty} P(\alpha_r^*(n),\alpha_{r'}^*(n),n) &= \Pr\{X_{r',\infty}^* = X_{r,\infty}\} + \Pr\{X_{r',\infty}^* = X_{r,\infty}^*+1\}] \\ &= \sum_{k=0}^{\infty} f(k|m_r^*)f(k|m_{r'}^*) + \sum_{k=0}^{\infty} f(k|m_r^*)f(k+1|m_{r'}^*) \\ &\equiv Q(m_r^*, m_{r'}^*).\end{aligned} \tag{8}$$

Together with Lemma 4, the equilibrium limiting turnouts, m_A^* and m_B^*, must then satisfy

$$\frac{1}{2}Q(m_r^*, m_{r'}^*) - \underline{c}_r \ \le 0 \ (= \underline{c}_r \text{ if } m_r^* > 0). \tag{9}$$

Proposition 2. *Without loss of generality, suppose* $\underline{c}_B \le \underline{c}_A$*. Then,*

(i) *there is a unique cost* $\underline{d}_A \in (0, \underline{c}_A)$ *such that*

$$\begin{cases} m_B^* > m_A^* = 0 & \text{if} \quad \underline{c}_B \leq \underline{d}_A \\ m_B^* > m_A^* \ > 0 & \text{if} \quad \underline{d}_A < \underline{c}_B < \underline{c}_A \\ m_B^* = m_A^* > 0 & \text{if} \quad \underline{c}_B = \underline{c}_A. \end{cases}$$

(ii) *Given* $\underline{c}_A$*,* m_B^* *is strictly decreasing and* m_A^* *is weakly increasing in* $\underline{c}_B$*.*

(iii) *Given* $\underline{c}_A$*, the limiting probability,* $\lim_{n\to\infty} \Pr\{B\ wins\}$*, is strictly decreasing in* $\underline{c}_B$*, and equal to* $\frac{1}{2}$ *for* $\underline{c}_B = \underline{c}_A$*.*

Proof. Without loss of generality, suppose $\underline{c}_B \leq \underline{c}_A$. Using the Poisson density,

$$Q(m_r^*, m_{r'}^*) = e^{-(m_A^* + m_B^*)} \left[\sum_{k=0}^{\infty} \frac{(m_A^* m_B^*)^k}{(k!)^2} + m_{r'}^* \sum_{k=0}^{\infty} \frac{(m_A^* m_B^*)^k}{k!(k+1)!} \right].$$

Hence, (9) implies $m_B^* \geq m_A^*$. Given $Q(0,0) = 1$ and $\underline{c}_r < \frac{1}{2}$, (9) also implies $m_B^* > 0$. Moreover, $m_A^* = 0$ if and only if $\frac{1}{2}Q(0, m_B^*) - \underline{c}_A \leq 0$ and $\frac{1}{2}Q(m_B^*, 0) - \underline{c}_B = 0$. Since $Q(0, m_B^*) = e^{-m_B^*}(1 + m_B^*)$ and $Q(m_B^*, 0) = e^{-m_B^*}$, this means $m_A^* = 0$ if and only if $2\underline{c}_B[1 - \ln(2\underline{c}_B)] \leq 2\underline{c}_A$. Note that for $x \in (0,1)$, the function $\varphi(x) = x(1 - \ln x)$ satisfies: $\lim_{x\to 0^+} \varphi(x) = 0$, $\lim_{x \to 1^-} \varphi(x) = 1$, $\varphi(x) > x$ and $\varphi'(x) > 0$. Hence, there exists a unique cost $\underline{d}_A \in (0, \underline{c}_A)$ that solves $2d[1 - \ln(2d)] = 2\underline{c}_A$. Clearly, $2\underline{c}_B[1 - \ln(2\underline{c}_B)] \leq 2\underline{c}_A$ for all $\underline{c}_B \leq \underline{d}_A$, and $m_A^* = 0$ as a result. For $\underline{c}_B \in (\underline{d}_A, \underline{c}_A]$, we have $m_A^* > 0$, and by (9), $m_B^* = m_A^*$ if and only if $\underline{c}_B = \underline{c}_A$, proving part (i).

Next, if $\underline{c}_B \leq \underline{d}_A$, then $m_A^* = 0$ and $\frac{1}{2}e^{-m_B^*} = \underline{c}_B$ by part (i). Thus, m_B^* is strictly decreasing in $\underline{c}_B$. Now, suppose $\underline{c}_B \in (\underline{d}_A, \underline{c}_A)$. Then, by part (i), $m_B^* > m_A^* \ > 0$ that solve $\frac{1}{2}Q(m_B^*, m_A^*) = \underline{c}_B$ and $\frac{1}{2}Q(m_A^*, m_B^*) = \underline{c}_A$. Simple algebra shows that $\frac{\partial}{\partial m_B} Q(m_B^*, m_A^*) < 0$; $\frac{\partial}{\partial m_A} Q(m_B^*, m_A^*) > 0$; $\frac{\partial}{\partial m_A} Q(m_A^*, m_B^*) < 0$; and $\frac{\partial}{\partial m_B} Q(m_A^*, m_B^*) < 0$. From here, it follows that m_B^* is strictly decreasing and m_A^* is strictly increasing in $\underline{c}_B$.

