Demand for Risky Assets/Portfolio Choice

Two assets: safe and risky

- the safe asset has a return of 1 dollar per dollar invested.

- the risky asset has a random return of \( z \in [a, b] \) dollars with distribution \( F \). Assume that \( E(z) > 1 \).

- An individual with utility function \( u(\cdot) \) has an initial amount of wealth \( w \) and invests an amount \( \alpha \) in the risky asset.

The individual’s end-of-period wealth would be:

\[
(w - \alpha) \cdot 1 + \alpha z = w + (z - 1)\alpha.
\]
Risk Averse People Invest in Risky Assets

The utility maximization problem of the individual is

$$\max_{\alpha} V(\alpha, w) = \int_{a}^{b} u(w + (z - 1)\alpha) \, dF(z) \text{ s.t. } 0 \leq \alpha \leq w$$

Let $\alpha^*$ be the amount that maximizes expected utility.

Goals:

1) show that $\alpha^* > 0$.

2) see how $\alpha^*$ varies with levels of risk aversion and with wealth.
\( \alpha^* \) must satisfy the following Kuhn-Tucker FOC:

\[
\phi(\alpha^*, w) = \int_a^b (z - 1)u'(w + (z - 1)\alpha^*) \ dF(z) < 0 \text{ if } \alpha^* = 0
\]

\[
= 0 \text{ if } 0 < \alpha^* < w
\]

\[
> 0 \text{ if } \alpha^* = w.
\]

The SOC is

\[
\int_a^b (z - 1)^2u''(w + (z - 1)\alpha^*) \ dF(z)
\]

which is satisfied because \((z - 1)^2 \geq 0\) and \(u'' < 0\).
Risk Averse People Invest in Risky Assets

**Proposition:** If a risk is actuarially favorable, then any risk averter will always accept at least a small amount of it.

**Proof:**

Define

\[ \phi(\alpha^*, w) = \int_a^b (z - 1)u'(w + (z - 1)\alpha^*) \, dF(z) \]

Note that \( \phi(0, w) = \int_a^b (z - 1)u'(w) \, dF(z) > 0 \), since \( E(z) > 1 \).

Hence \( \alpha^* = 0 \) cannot satisfy this FOC.
How $\alpha^*$ varies with Levels of Risk Aversion

**Proposition:** Consider two individuals with the same level of initial wealth. If individual 2 is strictly more risk averse than an individual 1, individual 2 will invest less wealth in a risky asset; i.e., $\alpha_1^* > \alpha_2^*$.

**Proof:**
Consider individuals 1 and 2, with $u_1(\cdot)$ and $u_2(\cdot)$, where 2 is strictly more risk averse than 1. Suppose that both have the same initial wealth $w$ and that $\alpha_1^* < w$ and $\alpha_2^* < w$.

The FOC’s for these two individuals are:

$$\phi_1(\alpha_1^*) = \int_a^b (z - 1)u_1'(w + (z - 1)\alpha_1^*)dF(z) = 0$$

and

$$\phi_2(\alpha_2^*) = \int_a^b (z - 1)u_2'(w + (z - 1)\alpha_2^*)dF(z) = 0$$

Since $u'' < 0$, $\phi_1$ and $\phi_2$ are decreasing functions, to prove that $\alpha_1^* > \alpha_2^*$, it is sufficient to show that $\phi_2(\alpha_1^*) < 0$. 

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Since individual 2 is strictly more averse than 1, there exists a strictly increasing concave function $\Psi(\cdot)$ such that $u_2(x) = \Psi(u_1(x))$.

We can write the individual 2’s objective function as

$$V_2(\alpha) = \int_a^b \Psi(u_1(w + (z - 1)\alpha)) \, dF(z)$$

Differentiating $V_2(\alpha)$ wrt $\alpha$ gives us

$$\phi_2(\alpha) =$$

$$\int_a^b (z - 1)\Psi'(u_1(w + (z - 1)\alpha)) u'_1(w + (z - 1)\alpha) \, dF(z),$$

which at $\alpha^*_1$ is

$$\int_a^b (z - 1)\Psi'(u_1(w + (z - 1)\alpha^*_1)) u'_1(w + (z - 1)\alpha^*_1) \, dF(z).$$

Note that $\Psi'(u_1(w + (z - 1)\alpha^*_1))$ is positive and decreasing in $z$. 

