Evaluating the Sample Likelihood of Linearized DSGE Models Without the Use of the Kalman Filter

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(Extremely Preliminary)

We derive a method for constructing the likelihood function of a general class of linearized dynamic general equilibrium models that does not require the application of the Kalman filter. The standard approach is based on a prediction-error decomposition, which expresses the likelihood as a function of unobservable states. By contrast, we view the observed sample as a single draw from a multivariate density, which allows for a representation of the likelihood in terms of observables alone.

Following Schmitt-Grohé and Uribe (2004), we consider a general class of linearized DSGE models where an $n_x \times 1$ state vector x_t and an $n_y \times 1$ control vector y_t evolve according to the law of motion

$$x_{t+1} = h(\theta)x_t + \eta(\theta)\epsilon_{t+1} \tag{1}$$

$$y_t = g(\theta)x_t, \tag{2}$$

where θ is an $n_\theta \times 1$ vector of deep structural parameters, which the econometrician wishes to estimate, $h(\theta)$ is an $n_x \times n_x$ transition matrix with roots inside the unit circle, $\eta(\theta)$ is an $n_x \times n_\epsilon$ matrix, and ϵ_t is an $n_\epsilon \times 1$ Gaussian vector with mean zero and variance-covariance matrix equal to an identity matrix of size $n_\epsilon \times n_\epsilon$. Without loss of generality, assume that $n_y \leq n_\epsilon$. The vector x_t may contain observable and unobservable endogenous and exogenous state variables. The vector y_t is assumed to be observable.

Suppose that the sample consists of T observations of the vector y_t. Let Y denote the $n_yT \times 1$ vector

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}.$$

We can interpret Y as a single draw from a $N(\mu, \Omega)$ distribution, where μ is a vector of order $n_yT \times 1$ and Ω is a matrix of order $n_yT \times n_yT$. Clearly,

$$\mu = \emptyset.$$

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In turn,

$$\begin{aligned}\Omega &= E(YY') \\ &= E\begin{bmatrix} y_1y_1' & y_1y_2' & \cdots & y_1y_T' \\ y_2y_1' & y_2y_2' & \cdots & y_2y_T' \\ \vdots & \vdots & \vdots & \vdots \\ y_Ty_1' & y_Ty_2' & \cdots & y_Ty_T' \end{bmatrix}.\end{aligned}$$

We next show how to compute Ω for a given value of θ. Start with

$$\begin{aligned}Ey_1y_1' &= Eg(\theta)x_1x_1'g(\theta)' \\ &= g(\theta)Ex_1x_1'g(\theta)' \\ &= g(\theta)\Sigma_x g(\theta)',\end{aligned}$$

where Σ_x is the covariance matrix of x_t, which, from the law of motion of x_t, must satisfy

$$\Sigma_x = h(\theta)\Sigma_x h(\theta)' + \eta(\theta)\eta(\theta)'.$$

Given θ, Σ_x can be readily computed.[1] In general,

$$Ey_iy_j' = \begin{cases} g(\theta)\Sigma_x[h(\theta)']^{j-i}g(\theta)' & \text{if } i \leq j \\ g(\theta)[h(\theta)]^{i-j}\Sigma_x g(\theta)' & \text{if } i > j \end{cases},$$

for $i, j = 1, \ldots, T$. It follows that, given θ, the covariance matrix Ω can be readily computed. The sample log likelihood can then be written immediately as

$$\mathcal{L}(\theta|Y) = (-T/2)\ln(2\pi) + \frac{1}{2}\ln|\Omega^{-1}| - \frac{1}{2}(Y-\mu)'\Omega^{-1}(Y-\mu).$$

This completes a procedure for evaluating the sample log likelihood for a linearized DSGE model with unobservable states without use of the Kalman filter.

Handling Measurement Error

Suppose y_t is observed with measurement error. Specifically, suppose that the econometricial observes a vector y_t^o given by

$$y_t^o = y_t + mw_t,$$

where the measurement error vector w_t is an autoregressive process of the form

$$w_t = nw_{t-1} + \nu\mu_t.$$

Note that variables in y_t that are observe without error, give rise to rows of m made up of zeros. Let $\tilde{\theta}$ be a new vector of parameters to be estimated with includes all of the elements of θ plus some elements of m, n, and ν that the econometrician wishes to estimate. Define

$$\tilde{x}_t = \begin{bmatrix} x_t \\ w_t, \end{bmatrix}; \quad \tilde{h}(\tilde{\theta}) = \begin{bmatrix} h(\theta) & \emptyset \\ \emptyset & m \end{bmatrix}; \quad \tilde{\eta}(\tilde{\theta})\begin{bmatrix} \eta & \emptyset \\ \emptyset & \nu \end{bmatrix}; \quad \tilde{\epsilon}_t = \begin{bmatrix} \epsilon_t \\ \mu_t \end{bmatrix}; \quad \tilde{g}(\tilde{\theta}) = \begin{bmatrix} g(\theta) & m \end{bmatrix}.$$

Then one can write

$$\begin{aligned}\tilde{x}_{t+1} &= \tilde{h}(\tilde{\theta})\tilde{x}_t + \tilde{\eta}(\tilde{\theta})\tilde{\epsilon}_{t+1} \\ y_t^o &= \tilde{g}(\tilde{\theta})\tilde{x}_t.\end{aligned}$$

This system has the same structure as (1) and (2), so its associated sample likelihood sample can be constructed applying the procedure described in the previous section.

[1]For example, by $\text{vec}(\Sigma_x) = (I - h(\theta)\otimes h(\theta))^{-1}\ \text{vec}(\eta(\theta)\eta(\theta)')$. For alternative algorithms for computing Σ_x, see the program mom.m on our websites.

Reference

Schmitt-Grohé, Stephanie and Martín Uribe, "Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function," *Journal of Economic Dynamics and Control 28,* January 2004, 755-775.