Finally, note that $\lim_{n\to\infty} \Pr\{B\ wins\} = \sum\limits_{k=0}^{\infty} \sum\limits_{k'=k+1}^{\infty} f(k'|m_B^*) f(k|m_A^*) + \frac{1}{2} \sum\limits_{k=0}^{\infty} f(k|m_B^*) f(k|m_A^*)$. It is easy to verify that the r.h.s. is strictly increasing in m_B^* and strictly decreasing in m_A^*. Part (iii) then follows from part (ii). ■

Proposition 2 is a key result of this paper. The most important observation is that the limit turnouts and the probability of winning are completely determined by the individuals with the lowest cost of voting in each group – *not* by the distributions of voting costs, G_r *or* political preferences, λ_r. This is because the free-rider problem in each group is amplified as the electorate size grows, leaving only the lowest cost agents to vote. As part (i) indicates however, one group may abstain altogether if the cost differential is sufficiently large, and such a cost differential *always* exists. Nonetheless, because the limit turnout for each group is finite, there is still a significant probability that the abstaining group will win.[17] If the cost differential is not too high, Proposition 2 leads to the intuitive observation that the group with the lowest possible cost is expected to turn out in larger number and thus more likely to win the election. In addition, as the cost differential increases, so does the probability of winning for the low-cost group.

Proposition 2 achieves the first two objectives set in the beginning of this section. Namely, the limit turnout does not depend on the initial distribution of political preferences. In addition, in the limit, the majority group loses its initial advantage, and a group benefits from a favorable cost distribution to the extent of its lowest cost vis-a-vis the rival's.

The distinction between a large and small election becomes most transparent when $\underline{c}_A = \underline{c}_B$. The advantage from being in the majority or from having a favorable cost distribution identified in Proposition 1 for a small election completely vanish in the limit, making each alternative *equally likely* to win. Moreover, two large elections one with $\lambda_r = .5$ and one with $\lambda_r \neq .5$ result in equal limiting turnouts.[18] Hence, the widely held intuition that elections with a more evenly split electorate should generate a greater expected turnout appears to be a property of small elections.

Proposition 2 also unifies various results pertaining to large electorates in the costly voting literature. For instance, it implies that an expected minority whose members are likely to have lower cost-benefit ratios may end up winning the election if the electorate size is sufficiently large. Indeed, this is what Campbell (1999), in rationalizing minority upsets in elections, finds, though he doesn't provide a complete asymptotic characterization. In a sense, it is the "quality" – not the "quantity" – of supporters that counts in order to win an election. In a more recent paper, Krasa and Polborn (2009), using a special case of the present model where $G_A = G_B$ and thus $\underline{c}_A = \underline{c}_B$, investigate socially optimal voting subsidies or nonvoting penalties, and have independently discovered that without such interventions, each alternative wins a large election with probability $\frac{1}{2}$. Proposition 2 shows that the assumption of equal cost lower bounds is both necessary and sufficient for

[17] In fact, if $\underline{c}_B \leq \underline{d}_A$, then $\lim_{n\to\infty} \Pr\{A \text{ wins}\} = \frac{1}{2}\Pr\{X_B^* = 0\} = \underline{c}_B$.

[18] In a previous version of this paper, we determined that if $\underline{c}_A = \underline{c}_B = \underline{c}$ and $\underline{c}$ is small, the aggregate limit turnout can be approximated by $\frac{1}{2\pi\underline{c}^2}$.

this $\frac{1}{2}$ result, pointing to its knife-edge nature. If the cost lower bounds are not equal, then one still obtains the intuitive result that the group with a cost advantage is strictly more likely to win a large election. Finally, Taylor and Yildirim (2010) who analyze the impact of public information about the distribution of political preferences on election outcomes and welfare, also uncover the $\frac{1}{2}$ result to be the probability of winning in a large electorate where each agent has a fixed and equal cost of voting, i.e., a degenerate cost distribution.[19] Proposition 2 reveals that while the result would remain true with the introduction of a nondegenerate cost distribution, if voting costs are fixed, then they must be equal in order for neutrality to obtain. Said differently, in a large election, a model with equal fixed voting costs and a model with symmetric cost uncertainty are strategically equivalent if and only if the fixed voting cost in the first model equals the lowest possible cost in the second one.

By utilizing the Poisson characterization, we can also link the costly voting model to Myerson's Poisson games, and in the process answer the third point made in the beginning of this section. Inspired mostly by large elections, Myerson (1998, 2000) introduced the concept of Poisson games, where the number of players (the electorate size, here) is distributed according to a Poisson distribution with an exogenous mean, rather than being fixed. However, as Myerson (2000, p.27) notes, because abstentions occur in equilibrium with costly voting, this interpretation is nontrivial. In particular, the expected number of active players, m_r^*, is endogenously determined by asymptotic equilibrium strategies. Moreover, within the set of active players, it is *incorrect* to assume that the probability that an agent votes for alternative r is λ_r. Rather, large elections analyzed here correspond to a Poisson game where the mean population is $m_A^* + m_B^*$, and each agent favoring alternative r votes with probability $\frac{m_r^*}{m_A^*+m_B^*}$, which is $\frac{1}{2}$ in the special case of $\underline{c}_A = \underline{c}_B$.

