

Answer key to problem set # 2  
 ECON 342  
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**Problem 1.** For  $T = 2$  consider the standard panel data model:

$$y_{it} = x_{it}\beta + \alpha_i + \epsilon_{it}$$

- a) Numerically compare the fixed effect and first difference estimates.  
 b) Compare the error variance estimates from the two methods.

*Solution.* The difference estimates are obtained by taking the difference across time periods to eliminate the unobservable. Hence, for individual  $i$  we have,

$$y_{i2} - y_{i1} = (x_{i2} - x_{i1})\beta + (\epsilon_{i2} - \epsilon_{i1}) \Rightarrow \Delta y_i = \Delta x_i\beta + \Delta \epsilon_i$$

Assuming that  $\mathbb{E}(\Delta x_i' \Delta \epsilon_i) = 0$ , the difference estimate of  $\beta$  is,

$$\hat{\beta}_{DE} = \left[ \sum_{i=1}^N \Delta x_i' \Delta x_i \right]^{-1} \sum_{i=1}^N \Delta x_i' \Delta y_i$$

The fixed effects estimator is obtained demeaning each equation. Let  $\bar{y}_i = \frac{y_{i1} + y_{i2}}{2}$ ,  $\bar{x}_i = \frac{x_{i1} + x_{i2}}{2}$ , and  $\bar{\epsilon}_i = \frac{\epsilon_{i1} + \epsilon_{i2}}{2}$  then:

$$y_{it} - \bar{y}_i = [x_{it} - \bar{x}_i]\beta + \epsilon_{it} - \bar{\epsilon}_i, \quad t = 1, 2$$

then, the fixed effect estimate of  $\beta$  is,

$$\hat{\beta}_{FE} = \left[ \sum_{i=1}^N \sum_{t=1}^2 [x_{it} - \bar{x}_i]' [x_{it} - \bar{x}_i] \right]^{-1} \sum_{i=1}^N \sum_{t=1}^2 [x_{it} - \bar{x}_i]' [y_{it} - \bar{y}_i]$$

Now, note that,

$$\begin{aligned} \sum_{t=1}^2 [x_{it} - \bar{x}_i]' [x_{it} - \bar{x}_i] &= \sum_{t=1}^2 \left[ x_{it} - \frac{x_{i1} + x_{i2}}{2} \right]' \left[ x_{it} - \frac{x_{i1} + x_{i2}}{2} \right] \\ &= \left[ \frac{x_{i1} - x_{i2}}{2} \right]' \left[ \frac{x_{i1} - x_{i2}}{2} \right] + \left[ \frac{x_{i2} - x_{i1}}{2} \right]' \left[ \frac{x_{i2} - x_{i1}}{2} \right] \\ &= \frac{\Delta x_i' \Delta x_i}{2} \end{aligned}$$

and,

$$\begin{aligned}\sum_{t=1}^2 [x_{it} - \bar{x}_i]' [y_{it} - \bar{y}_i] &= \sum_{t=1}^2 \left[ x_{it} - \frac{x_{i1} + x_{i2}}{2} \right]' \left[ y_{it} - \frac{y_{i1} + y_{i2}}{2} \right] \\ &= \left[ \frac{x_{i1} - x_{i2}}{2} \right]' \left[ \frac{y_{i1} - y_{i2}}{2} \right] + \left[ \frac{x_{i2} - x_{i1}}{2} \right]' \left[ \frac{y_{i2} - y_{i1}}{2} \right] \\ &= \frac{\Delta x_i' \Delta y_i}{2}\end{aligned}$$

Replacing these results into the FE estimator we have:

$$\hat{\beta}_{FE} = \left[ \sum_{i=1}^N \frac{\Delta x_i' \Delta x_i}{2} \right]^{-1} \sum_{i=1}^N \frac{\Delta x_i' \Delta y_i}{2} = \hat{\beta}_{DE}$$

