1. (Total 34 points) A random sample $X_1, X_2, ..., X_n$ is taken from an exponential distribution with common probability density function:

$$f_X(x|\lambda) = \frac{1}{\lambda} e^{-x/\lambda} \quad x \geq 0 \quad \lambda > 0;$$

Our aim is to estimate $\lambda$ from the random sample.

(a) (10) Write down the log-likelihood function, defined as the log of the joint probability function. Recall that since the random variables in the random sample are mutually independent, the joint probability function is simply the product of the marginal probability functions.

(b) (12) Derive the form of the MLE for $\lambda$ and derive its asymptotic distribution.

(c) (12) Derive the analytic form of $E[X_i^2]$ . Use this result construct a method of moments estimator for $\lambda$, based on this specific moment, and prove it is consistent.

2. (Total 25 points) Suppose $(X_i, Y_i)$ is a serially independent sample from a sequence of jointly normal distributions with $E[X_i] = E[Y_i] = \mu_i$, $\text{Var}[X_i] = \text{Var}[Y_i] = \sigma^2$, and $\text{Cov}[X_i, Y_i] = 0$ (i.e. $X_i$ and $Y_i$ are independent with common but varying means and constant common variance). Derive the MLE of $\sigma^2$ and show that it is inconsistent. Propose an estimator that is consistent.

3. (Total 25 points.) For the MA(1) process defined by

$$y_t = \epsilon_t - \rho \epsilon_{t-1}$$

where $\epsilon_t$ is i.i.d. with mean 0 and variance $\sigma^2$. Define

$$y^*_t = \epsilon^*_t - \rho^{-1} \epsilon^*_{t-1}$$
where $\epsilon_t^*$ are i.i.d. with mean 0 and variance $\rho \sigma^2$. Compare the autocovariances for $y_t$ and $y_t^*$.

4. (Total 28 points) Consider the AR(2) process:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \epsilon_t$$

where $\epsilon_t$ is i.i.d, mean 0, variance $\sigma^2$.

(a) (14) Determine if the stability (stationarity) condition is satisfied if

i. $\rho_1 = \rho_2 = 1/2$

ii. $\rho_1 = \rho_2 = -1/2$

(b) (14) Assuming $\rho_1, \rho_2$ satisfy the stability condition. Derive $\gamma_0$ and $\gamma_1$.

5. (Total 28 points.) Consider the following 2 equation model:

\begin{align*}
y_{i1} &= \gamma_1 y_{i2} + u_{i1} \quad (1) \\
y_{i2} &= \gamma_2 y_{i1} + \beta_1 x_{i1} + \beta_2 x_{i2} + u_{i2} \quad (2)
\end{align*}

Compute the 2SLS estimates of $\gamma_1$ given the following moment conditions: $E[x_i'x_i] = I_2$,

\begin{align*}
E[y_i'y_i] &= \begin{bmatrix} 7/2 & 2 \\ 2 & 3/2 \end{bmatrix} \\
E[x_i'y_i] &= \begin{bmatrix} 1 & 1 \\ 3/2 & 1/2 \end{bmatrix}
\end{align*}

where $x_i = x_{i1}, x_{i2}, y_i = y_{i1}, y_{i2}$.

6. (Total 40 points) In this question we will explore a right censored regression model, with varying censoring points. Consider a random sample of size $n$ from the following right censored regression model:

$$y_i = d_i(x_i'\beta_0 + \epsilon_i) + (1-d_i)k_i \quad i = 1, 2, ... n$$

Where $d_i = I[x_i'\beta_0 + \epsilon_i \leq k_i]$ is an observed censoring indicator; the censoring threshold is $k_i$ and varies across observations but is observed to the econometrician for all observations. Therefore, the observed random variables are $(y_i, d_i, x_i, k_i)$. Assume that $(k_i, x_i, \epsilon_i)$ are jointly i.i.d. as well as mutually independent, and $\epsilon_i$ is normally distributed with mean 0 and variance $\sigma^2$. 

2
(a) (10) Write down the log-likelihood for this model.

(b) (15) Propose a Wald statistic for the null $H_0: 1'\beta_0 = 0$ where 1 is a vector of ones the same dimension as $x_i$, so $1'x_i$ is a scalar. Derive the limiting distribution of your test statistic when the null is true.

(c) (15) Now assume that $k_i$ is observed, but only for censored observations, and again assume that $(k_i, x_i, \epsilon_i)$ are mutually independent. We’ll refer to this as a randomly censored regression model - it’s very popular in duration analysis. Write down the likelihood function for this model; how does it differ from what you got in the first question?