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How $\alpha^*$ varies with Risk Aversion (cont.)

Compare the following two expressions:

$$\phi_2(\alpha_1^*) =$$

$$\int_a^b (z-1)\psi'\left(u_1\left(w + (z-1)\alpha_1^*\right)\right) u'_1\left(w + (z-1)\alpha_1^*\right) dF(z).$$

To

$$\phi_1(\alpha_1^*) = \int_a^b (z-1)u'_1\left(w + (z-1)\alpha_1^*\right) dF(z) = 0$$

The term $\psi'\left(u_1\left(w + (z-1)\alpha_1^*\right)\right)$ is higher for $z$’s less than 1 than for $z$’s greater than 1. So it biases the weights from $u'_1\left(w + (z-1)\alpha_1^*\right)$ which are perfectly balanced in individual 1’s FOC. Therefore $\phi_2(\alpha_1^*) < 0$. Hence, $\alpha_1^* > \alpha_2^*$. ■
How Investment Level Varies with Wealth

Assume $0 < \alpha^* < w$

**Proposition:** If the utility function exhibits DARA, then $\frac{d\alpha^*}{dw} > 0$; if CARA, then $\frac{d\alpha^*}{dw} = 0$; if increasing absolute risk aversion, then $\frac{d\alpha^*}{dw} < 0$.

By standard comparative statics results, examine the sign of the cross-partial of the objective function:

$$
sign \left[ \frac{d\alpha^*}{dw} \right] = sign \left[ \frac{\partial^2 V(\alpha, w)}{\partial \alpha \partial w} \right] \text{ evaluated at } \alpha = \alpha^*.
$$

We have

$$
\frac{\partial^2 V(\alpha^*, w)}{\partial \alpha \partial w} = \frac{\partial \phi(\alpha^*, w)}{\partial w} = \int_{a}^{b} (z - 1)u''(w + (z - 1)\alpha^*) dF(z)
$$

let $y(\alpha^*, z) \equiv w + (z - 1)\alpha^*$
How Investment Varies with Wealth (cont.)

\[
\frac{\partial^2 V(\alpha, w)}{\partial \alpha \partial w} = \int_a^b (z - 1)u''(y(\alpha^*, z)) \, dF(z) \\
= \int_a^b (z - 1)u''(y(\alpha^*, z)) \left[ \frac{u'(y(\alpha^*, z))}{u'(y(\alpha^*, z))} \right] \, dF(z) \\
= -\int_a^b [(z - 1)u'(y(\alpha^*, z))] \left[ r_A(y(\alpha^*, z)) \right] \, dF(z)
\]

Suppose \([r_A(y(\alpha^*, z))]\) is decreasing in \(z\) (DARA). Compared to the weight on \(z = 1\), we put higher weights \(z < 1\), and lower weights on levels of \(z > 1\). Therefore, for DARA, \(\frac{d\alpha^*}{dw} > 0\).

Suppose \([r_A(y(\alpha^*, z))]\) is a constant with respect to \(z\) (CARA). Then \(\frac{d\alpha^*}{dw} = 0\).

Suppose \([r_A(y(\alpha^*, z))]\) is increasing in \(z\) (IARA). Then \(\frac{d\alpha^*}{dw} < 0\).
Summary of results:

- A risk-averse individual will fully insure when able to buy insurance that is actuarily fair.

- A risk-averse individual will always buy some of a risky asset if it is actuarily favorable.

- An individual that is strictly more risk averse than another will invest less in the risky asset (given the same level of wealth).

- An individual that exhibits decreasing (constant) (increasing) absolute risk aversion will invest more (the same) (less) in the risky asset at higher levels of wealth.