Thus,

$$\hat{\beta}_{FE} = \hat{\beta}_{DE}.$$

The variance covariance matrix of  $\hat{\beta}_{DE}$  equals to,

$$\hat{\sigma}_{DE}^2 \left[ \sum_{i=1}^N \Delta x_i' \Delta x_i \right]^{-1}$$

where the residuals from the difference estimator,  $\hat{e}_i^{DE} = \Delta y_i - \Delta x_i \hat{\beta}_{DE}$ , are used to calculate,

$$\hat{\sigma}_{DE}^2 = \frac{\sum_{i=1}^N (\hat{e}_i^{DE})^2}{N - k}$$

On the other hand, variance covariance matrix of  $\hat{\beta}_{FE}$  equals to,

$$\hat{\sigma}_{FE}^2 \left[ \sum_{i=1}^N \sum_{t=1}^2 [x_{it} - \bar{x}_i]' [x_{it} - \bar{x}_i] \right]^{-1}$$

where the residuals from the fixed-effect estimator,

$$\hat{e}_{it}^{FE} = [y_{it} - \bar{y}_i] - [x_{it} - \bar{x}_i] \hat{\beta}_{FE} \quad t = 1, 2,$$

are used to calculate,

$$\hat{\sigma}_{FE}^2 = \frac{\sum_{i=1}^N (\hat{e}_{i1}^{FE} + \hat{e}_{i2}^{FE})^2}{N - k}$$

Using the fact that  $\hat{\beta}_{DE} = \hat{\beta}_{FE}$ :

$$\begin{aligned}\hat{e}_{i1}^{FE} &= \left[ \frac{y_{i1} - y_{i2}}{2} \right] - \left[ \frac{x_{i1} - x_{i2}}{2} \right] \hat{\beta}_{FE} \\ &= \frac{-\Delta y_i + \Delta x_i \hat{\beta}_{DE}}{2} = -\frac{\hat{e}_i^{DE}}{2} \\ \hat{e}_{i2}^{FE} &= \left[ \frac{y_{i2} - y_{i1}}{2} \right] - \left[ \frac{x_{i2} - x_{i1}}{2} \right] \hat{\beta}_{FE} \\ &= \frac{\Delta y_i - \Delta x_i \hat{\beta}_{DE}}{2} = \frac{\hat{e}_i^{DE}}{2}\end{aligned}$$

Then, the sum of squared residuals from the fixed effect estimator is,

$$\begin{aligned}\sigma^2(FE) &= \frac{\sum_i [(\hat{e}_{i1}^{FE})^2 + (\hat{e}_{i2}^{FE})^2]}{N - k} \\ &= \frac{\sum_i [(-\hat{e}_i^{DE})^2 + (\hat{e}_i^{DE})^2]}{4(N - k)} \\ &= \frac{1}{2} \frac{\sum_i [(\hat{e}_i^{DE})^2]}{N - k} = \frac{1}{2} \sigma^2(DE)\end{aligned}$$

Therefore,

$$\begin{aligned}V(\hat{\beta}_{DE}) &= \hat{\sigma}_{DE}^2 \left[ \sum_{i=1}^N \Delta x_i' \Delta x_i \right]^{-1} \\ &= \frac{\hat{\sigma}_{DE}^2}{2} \left[ \sum_{i=1}^N \frac{\Delta x_i' \Delta x_i}{2} \right]^{-1} \\ &= \sigma^2(FE) \left[ \sum_{i=1}^N \sum_{t=1}^2 [x_{it} - \bar{x}_i]' [x_{it} - \bar{x}_i] \right]^{-1} \\ &= V(\hat{\beta}_{FE})\end{aligned}$$

◇

**Problem 2.** Consider the following panel data model:

$$y_{it} = \alpha_i + x_{it}\beta + z_i\gamma + \varepsilon_{it}$$

Let  $x_i = (x_{i1}, \dots, x_{iT})$ , and assume  $\mathbb{E}(\varepsilon_{it}|x_i, z_i, \alpha_i) = 0$ . Let  $\sigma_\alpha^2 = V(\alpha_i)$  and  $\sigma_\varepsilon^2 = V(\varepsilon_{it})$ .

- a) Let  $c_i = \alpha_i + z_i\gamma$ . Find  $V(c_i)$  and compare it to  $\sigma_\alpha^2$ .
- b) Compare the estimated variance of the unobserved effect when estimating the model by fixed effects to the estimated variance of the unobserved effect to if we estimated the model by random effects..