5 On the Uniqueness of Type-Symmetric Equilibrium

Armed with the characterization of the pivot probability in Lemma 1, we now establish a sufficient condition for the uniqueness of type-symmetric equilibrium, which received some attention in the costly voting literature. Börgers (2004) showed that when all agents are *ex ante* symmetric, i.e., $\lambda_A = \lambda_B$ and $G_A = G_B$, then the type-symmetric equilibrium is unique. Goeree and Grosser (2007), and Taylor and Yildirim (2010) proved the uniqueness of type-symmetric equilibrium in totally mixed strategies when $\lambda_A \neq \lambda_B$ and each agent

[19] As mentioned in the previous section, Goeree and Grosser (2007) also find the $\frac{1}{2}$ result, but only for a small electorate.

has a fixed and equal cost of voting. Given the special structures in these investigations, however, it is difficult to understand what drives the uniqueness result and whether or not it is robust to (at least) small perturbations. In particular, all of these studies have utilized two observations: $\alpha_A = \alpha_B = \alpha$ at an equilibrium, and the pivot probability along this path, namely $P(\alpha, \alpha, n)$, is strictly decreasing in α. Neither of these observations is true in general, as we now know from Lemma 1 and Proposition 1 above. Intuitively though, the uniqueness result should continue to hold if α_A and α_B are sufficiently close in equilibrium.[20]

PROPOSITION 3. *There is at most one type-symmetric equilibrium that satisfies:* $1-\frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\alpha_B^*}{\alpha_A^*} \leq 1$. *Moreover, if* $G_A = G_B$ *and* $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\lambda_B}{\lambda_A} \leq 1$, *then there exists a unique type-symmetric equilibrium.*

PROOF. We first make some preliminary observations. Fixing $\alpha_{r'} \in [0, \lambda_r]$, let $\widehat{\alpha}_r \equiv R_r(\alpha_{r'}) \in [0, \lambda_r]$ be a solution to $\Phi_r(\alpha_r, \alpha_{r'}) = 0$. $R_r(\alpha_{r'})$ exists because $\Phi_r(0, \alpha_{r'}) > 0$, $\Phi_r(\lambda_r, \alpha_{r'}) < 0$, and Φ_r is continuous. Next, note that if $\left(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor}\right) \alpha_{r'} \leq R_r(\alpha_{r'})$ for some region of $\alpha_{r'}$, then $R_r(\alpha_{r'})$ is single-valued and differentiable in this region; because, by part (iii) of Lemma 1, $\Phi_r(\alpha_r, \alpha_{r'})$ is strictly decreasing in α_r whenever $\left(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor}\right) \alpha_{r'} \leq \alpha_r$. More importantly, $R_r'(\alpha_{r'}) =^{sign} \frac{\partial}{\partial \alpha_{r'}} P(\alpha_r, \alpha_{r'}, n)$, which, by part (ii) of Lemma 1, means $R_r'(\alpha_{r'}) =^{sign} \alpha_r - \alpha_{r'}$.

To prove the first part of the proposition, suppose there are two equilibria $(\alpha_A^*, \alpha_B^*) \neq (\alpha_A^{**}, \alpha_B^{**})$ such that $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\alpha_B^*}{\alpha_A^*} \leq 1$ and $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\alpha_B^{**}}{\alpha_A^{**}} \leq 1$. Since, by definition of an equilibrium, $\alpha_r^* = R_r(\alpha_{r'}^*)$ and $\alpha_r^{**} = R_r(\alpha_{r'}^{**})$, we have $\left(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor}\right) \alpha_{r'}^* \leq R_r(\alpha_{r'}^*)$ and $\left(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor}\right) \alpha_{r'}^{**} \leq R_r(\alpha_{r'}^{**})$. This means that both equilibria are in the region of $(\alpha_r, \alpha_{r'})$ in which $R_r(\alpha_{r'})$ is single-valued and differentiable. Moreover, since both equilibria are also in the region with $\alpha_A \geq \alpha_B$, it follows that $R_A'(\alpha_B) \geq 0$ and $R_B'(\alpha_A) \leq 0$, where equalities hold only when $\alpha_A = \alpha_B$. Without loss of generality, suppose $\alpha_A^* > \alpha_A^{**}$. Then, $R_A(\alpha_B^*) > R_A(\alpha_B^{**})$, implying that $\alpha_B^* \geq \alpha_B^{**}$. But, this means $R_B(\alpha_A^*) \geq R_B(\alpha_A^{**})$ and thus $\alpha_A^* \leq \alpha_A^{**}$ – a contradiction. Hence, $\alpha_A^* = \alpha_A^{**}$. This implies $\alpha_B^* = \alpha_B^{**}$, because $R_B(\alpha_A)$ is decreasing, yielding a contradiction to $(\alpha_A^*, \alpha_B^*) \neq (\alpha_A^{**}, \alpha_B^{**})$. Hence, $(\alpha_A^*, \alpha_B^*) = (\alpha_A^{**}, \alpha_B^{**})$.

[20]For $n = 2$, uniqueness obtains without any additional condition because, by Lemma 1, $P(\alpha_r, \alpha_{r'}, 2)$ is independent of $\alpha_{r'}$ and so is $\Phi_r(\alpha_r, \alpha_{r'}, 2)$.

