Answer key to problem set # 5

ECON 342

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Problem (Hayashi Chapter 6, page 375 #1). *Solution.* If $\{\gamma_j\}$ is absolutely summable then:

$$\sum_{j=-\infty}^{\infty} |\gamma_j| < \infty$$

then,

$$\sum_{j=1}^{n} |\gamma_j| < \infty \text{ and } \sum_{j=1}^{m} |\gamma_j| < \infty$$

Since these two sequences are bounded, and convergent, then they're Cauchy. Thus,

$$\sum_{j=1}^{n} |\gamma_j| - \sum_{j=1}^{m} |\gamma_j| = \sum_{j=n+1}^{m} |\gamma_j| \to 0$$

$\diamond$

Problem (Hayashi Chapter 6, page 385 #1). *Solution.* From the book we have,

$$\hat{\mathbb{E}}(y_t|1, y_{t-1}) = a + by_{t-1}$$

and

$$b = \frac{\text{cov}\,(y_t, y_{t-1})}{\text{V}\,(y_{t-1})} = \frac{\phi\gamma_0}{\gamma_0} = \phi,$$

and $a = \mathbb{E}\,(y_t) - b\mathbb{E}\,(y_{t-1}) \Rightarrow a = \mathbb{E}\,(y_t)\,(1-\phi)$ when $|\phi| < 1$.

In the AR(1) model,

$$y_t = \mu + \phi y_{t-1} + \varepsilon_t$$

then,

$$\mathbb{E}\,(y_t|y_{t-1}) = c + \phi y_{t-1} + \mathbb{E}\,(\varepsilon_t|y_{t-1}) = c + \phi y_{t-1} \qquad \text{(Assuming } \varepsilon_t \text{ is independent)}$$

and $c = \mathbb{E}\,(y_t)\,(1-\phi)$. Then, we can conclude,

$$\hat{\mathbb{E}}(y_t|1, y_{t-1}) = \mathbb{E}\,(y_t|y_{t-1})$$

Similarly,

$$\hat{\mathbb{E}}(y_t|1, y_{t-1}, y_{t-2}) = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2}$$

where,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \mathrm{V}\left(y_{t-1}\right) & \mathrm{cov}\left(y_{t-1}, y_{t-2}\right) \\ \mathrm{cov}\left(y_{t-1}, y_{t-2}\right) & \mathrm{V}\left(y_{t-2}\right) \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{cov}\left(y_{t-1}, y_t\right) \\ \mathrm{cov}\left(y_{t-2}, y_t\right) \end{pmatrix}$$

When $|\phi| < 1$ we have:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$$

Note that,

$$\gamma_0^{-1} \begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix}^{-1} = \frac{1}{\gamma_0 - \gamma_0 \phi^2} \begin{pmatrix} 1 & -\phi \\ -\phi & 1 \end{pmatrix}$$

then,

$$\begin{aligned} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} &= \frac{1}{\gamma_0 - \gamma_0 \phi^2} \begin{pmatrix} 1 & -\phi \\ -\phi & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \\ &= \frac{1}{\gamma_0 - \gamma_0 \phi^2} \begin{pmatrix} \gamma_1 - \phi\gamma_2 \\ \gamma_2 - \phi\gamma_1 \end{pmatrix} = \frac{1}{\gamma_0 - \gamma_0 \phi^2} \begin{pmatrix} \phi\gamma_0 - \phi^3\gamma_0 \\ \phi^2\gamma_0 - \phi^2\gamma_0 \end{pmatrix} = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \end{aligned}$$

then,

$$\hat{\mathbb{E}}(y_t | 1, y_{t-1}, y_{t-2}) = c + \phi y_{t-1}$$

When $|\phi| > 1$, we solve the differential equation forward giving,

$$y_t = \mu - \sum_{j=-\infty}^{\infty} \phi^{-j} \varepsilon_{t+j} \Rightarrow$$

then,

$$\mathbb{E}\left(y_{t-1}\varepsilon_t\right) = \mathbb{E}\left(\mu\varepsilon_t - phi^{-1}\varepsilon_t^2 - phi^{-2}\varepsilon_{t+1}\varepsilon_t - \cdots\right) = -\phi^{-1}\sigma^2 \neq 0$$

then,

$$\mathbb{E}\left(y_t | y_{t-1}\right) = c + \phi y_{t-1} + \mathbb{E}\left(\varepsilon_t | y_{t-1}\right) \neq c + \phi y_{t-1}$$

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Problem (Hayashi Chapter 6, page 399 #1)**.** *Solution.* Re-write the AR(1) process as,

$$y_t - \mu = \phi(y_{t-1} - \mu) + \varepsilon_t$$

then, the OLS estimate fro ϕ is,

$$\hat{\phi} = \frac{\sum_{t=1}^{T} (y_{t-1} - \mu)(y_t - \mu)}{\sum_{t=1}^{T} (y_{t-1} - \mu)^2}$$

the sampling error yields,

$$\sqrt{T}(\hat{\phi} - \phi) = \frac{T^{-1/2} \sum_{t=1}^{T} (y_{t-1} - \mu)\varepsilon_t}{T^{-1} \sum_{t=1}^{T} (y_{t-1} - \mu)^2}$$

Note that,

$$T^{-1}\sum_{t=1}^{T}(y_{t-1}-\mu)^2 \xrightarrow{p} \mathrm{V}\left(y_{t-1}\right)=\gamma_0=\frac{\sigma^2}{1-\phi^2}$$

$$T^{-1/2}\sum_{t=1}^{T}(y_{t-1}-\mu)\varepsilon_t \xrightarrow{d} N(0,\sigma^2\gamma_0)$$

thus,

$$\sqrt{T}(\hat{\phi}-\phi)\xrightarrow{d} N\left(0,\sigma^2\frac{\gamma_0}{\gamma_0^2}\right)=N(0,1-\phi^2)$$