*Solution.* Given the general assumptions we have:

$$\begin{aligned}\mathbb{E}(c_i|x_i, z_i) &= \mathbb{E}(\alpha_i + z_i\gamma|x_i, z_i) \\ &= \underbrace{\mathbb{E}(\alpha_i|x_i, z_i)}_0 + z_i\gamma \\ &= z_i\gamma \\ \mathbb{E}(c_i^2|x_i, z_i) &= \mathbb{E}(\alpha_i^2 + z_i^2\gamma^2 + \alpha_i z_i\gamma|x_i, z_i) \\ &= \mathbb{E}(\alpha_i^2|x_i, z_i) + z_i^2\gamma^2 + \mathbb{E}(\alpha_i|x_i, z_i) z_i\gamma \\ &= \sigma_\alpha^2 + z_i^2\gamma^2\end{aligned}$$

then,

$$V(c_i|x_i, z_i) = \sigma_\alpha^2 + z_i^2\gamma^2 - z_i^2\gamma^2 = \sigma_\alpha^2$$

Using the conditional variance identity we have:

$$\begin{aligned} V(c_i) &= V(\mathbb{E}(c_i|x_i, z_i)) + \mathbb{E}(V(c_i|x_i, z_i)) \\ &= \gamma^2 V(z_i) + \sigma_\alpha^2 \\ &> \sigma_\alpha^2 \end{aligned}$$

For (b) note that when you estimate a fixed effects model, the unobserved effect that is estimated is  $c_i$ , while when you estimate a random effects model  $\alpha_i$  is the unobserved effect as  $z_i$  can enter as an explanatory variable. From (a) it follows that the estimated variance of the unobserved effect is larger when it is estimated using fixed-effects.  $\diamond$

**Problem** (Hayashi Analytical Exercise #1). *Solution.* (a) Following the hint in the book, let,

$$M_D = I_{Mn} - D(D'D)^{-1}D'$$

then, we can pre-multiply the original model by this annihilator matrix associated to  $D$  and obtain,

$$M_D y = \cancel{M_D D} \alpha + M_D F \beta + M_D \eta$$

The estimate of  $\beta$  is,

$$\hat{\beta} = [F' M_D M_D F]^{-1} F' M_D M_D y$$

To show that this estimate equals the fixed effects estimator it must be true that  $M_D F = \tilde{F}$  which is true if  $M_D = I_n \otimes Q$ , where  $Q = I_M - \iota_M (\iota_M' \iota_M)^{-1} \iota_M'$ ,

$$\begin{aligned} M_D &= I_{Mn} - (I_n \otimes \iota_M) ((I_n \otimes \iota_M)' (I_n \otimes \iota_M))^{-1} (I_n \otimes \iota_M)' \\ &= I_n \otimes I_M - (I_n \otimes \iota_M) ((I_n \otimes \iota_M)' (I_n \otimes \iota_M))^{-1} (I_n \otimes \iota_M)' \\ &= I_n \otimes I_M - (I_n \otimes \iota_M) [(I_n' I_n)^{-1} \otimes (\iota_M' \iota_M)^{-1}] (I_n \otimes \iota_M)' \\ &= I_n \otimes I_M - (I_n \otimes \iota_M) (I_n \otimes M^{-1}) (I_n \otimes \iota_M)' \\ &= I_n \otimes I_M - (I_n \otimes \iota_M M^{-1}) (I_n \otimes \iota_M)' \\ &= I_n \otimes I_M - (I_n \otimes \iota_M \iota_M' M^{-1}) \\ &= I_n \otimes [I_M - \iota_M \iota_M' M^{-1}] \\ &= I_n \otimes Q \end{aligned}$$

as we wanted to show.

(b) We have that,

$$\hat{\alpha} = (D'D)^{-1} (D'y - D'F\hat{\beta})$$

then,

$$\begin{aligned} D'D &= (I_n \otimes \iota_M)'(I_n \otimes \iota_M) = (I_n' I_n) \otimes (\iota_M' \iota_M) = MI_n \\ D'y &= (I_n \otimes \iota_M)'y \\ D'F &= (I_n \otimes \iota_M)'F \end{aligned}$$

The  $i$  –  $th$  element of  $\hat{\alpha}$  can be written as,

$$\hat{\alpha}_i = \frac{1}{M}(\iota_M' y_i - \iota_M' F_i \hat{\beta}) = \bar{y}_i - \frac{1}{M} \iota_M' F_i \hat{\beta}$$

as we wanted to show.