To prove the second part, note that Proposition 1 guarantees the existence of a type-symmetric equilibrium, (α_A^*, α_B^*). If, in addition, $G_A = G_B$ and $\frac{\lambda_B}{\lambda_A} \leq 1$, then Proposition 1 reveals that $0 < \phi_A^* \leq \phi_B^*$ and $\alpha_A^* \geq \alpha_B^*$. Thus, for any type-symmetric equilibrium,

$$\frac{\lambda_B}{\lambda_A} \leq \frac{\lambda_B \phi_B^*}{\lambda_A \phi_A^*} = \frac{\alpha_B^*}{\alpha_A^*} \leq 1.$$

If $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\lambda_B}{\lambda_A}$, then we have $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\alpha_B^*}{\alpha_A^*} \leq 1$ for *any* type-symmetric equilibrium, which, by the first part of the proposition, must be unique. ■

As suggested above, the potential source of multiple equilibria is that members of *some* political group view their votes as complements rather than substitutes. In light of Lemma 1, such complementarity between the votes can occur only in the group whose members' *ex ante* probability of voting is far below the rival's so that the free-rider incentive is not strong enough to overwhelm the coordination incentive. The first part of Proposition 3 simply says that when equilibrium voting strategies are sufficiently symmetric across the groups, the free-rider incentive dominates for *all* individuals. The second part of Proposition 3 provides a parametric condition under which a unique type-symmetric equilibrium obtains. In particular, it demonstrates that Börgers' uniqueness result derived under complete symmetry, i.e., $G_A = G_B$ and $\lambda_A = \lambda_B$ is robust to (at least) small perturbations. That is, if agents are sufficiently symmetric, then their equilibrium strategies are sufficiently close for the free-rider incentive to dominate and make the equilibrium unique.

6 Concluding Remarks

There are two ways to interpret the contribution of this paper. First, it deepens our understanding of the rational choice theory of voting in its purest form, and second, by doing so, it allows for richer and better grounded empirical and experimental investigation. Some prominent recent developments in voting theory have been concentrated around a model involving "group-based ethical voters" who care not only about their own payoff but also the payoffs of others with similar political preferences (Feddersen (2004)). While we believe the group-based approach shows some promise, we also believe there are further directions in which rational voting theory can be fruitfully extended to better reflect reality.

For one, it would be useful to expand the notion of a pivotal vote to recognize the fact that the vote counting process is imperfect.[21] Hence, it would be edifying to extend the basic

[21] In fact, in many states in the U.S. if a vote count is too close, then a recount is either triggered

model to explicitly account for the vote counting technology, and to study its implications on voter behavior. Another important assumption of the basic theory that could be profitably relaxed is that costs of voting are independently distributed across citizens. There are many factors such as weather and security concerns that influence costs of voting for large groups of citizens. Hence, an extension that allows for cost correlation would also be valuable. We leave these extensions for future work and conclude by remarking simply that there is still a lot to discover in the context of rational voting theory.

automatically or may be demanded by the potential loser.

A Appendix

Proof of Lemma 1: First, note that a type r agent's expected payoffs from voting and abstaining are given, respectively by

$$V_r^1 = \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda_r^k (1-\lambda_r)^{n-1-k} \sum_{k_r=0}^{k} \binom{k}{k_r} \phi_r^{k_r} (1-\phi_r)^{k-k_r}$$
$$\times \left[\sum_{k_{r'}=0}^{k_r-1} \binom{n-1-k}{k_{r'}} \phi_{r'}^{k_{r'}} (1-\phi_{r'})^{n-1-k-k_{r'}} + \binom{n-1-k}{k_r} \phi_{r'}^{k_r} (1-\phi_{r'})^{n-1-k-k_r} \right.$$
$$\left. + \frac{1}{2} \binom{n-1-k}{k_r+1} \phi_{r'}^{k_r+1} (1-\phi_{r'})^{n-2-k-k_r} \right] - c,$$

and

$$V_r^0 = \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda_r^k (1-\lambda_r)^{n-1-k} \sum_{k_r=0}^{k} \binom{k}{k_r} \phi_r^{k_r} (1-\phi_r)^{k-k_r}$$
$$\times \left[\sum_{k_{r'}=0}^{k_r-1} \binom{n-1-k}{k_{r'}} \phi_{r'}^{k_{r'}} (1-\phi_{r'})^{n-1-k-k_{r'}} + \frac{1}{2} \binom{n-1-k}{k_r} \phi_{r'}^{k_r} (1-\phi_{r'})^{n-1-k-k_r} \right].$$

To understand these expected payoffs, fix a type r agent, and let k_r be the number of votes for alternative r excluding his, and $k_{r'}$ be the number of all votes for alternative r'. Clearly, if $k_{r'} \leq k_r - 1$ and $k_{r'} \geq k_r + 2$, then alternative r respectively wins and loses with probability 1, regardless of the type r agent's action. If $k_{r'} = k_r$, alternative r wins with probability 1 if the type r agent in question votes, and wins with probability $\frac{1}{2}$ if he abstains and leaves the tie. Finally, if $k_{r'} = k_r + 1$, alternative r loses with probability 1 if the type r agent abstains; but may win with probability $\frac{1}{2}$ if he votes. These events explain the expressions in parentheses above. The first two summations in V_r^1 and V_r^0 account for the distribution of preferences.

Next, subtracting V_r^0 from V_r^1, the third summation inside parentheses cancel out, reducing the net expected payoff to

$$\Delta_r = V_r^1 - V_r^0 = \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda_r^k (1-\lambda_r)^{n-1-k} \sum_{k_r=0}^{k} \binom{k}{k_r} \phi_r^{k_r} (1-\phi_r)^{k-k_r} \tag{A-1}$$
$$\times \left[\frac{1}{2} \binom{n-1-k}{k_r} \phi_{r'}^{k_r} (1-\phi_{r'})^{n-1-k-k_r} \right.$$
$$\left. + \frac{1}{2} \binom{n-1-k}{k_r+1} \phi_{r'}^{k_r+1} (1-\phi_{r'})^{n-2-k-k_r} \right] - c.$$