Similarly, re-write the AR(p) process as,

$$y_t-\mu=\phi_1(y_{t-1}-\mu)+\cdots+\phi_p(y_{t-p}-\mu)+\varepsilon_t=\mathbf{x}_t'\beta+\varepsilon_t$$

with $\beta=(\phi_1,\ldots,\phi_p)'$, and $\mathbf{x}_t=(y_{t-1}-\mu,\ldots,y_{t-p}-\mu)'$. Thus,

$$\hat{\beta}=\left(\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t'\right)^{-1}\sum_{t=1}^{T}\mathbf{x}_t y_t$$

then,

$$\sqrt{T}(\hat{\beta}-\beta)\xrightarrow{d} N\left(0,\sigma^2\left[\mathbb{E}\left(\mathbf{x}_t\mathbf{x}_t'\right)\right]^{-1}\right)$$

You can now show the desired result easily.

$\diamondsuit$

Problem (2). *Solution.* As in the previous problem, re-write the AR(1) process as,

$$y_t-\mu=\phi(y_{t-1}-\mu)+\varepsilon_t$$

then, the OLS estimate fro ϕ is,

$$\hat{\phi}=\frac{\sum_{t=1}^{T}(y_{t-1}-\mu)(y_t-\mu)}{\sum_{t=1}^{T}(y_{t-1}-\mu)^2}$$

thus,

$$\sqrt{T}(\hat{\phi}-\phi)\xrightarrow{d} N\left(0,\sigma^2\frac{\gamma_0}{\gamma_0^2}\right)=N(0,1-\phi^2)$$

$\diamondsuit$

Problem (3). *Solution.* Let the AR(2) process be,

$$y_t=c+\phi_1 y_{t-1}+\phi_2 y_{t-2}+\varepsilon_t \tag{1}$$

which can be re-written using the lag operator as:

$$\left(1-\phi_1 L-\phi_2 L^2\right) y_t=c+\varepsilon_t \tag{2}$$

This differential equation is stable provided that the roots of

$$1-\phi_1 z-\phi_2 z^2=0$$

which in our case is,

$$1+0.5z-0.25z^2=0$$

You can show that $z=-1.24<-1$ and $z=3.24>1$ then $|z|>1$, thus the system is stationary.

To find the first moment y_t take expectations directly,

$$\mathbb{E}\left(y_t\right)=c+\phi_1 \mathbb{E}\left(y_{t-1}\right)+\phi_2 \mathbb{E}\left(y_{t-2}\right) \Rightarrow \mu=\mathbb{E}\left(y_t\right)=\frac{c}{1-\phi_1-\phi_2}$$

To obtain the second moments re-write (1) as,

$$y_t=\mu(1-\phi_1-\phi_2)+\phi_1 y_{t-1}+\phi_2 y_{t-2}+\varepsilon_t$$

or,

$$y_t-\mu=\phi_1(y_{t-1}-\mu)+\phi_2(y_{t-2}-\mu)+\varepsilon_t$$

then:

$$\begin{aligned}
\gamma_0=\mathrm{V}\left(y_t\right)&=\mathbb{E}\left(y_t-\mu\right)^2\\
&=\mathbb{E}\left((\phi_1(y_{t-1}-\mu)+\phi_2(y_{t-2}-\mu)+\varepsilon_t)(y_t-\mu)\right)\\
&=\phi_1\gamma_1+\phi_2\gamma_2+\mathcal{E}\left(\varepsilon_t(y_t-\mu)\right)\\
&=\phi_1\gamma_1+\phi_2\gamma_2+\sigma^2
\end{aligned}$$

Higher order-autocovariances are given by:

$$\begin{aligned}
\gamma_1&=\mathbb{E}\left((y_t-\mu)(y_{t-1}-\mu)\right)\\
&=expt(\phi_1(y_{t-1}-\mu)+\phi_2(y_{t-2}-\mu)+\varepsilon_t)(y_{t-1}-\mu)\\
&=\phi_1\gamma_0+\phi_2\gamma_1\\
\gamma_2&=\mathbb{E}\left((y_t-\mu)(y_{t-2}-\mu)\right)\\
&=expt(\phi_1(y_{t-1}-\mu)+\phi_2(y_{t-2}-\mu)+\varepsilon_t)(y_{t-2}-\mu)\\
&=\phi_1\gamma_1+\phi_0\gamma_1\\
&\vdots\\
\gamma_j&=\phi_1\gamma_{j-1}+\phi_2\gamma_{j-2}
\end{aligned}$$

Then, the autocorrelations are:

$$\rho_j=\phi_1\rho_{j-1}+\phi_2\rho_{j-2} \qquad j=1,2,\cdots$$

in particular,

$$\rho_0 = 1$$
$$\rho_1 = \phi_1 + \phi_2\rho_1 \;\Rightarrow\; \rho_1 = \frac{\phi_1}{1-\phi_2}$$
$$\rho_2 = \phi_1\rho_1 + \phi_2 = \frac{\phi_1^2}{1-\phi_2} + \phi_2$$

Using these results, our expression for γ_0,

$$\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2 = \phi_1\rho_1\gamma_0 + \phi_2\rho_2\gamma_0 + \sigma^2$$

can be expressed as

$$\gamma_0 = \frac{(1-\phi_2)\,\sigma^2}{(1+\phi_2)\left[(1-\phi_2)^2 - \phi_1^2\right]}$$

$\Diamond$