(c) Given the assumptions of the model, we have that:

$$\begin{aligned} \mathbb{E}(\eta_i | W) &= \mathbb{E}(\eta_i | F, D) \\ &= \mathbb{E}(\eta_i | F, D) = 0 && \text{(since D is full of constants)} \\ &= \mathbb{E}(\eta_i | F_i) && \text{(by assumption (i))} \\ &= \mathbf{0} && \text{(by assumption (ii))} \end{aligned}$$

thus, the assumption of strict exogeneity holds.

$$\begin{aligned} \mathbb{E}(\eta_i \eta_i' | W) &= \mathbb{E}(\eta_i \eta_i' | F, D) \\ &= \mathbb{E}(\eta_i \eta_i' | F) = \sigma_\eta^2 I_M \text{ by assumption (iii)} \end{aligned}$$

and,

$$\begin{aligned} \mathbb{E}(\eta_i \eta_j' | W) &= \mathbb{E}(\eta_i \eta_j' | F, D) \\ &= \mathbb{E}(\eta_i \mathbb{E}(\eta_j' | F_j) | F) = \mathbf{0} \text{ by assumption (iii)} \end{aligned}$$

therefore, the residual is spherical, and the assumptions of the classical regression hold.  $\diamond$

**Problem** (Hayashi Analytical Exercise #2). *Solution.* (a) To show this is true consider,

$$C = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$$

then,

$$C'I_M = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+1+0 \\ 1-1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In the case where  $C$  is created from  $Q$ , the identity follows directly.

(b) The model is,

$$y_i = F_i\beta + \iota_M b'_i \gamma + \iota_m \alpha_i + \eta_i$$

which multiplied by  $C'$  yields,

$$C' y_i = C' F_i \beta + C' \iota_M b'_i \gamma + C' \iota_m \alpha_i + C' \eta_i$$

$$C' y_i = C' F_i \beta + C' \eta_i$$

since  $C' \iota_M = 0$ .

(d) We have that,

$$S_{xz} = \frac{1}{n} \sum_i (C' F_i) \otimes x_i = (C' \otimes I_K) \left( \frac{1}{n} \sum_i F_i \otimes x_i \right)$$

$$s_{xy} = (C' \otimes I_K) \left( \frac{1}{n} \sum_i y_i \otimes x_i \right)$$

Using  $\hat{W} = (CC')^{-1} \otimes \left( \frac{1}{n} \sum_i x_i x'_i \right)^{-1}$  we have,

$$\begin{aligned} \beta_{GMM} &= \left( S'_{xz} \hat{W} S_{xz} \right)^{-1} S_{xz} \hat{W} s_{xy} \\ &= \left[ \left( \frac{1}{n} \sum_i F_i \otimes x_i \right)' (C' \otimes I_K)' (CC')^{-1} \otimes \left( \frac{1}{n} \sum_i x_i x'_i \right)^{-1} (C' \otimes I_K) \left( \frac{1}{n} \sum_i F_i \otimes x_i \right) \right]^{-1} \\ &\quad \times \left( \frac{1}{n} \sum_i F_i \otimes x_i \right)' (C' \otimes I_K)' (CC')^{-1} \otimes \left( \frac{1}{n} \sum_i x_i x'_i \right)^{-1} (C' \otimes I_K) \left( \frac{1}{n} \sum_i y_i \otimes x_i \right) \\ &= \left[ \left( \frac{1}{n} \sum_i F'_i Q F_i \right) \otimes \frac{1}{n} \sum_i x'_i \left( \frac{1}{n} \sum_i x_i x'_i \right)^{-1} x_i \right]^{-1} \\ &\quad \times \left( \frac{1}{n} \sum_i F'_i Q y_i \right) \otimes \frac{1}{n} \sum_i x'_i \left( \frac{1}{n} \sum_i x_i x'_i \right)^{-1} x_i \\ &= \left[ \left( \frac{1}{n} \sum_i F'_i Q F_i \right) \right]^{-1} \times \left( \frac{1}{n} \sum_i F'_i Q y_i \right) \end{aligned}$$

which is the fixed effect estimator.