Now, recall $\alpha_r = \lambda_r \phi_r$ and define $\beta_r = \lambda_r(1-\phi_r)$. By substituting for these terms into (A-1), and noting the following facts,

$$\begin{aligned} \lambda_r^k &= \lambda_r^{k_r} \lambda_r^{k-k_r} \\ (1-\lambda_r)^{n-1-k} &= (1-\lambda_r)^{k_r}(1-\lambda_r)^{n-1-k-k_r} \\ (1-\lambda_r)^{n-1-k} &= (1-\lambda_r)^{k_r+1}(1-\lambda_r)^{n-2-k-k_r}, \end{aligned}$$

(A-1) further reduces to

$$\begin{aligned} \Delta_r &= \frac{1}{2}\sum_{k=0}^{n-1}\left[\binom{n-1}{k}\sum_{k_r=0}^{k}\binom{k}{k_r}\alpha_r^{k_r}\beta_r^{k-k_r} \right. \\ &\quad \left. \times\left(\binom{n-1-k}{k_r}\alpha_{r'}^{k_r}\beta_{r'}^{n-1-k-k_r} + \binom{n-1-k}{k_r+1}\alpha_{r'}^{k_r+1}\beta_{r'}^{n-2-k-k_r}\right)\right] - c \\ &= \frac{1}{2}\sum_{k_r=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{k}{k_r}\binom{n-1-k}{k_r}\alpha_r^{k_r}\alpha_{r'}^{k_r}\beta_r^{k-k_r}\beta_{r'}^{n-1-k-k_r} \\ &\quad + \frac{1}{2}\sum_{k_r=0}^{\lfloor\frac{n-2}{2}\rfloor}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{k}{k_r}\binom{n-1-k}{k_r+1}\alpha_r^{k_r}\alpha_{r'}^{k_r+1}\beta_r^{k-k_r}\beta_{r'}^{n-2-k-k_r} - c. \end{aligned}$$

Using the following two combinatorial identities:

$$\binom{n-1}{k}\binom{k}{k_r}\binom{n-1-k}{k_r} = \binom{n-1}{k_r, k_r, n-1-2k_r}\binom{n-1-2k_r}{k-k_r}$$

and

$$\binom{n-1}{k}\binom{k}{k_r}\binom{n-1-k}{k_r+1} = \binom{n-1}{k_r, k_r+1, n-2-2k_r}\binom{n-2-2k_r}{k-k_r},$$

Δ_r becomes

$$\begin{aligned} \Delta_r &= \frac{1}{2}\sum_{k_r=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1}{k_r, k_r, n-1-2k_r}\alpha_r^{k_r}\alpha_{r'}^{k_r}\sum_{k=0}^{n-1}\binom{n-1-2k_r}{k-k_r}\beta_r^{k-k_r}\beta_{r'}^{n-1-k-k_r} \\ &\quad + \frac{1}{2}\sum_{k_r=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k_r, k_r+1, n-2-2k_r}\alpha_r^{k_r}\alpha_{r'}^{k_r+1}\sum_{k=0}^{n-1}\binom{n-2-2k_r}{k-k_r}\beta_r^{k-k_r}\beta_{r'}^{n-2-k-k_r} - c \\ &= \frac{1}{2}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1}{k, k, n-1-2k}\alpha_r^k\alpha_{r'}^k(\beta_r+\beta_{r'})^{n-1-2k} \\ &\quad + \frac{1}{2}\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k, k+1, n-2-2k}\alpha_r^k\alpha_{r'}^{k+1}(\beta_r+\beta_{r'})^{n-2-2k} - c, \end{aligned}$$

where we also use the facts that

$$\sum_{k=0}^{n-1}\binom{n-1-2k_r}{k-k_r}\beta_r^{k-k_r}\beta_{r'}^{n-1-k-k_r}=\sum_{k=0}^{n-1}\binom{n-1-2k_r}{k-k_r}\beta_r^{k-k_r}\beta_{r'}^{n-1-2k_r-(k-k_r)}=(\beta_r+\beta_{r'})^{n-1-2k_r}$$

and

$$\sum_{k=0}^{n-1}\binom{n-2-2k_r}{k-k_r}\beta_r^{k-k_r}\beta_{r'}^{n-2-k-k_r}=\sum_{k=0}^{n-1}\binom{n-2-2k_r}{k-k_r}\beta_r^{k-k_r}\beta_{r'}^{n-2-2k_r-(k-k_r)}=(\beta_r+\beta_{r'})^{n-2-2k_r}$$

and, without loss of generality, change index of summations to k in the last equality. The expressions in (1) and (2) then follows by simply observing that $\beta_A+\beta_B=1-\alpha_A-\alpha_B$.

Next, we proceed to Lemma 1. Let $(\alpha_r,\alpha_{r'})\in(0,\lambda_r)\times(0,\lambda_{r'})$. Using the definition of $P(\alpha_r,\alpha_{r'},n)$ in (2) and canceling the first terms, part (i) follows because

$$P(\alpha_r,\alpha_{r'},n)-P(\alpha_{r'},\alpha_r,n)=(\alpha_{r'}-\alpha_r)\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k,k+1,n-2-2k}\alpha_r^k\alpha_{r'}^k(1-\alpha_r-\alpha_{r'})^{n-2-2k}.$$