(e) In this case the efficient weighting matrix  $W = S^{-1}$ , with  $S = \mathcal{E}[\eta_i \eta'_i] \otimes \mathcal{E}[x_i x'_i]$ . Therefore the efficient weighting matrix is given by,

$$\hat{W} = \hat{\Psi}^{-1} \otimes \left( \frac{1}{n} \sum_i x_i x'_i \right)^{-1}$$

The efficient GMM estimator is,

$$\begin{aligned}
\beta_{GMM} &= \left[ \frac{1}{n} \sum_i \tilde{F}_i' \otimes x_i' \hat{\Psi}^{-1} \otimes \left( \frac{1}{n} \sum_i x_i x_i' \right)^{-1} \frac{1}{n} \sum_i \tilde{F}_i \otimes x_i \right]^{-1} \\
&\quad \times \frac{1}{n} \sum_i \tilde{F}_i' \otimes x_i' \hat{\Psi}^{-1} \otimes \left( \frac{1}{n} \sum_i x_i x_i' \right)^{-1} \frac{1}{n} \sum_i \tilde{y}_i \otimes x_i \\
&= \left[ \frac{1}{n} \sum_i \tilde{F}_i' \hat{\Psi}^{-1} \tilde{F}_i \otimes \frac{1}{n} \sum_i x_i' \otimes \left( \frac{1}{n} \sum_i x_i x_i' \right)^{-1} \right]^{-1} \\
&\quad \frac{1}{n} \sum_i \tilde{F}_i' \hat{\Psi}^{-1} \tilde{y}_i \frac{1}{n} \sum_i x_i' \otimes \left( \frac{1}{n} \sum_i x_i x_i' \right)^{-1} x_i \\
&= \left[ \frac{1}{n} \sum_i \tilde{F}_i' \hat{\Psi}^{-1} \tilde{F}_i \right]^{-1} \frac{1}{n} \sum_i \tilde{F}_i' \hat{\Psi}^{-1} \tilde{y}_i
\end{aligned}$$

To obtain the asymptotic variance of the GMM estimator note that,

$$\begin{aligned}
\beta_{GMM} &= \left[ \frac{1}{n} \sum_i \tilde{F}_i' \hat{\Psi}^{-1} \tilde{F}_i \right]^{-1} \frac{1}{n} \sum_i \tilde{F}_i' \hat{\Psi}^{-1} [\tilde{F}_i \beta + \tilde{\eta}_i] \\
&= \beta + \left[ \frac{1}{n} \sum_i \tilde{F}_i' \hat{\Psi}^{-1} \tilde{F}_i \right]^{-1} \frac{1}{n} \sum_i \tilde{F}_i' \hat{\Psi}^{-1} \tilde{\eta}_i
\end{aligned}$$

Using the sampling error equation we can easily obtain the asymptotic variance of this estimator.

(f) The proposed estimate of  $\Psi$  satisfies the conditions of Proposition 4.1. Most importantly, the residuals are calculated using a consistent estimate of  $\beta$ , and the cross moment correlation between the regressors exists and is of column full rank. Hence, it is a consistent estimate of  $\Psi$ .

(g) Sargan test in this case equals,

$$\begin{aligned}
J &= n g_n(\hat{\beta}) \hat{S}^{-1} g_n(\hat{\beta}) \\
&= n \left[ \frac{1}{n} \sum_i \tilde{F}_i \otimes \tilde{y}_i - \frac{1}{n} \sum_i \tilde{F}_i \otimes x_i \hat{\beta} \right] \left[ \hat{\Psi}^{-1} \otimes \left( \frac{1}{n} \sum_i x_i x_i' \right)^{-1} \right] \left[ \frac{1}{n} \sum_i \tilde{F}_i \otimes \tilde{y}_i - \frac{1}{n} \sum_i \tilde{F}_i \otimes x_i \hat{\beta} \right]
\end{aligned}$$

(h) The first result that has to be verified follows from the fact that,

$$\Psi = \mathcal{E} [\tilde{\eta}_i \tilde{\eta}_i'] = \mathcal{E} [C' \eta_i \eta_i' C] = \sigma_\eta^2 C' I_M C = \sigma_\eta^2 C' C$$

hence, a consistent estimator of  $\Psi$  is,

$$\hat{\Psi} = \hat{\sigma}_\eta^2 C' C$$

Replacing this value of the estimator found in (e) we obtain,

$$\begin{aligned} \hat{\beta} &= \left[ \frac{1}{n} \sum_i \tilde{F}'_i [\hat{\sigma}_\eta^2]^{-1} (C' C)^{-1} \tilde{F}_i \right]^{-1} \frac{1}{n} \sum_i \tilde{F}'_i [\hat{\sigma}_\eta^2]^{-1} (C' C)^{-1} \tilde{y}_i \\ &= \left[ \frac{1}{n} \sum_i \tilde{F}'_i (C' C)^{-1} \tilde{F}_i \right]^{-1} \frac{1}{n} \sum_i \tilde{F}'_i (C' C)^{-1} \tilde{y}_i \\ &= \left[ \frac{1}{n} \sum_i F'_i C (C' C)^{-1} C' F_i \right]^{-1} \frac{1}{n} \sum_i F'_i C (C' C)^{-1} C y_i \\ &= \left[ \frac{1}{n} \sum_i F'_i Q F_i \right]^{-1} \frac{1}{n} \sum_i F'_i Q y_i \\ &= \left[ \frac{1}{n} \sum_i F'_i Q' Q F_i \right]^{-1} \frac{1}{n} \sum_i F'_i Q' Q y_i \end{aligned}$$

which if the fixed effects estimator. This was derived using the fact that  $Q = C(C' C)^{-1} C'$  and that  $Q$  is an idempotent matrix. ◇

**Problem** (Hayashi Analytical Exercise #4). *Solution.* (b) Let  $y_{i0}$  be given, then:

$$\begin{aligned} y_{i1} &= \alpha_i + \rho y_{i0} + \eta_{i1} \\ y_{i2} &= \alpha_i + \rho(\alpha_i + \rho y_{i0} + \eta_{i1}) + \eta_{i2} \\ &= \alpha_i(1 + \rho) + \rho^2 y_{i0} + \rho \eta_{i1} + \eta_{i2} \\ &\vdots \\ y_{im} &= \alpha_i \frac{1 + \rho^m}{1 - \rho} + \rho^m y_{i0} + \eta_{im} + \rho \eta_{i,m-1} + \cdots + \rho^{m-1} \eta_{i1} \end{aligned}$$

Multiplying this last equation by  $\eta_{ih}$  and taking expectations we have:

$$\begin{aligned} \mathbb{E}(y_{im} \eta_{ih}) &= \mathbb{E}(\alpha_i \eta_{ih}) \frac{1 + \rho^m}{1 - \rho} + \rho^m \mathbb{E}(y_{i0} \eta_{ih}) + \mathbb{E}(\eta_{ih} \eta_{im}) + \rho \mathbb{E}(\eta_{ih} \eta_{i,m-1}) + \cdots \\ &\quad + \rho^{m-1} \mathbb{E}(\eta_{i1} \eta_{ih}) \end{aligned}$$

By assumption we have that  $\mathbb{E}(\alpha_i \eta_{ih}) = 0$ ,  $\mathbb{E}(y_{i0} \eta_{ih}) = 0$  and  $\mathbb{E}(\eta_{ih} \eta_{i,m-j}) = 0$ , then

$$\mathbb{E}(y_{im} \eta_{ih}) = 0.$$

(c) We use again the recursion, and multiply this last equation by  $\eta_{i,m-j}$ ,

$$\begin{aligned} \mathbb{E}(y_{im}\eta_{i,m-j}) = & \mathbb{E}(\alpha_i\eta_{i,m-j}) \frac{1 + \rho^m}{1 - \rho} + \rho^m \mathbb{E}(y_{i0}\eta_{i,m-j}) + \mathbb{E}(\eta_{i,m-j}\eta_{im}) + \rho \mathbb{E}(\eta_{i,m-j}\eta_{i,m-1}) \\ & + \dots + \rho^{m-j} \mathbb{E}(\eta_{i,m-j}^2) + \dots + \rho^{m-1} \mathbb{E}(\eta_{i1}\eta_{ih}) \end{aligned}$$

then,

$$\mathbb{E}(y_{im}\eta_{i,m-j}) = \rho^{m-j} \sigma_\eta^2$$

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