Differentiating $P(\alpha_r,\alpha_{r'},n)$ with respect to $\alpha_{r'}$,

$$\begin{aligned}\frac{\partial}{\partial\alpha_{r'}}P(\alpha_r,\alpha_{r'},n) &= \sum_{k=0}^{\lfloor\frac{n-3}{2}\rfloor}\frac{(n-1)!}{k!(k+1)!(n-3-2k)!}\alpha_r^{k+1}\alpha_{r'}^k(1-\alpha_r-\alpha_{r'})^{n-3-2k}\\ &\quad-\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\frac{(n-1)!}{k!k!(n-2-2k)!}\alpha_r^k\alpha_{r'}^k(1-\alpha_r-\alpha_{r'})^{n-2-2k}\\ &\quad+\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\frac{(n-1)!}{k!k!(n-2-2k)!}\alpha_r^k\alpha_{r'}^k(1-\alpha_r-\alpha_{r'})^{n-2-2k}\\ &\quad-\sum_{k=0}^{\lfloor\frac{n-3}{2}\rfloor}\frac{(n-1)!}{k!(k+1)!(n-3-2k)!}\alpha_r^k\alpha_{r'}^{k+1}(1-\alpha_r-\alpha_{r'})^{n-3-2k}.\end{aligned}$$

Note that the second and third terms on the r.h.s. cancel out. The remaining two terms can be rewritten

$$\frac{\partial}{\partial\alpha_{r'}}P(\alpha_r,\alpha_{r'},n)=(\alpha_r-\alpha_{r'})\sum_{k=0}^{\lfloor\frac{n-3}{2}\rfloor}\frac{(n-1)!}{k!(k+1)!(n-3-2k)!}\alpha_r^k\alpha_{r'}^k(1-\alpha_r-\alpha_{r'})^{n-3-2k},$$

where the summation is 0 for $n=2$ for all $(\alpha_r,\alpha_{r'})$.

To prove part (iii), suppose n is odd and differentiate $P(\alpha_r, \alpha_{r'}, n)$ with respect to α_r to obtain

$$\begin{aligned}\frac{\partial}{\partial \alpha_r} P(\alpha_r, \alpha_{r'}, n) &= \sum_{k=1}^{\frac{n-1}{2}} \frac{(n-1)!}{(k-1)!k!(n-1-2k)!} \alpha_r^{k-1} \alpha_{r'}^{k} (1-\alpha_r-\alpha_{r'})^{n-1-2k} \\ &\quad - \sum_{k=0}^{\frac{n-3}{2}} \frac{(n-1)!}{k!k!(n-2-2k)!} \alpha_r^{k} \alpha_{r'}^{k} (1-\alpha_r-\alpha_{r'})^{n-2-2k} \\ &\quad + \sum_{k=1}^{\frac{n-3}{2}} \frac{(n-1)!}{(k-1)!(k+1)!(n-2-2k)!} \alpha_r^{k-1} \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-2-2k} \\ &\quad - \sum_{k=0}^{\frac{n-3}{2}} \frac{(n-1)!}{k!(k+1)!(n-3-2k)!} \alpha_r^{k} \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-3-2k}.\end{aligned}$$

Note that the first and the last terms cancel out. Separating the term for $k=0$, the second term can be re-written $\sum_{k=0}^{\frac{n-3}{2}} (.) = (n-1)(1-\alpha_r-\alpha_{r'})^{n-2} + \sum_{k=1}^{\frac{n-3}{2}} (.)$. This implies

$$\begin{aligned}\frac{\partial}{\partial \alpha_r} P(\alpha_r, \alpha_{r'}, n) &= \sum_{k=1}^{\frac{n-3}{2}} \left(\frac{k}{k+1} \alpha_{r'} - \alpha_r \right) \frac{(n-1)!}{k!k!(n-2-2k)!} \alpha_r^{k-1} \alpha_{r'}^{k} (1-\alpha_r-\alpha_{r'})^{n-2-2k} \\ &\quad -(n-1)(1-\alpha_r-\alpha_{r'})^{n-2}.\end{aligned}$$

A similar line of derivation shows that for an even n, only the upper bound in the above summation switches to $\frac{n-2}{2}$. Hence, for any $n \geq 2$,

$$\begin{aligned}\frac{\partial}{\partial \alpha_r} P(\alpha_r, \alpha_{r'}, n) &= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} \left(\frac{k}{k+1} \alpha_{r'} - \alpha_r \right) \frac{(n-1)!}{k!k!(n-2-2k)!} \alpha_r^{k-1} \alpha_{r'}^{k} (1-\alpha_r-\alpha_{r'})^{n-2-2k} \\ &\quad -(n-1)(1-\alpha_r-\alpha_{r'})^{n-2}.\end{aligned}$$

Note that if $\alpha_r \geq \left(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor}\right) \alpha_{r'}$, then $\frac{k}{k+1}\alpha_{r'} - \alpha_r \leq 0$ for each $k \in \{1, ..., \lfloor \frac{n}{2} \rfloor - 1\}$. Together with $1 - \alpha_r - \alpha_{r'} \neq 0$, it follows that $\frac{\partial}{\partial \alpha_{r'}} P(\alpha_r, \alpha_{r'}, n) < 0$.

To prove part (iv), we begin by noting that

$$P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-1-2k} \tag{A-2}$$

$$+ \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \frac{(n-1)!}{k!(k+1)!(n-2-2k)!} \alpha_r^k \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-2-2k}$$

$$- \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(k!)^2(n-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-2k}$$

$$- \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n!}{k!(k+1)!(n-1-2k)!} \alpha_r^k \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-1-2k}.$$

Before signing this expression, we suppose that n is odd, and re-write the third summation:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(k!)^2(n-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-2k}$$

$$= \sum_{k=0}^{\frac{n-1}{2}} \left[1 + \frac{2k}{n-2k}\right] \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-2k}$$

$$= (1-\alpha_r-\alpha_{r'}) \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-1-2k}$$

$$+ 2 \sum_{k=1}^{\frac{n-1}{2}} \frac{(n-1)!}{(k-1)!k!(n-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-2k}$$

$$= (1-\alpha_r-\alpha_{r'}) \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-1-2k}$$

$$+ 2 \sum_{k=0}^{\frac{n-1}{2}-1} \frac{(n-1)!}{k!(k+1)!(n-2-2k)!} \alpha_r^{k+1} \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-2-2k}.$$

Inserting this into (A-2) and canceling terms yield

$$
\begin{aligned}
& P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) \\
= \quad & (\alpha_r + \alpha_{r'}) \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-1-2k} \\
& +(1-2\alpha_r) \sum_{k=0}^{\frac{n-3}{2}} \frac{(n-1)!}{k!(k+1)!(n-2-2k)!} \alpha_r^k \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-2-2k} \\
& -\sum_{k=0}^{\frac{n-1}{2}} \frac{n!}{k!(k+1)!(n-1-2k)!} \alpha_r^k \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-1-2k}.
\end{aligned}
$$

Now, noting $\frac{n!}{k!(k+1)!(n-1-2k)!} = \left(1 + \frac{k}{n-1-2k} + \frac{k+1}{n-1-2k}\right) \frac{(n-1)!}{k!(k+1)!(n-2-2k)!}$, we re-write the last summation in three terms. Moreover, we expand the first and second summations by multiplying with $(\alpha_r + \alpha_{r'})$ and $(1-2\alpha_r)$, respectively. Canceling and collecting terms then reveal

$$
\begin{aligned}
& P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) \\
= \quad & \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{k!(k+1)!(n-1-2k)!} \alpha_r^k \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-1-2k} \\
& +(\alpha_r - \alpha_{r'}) \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-1-2k} \\
& -(\alpha_r - \alpha_{r'}) \sum_{k=0}^{\frac{n-3}{2}} \frac{(n-1)!}{k!(k+1)!(n-2-2k)!} \alpha_r^k \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-2-2k}.
\end{aligned}
$$

For $\alpha_r = \alpha_{r'}$, clearly $P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) > 0$. For $\alpha_r \neq \alpha_{r'}$, note that

$$
\begin{aligned}
P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+2) \quad = \quad & [P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1)] \\
& +[P(\alpha_r, \alpha_{r'}, n+1) - P(\alpha_r, \alpha_{r'}, n+2)]
\end{aligned}
$$

Performing similar decompositions to those above, it follows that $P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+2) > 0$. ■

Before proving Lemmas 2 and 3, we note the following useful result.

LEMMA A1. *Fix a pair* $(\alpha_A, \alpha_B) \in [0, \lambda_A] \times [0, \lambda_B]$ *such that* $(\alpha_A, \alpha_B) \neq (0, 0)$*. Then,* $\lim_{n\to\infty} P(\alpha_A, \alpha_B, n) = \lim_{n\to\infty} P(\alpha_A, \alpha_B, n) = 0$.

Proof of Lemma A1. Fix a pair $(\alpha_A, \alpha_B) \in [0, \lambda_A] \times [0, \lambda_B]$ such that $(\alpha_A, \alpha_B) \neq (0,0)$. Let $X_{A,n}$ and $X_{B,n}$ be the number of votes for alternatives A and B, respectively, and $X_{0,n} = n - X_{A,n} - X_{B,n}$ be the number abstentions. Clearly, $(X_{A,n}, X_{B,n}, X_{0,n}) \sim$Multinomial$(\alpha_A, \alpha_B, 1 - \alpha_A - \alpha_B | n)$. By definition of the pivot probability in (2), this means

$$P(\alpha_A, \alpha_B, n) = \Pr\{W_{BA,n} = 0\} + \Pr\{W_{BA,n} = 1\},$$

where $W_{BA,n} \equiv X_{B,n} - X_{A,n}$ such that $E[W_{BA,n}] = n(\alpha_B - \alpha_A)$ and $Var[W_{BA,n}] = n[\alpha_A(1 - \alpha_A) + \alpha_B(1 - \alpha_B) + 2\alpha_A \alpha_B]$. It is well-known (see, e.g., Arnold (1990), Th. 5.8) that

$$\frac{W_{BA,n} - E[W_{BA,n}]}{\sqrt{Var[W_{BA,n}]}} \to^D N(0,1),$$

which implies $\Pr\{W_{BA,n} = 0\} \to 0$ and $\Pr\{W_{BA,n} = 1\} \to 0$ as $n \to \infty$. Hence, $P(\alpha_A, \alpha_B, n) \to 0$. Re-labeling, it also follows that $P(\alpha_B, \alpha_A, n) \to 0$. ■

Proof of Lemma 3. Suppose, to the contrary, $\lim_{n\to\infty} \alpha_r^*(n) > 0$. Since $\alpha_r^*(n) \in [0, \lambda_r]$, by Bolzano-Weierstrass theorem, there is a subsequence $\widehat{\alpha}_r^*(n)$ that converges to some $\ell > 0$. This implies: $\widehat{\alpha}_r^*(n) > 0$ for a sufficiently large n, and together with Lemma A1, $P(\widehat{\alpha}_r^*(n), \alpha_{r'}^*(n), n) \to 0$ as $n \to \infty$. Using (4), the latter further implies $\Phi_r(\widehat{\alpha}_r^*(n), \alpha_{r'}^*(n)) < 0$ for a sufficiently large n, and thus $\widehat{\alpha}_r^*(n) = 0$ – a contradiction. Hence, $\lim_{n\to\infty} \alpha_r^*(n) = 0$.

To prove the second part, suppose, to the contrary, $\lim_{n\to\infty}[n\alpha_r^*(n)] = \infty$. Then, clearly $\alpha_r^*(n) > 0$ for a large n and thus $\Phi_r(\alpha_r^*(n), \alpha_{r'}^*(n)) = 0$. Moreover, for a fixed n, we can use a multinomial decomposition for the pivot probability as in Lemma A1 above, and find that $P(\alpha_r^*(n), \alpha_{r'}^*(n), n)$ becomes arbitrarily small as n gets large. In particular, $\frac{1}{2}P(\alpha_r^*(n), \alpha_{r'}^*(n), n) < \underline{c}_r$ and $\Phi_r(\alpha_r^*(n), \alpha_{r'}^*(n)) < 0$ for a sufficiently large n, yielding a contradiction. Hence, $\lim_{n\to\infty}[n\alpha_r^*(n)] < \infty$. ■

Proof of Lemma 4. Immediately follows from Lemma 3 and eq.(3). ■

Proof of Lemma 5. Note first that the marginal distribution of $X_{A,n-1}^*$ conditional on X_B is $X_{A,n-1}^* | X_B \sim$Binomial$(n - 1 - X_B, \frac{\alpha_A^*(n)}{1 - \alpha_B^*(n)})$. Since, by Lemma 4, $\alpha_r^*(n) \to 0$ and $n\alpha_r^*(n) \to m_r^* < \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} E[X_{A,n-1}^* | X_B] = m_A^*.$$

Hence, (see, Arnold (1990), Th. 5.5)

$$X_{A,n-1}^* | X_B \xrightarrow{D} \text{Poisson}(m_A^*),$$

which is independent of X_B. The same argument shows

$$X^*_{B,n-1}|X_A \xrightarrow{D} \text{Poisson}(m^*_B).$$

As a result, the limiting distributions, of $X^*_{A,\infty}$ and $X^*_{B,\infty}$ are independent Poissons, and

$$\left(X^*_{A,\infty} + X^*_{B,\infty}\right) \sim \text{Poisson}(m^*_A + m^*_B). \quad \blacksquare$$

References

[1] Aldrich, J., 1997. When is it rational to vote? In: D.C. Mueller, Perspectives on Public Choice: A Handbook. Cambridge University Press, pp. 373-90.

[2] Arnold, S. F., 1990. Mathematical Statistics. Englewood Cliffs, NJ: Prentice Hall.

[3] Blais, A., 2000. To Vote or Not to Vote: The Merits and Limits of Rational Choice Theory, University of Pittsburgh Press.

[4] Borgers, T. , 2004. Costly voting. Amer. Econ. Rev. 94, 57-66.

[5] Campbell, C., 1999. Large electorates and decisive minorities. J. Polit. Economy 107, 1199-1217.

[6] Downs, A. 1957. An Economic Theory of Democracy. New York: Harper and Row.

[7] Feddersen, T. 2004. Rational choice theory and the paradox of not voting. J. Econ. Perspect. 18(1), 99-112.

[8] Feddersen, T., Pesendorfer, W., 1997. Voting behavior and information aggregation in elections with private information. Econometrica 65, 1029-1058.

[9] Ghosal, S., Lockwood, B., forthcoming. Costly voting when both information and preferences differ: is turnout too high or too low? Soc. Choice Welfare.

[10] Goeree, J., Grosser, J., 2007. Welfare reducing polls. Econ. Theory 31(1), 51-68.

[11] Hansen, S., Palfrey, T., Rosenthal, H., 1987. The Downsian model of electoral participation: formal theory and empirical analysis of the constituency size effect. Public Choice 52, 15-33.

[12] Krasa, S., Polborn, M., 2009. Is mandatory voting better than voluntary voting? Games Econ. Behav. 66, 275-291.

[13] Krishna, V., Morgan, J., 2008. Voluntary voting: costs and benefits. mimeo.

[14] Ledyard, J. 1981. The Paradox of voting and candidate Competition: a general equilibrium analysis. In: G. Horwich and J. Quirk (Eds.). Essays in Contemporary Fields of Economics. Purdue University Press.

[15] Ledyard, J., 1984. The pure theory of large two-candidate elections. Public Choice 44(1), 7-41.

[16] Levine, D., Palfrey, T., 2007. The paradox of voter participation? A laboratory study. Amer. Polit. Sci. Rev. 101(1), 143-58.

[17] Merlo, A. 2006. Whither political economy? Theories, facts and issues. In: R. Blundell, W. Newey and T. Persson (eds.), Advances in Economics and Econometrics, Theory and Applications: Cambridge: Cambridge University Press, pp. 381-421.

[18] Myerson, R. 1998. Population uncertainty and poisson games. Int. J. Game Theory 27, 375-392.

[19] Myerson, R., 2000. Large poisson games. J. Econ. Theory 94, 7-45.

[20] Palfrey, T., Rosenthal, H., 1983. A strategic calculus of voting. Public Choice 41, 7-53.

[21] Palfrey, T., Rosenthal, H., 1985. Voter participation and strategic uncertainty. Amer. Polit. Sci. Rev. 79, 62-78.

[22] Taylor, C., Yildirim, H., 2010. Public information and electoral bias. Games Econ. Behav. 68, 353-375.

[23] Razin, R., 2003. Signalling and election motivations in a voting model with common values and responsive candidates. Econometrica 71(4), 1083-1119.