For most of the book, we have assumed that individuals are “small” — that they are unable to alter the economic environment that emerges from individual decisions in a competitive equilibrium and therefore have no reason to think about their role in the world “strategically”.¹ In Chapter 23, we began to deviate from this assumption by considering the case of “large” firms that constitute monopolies, and we found that such firms become “price setters” who deliberately manipulate the economic environment in which they operate. But the case of monopolies is just one example of a large set of possible economic settings in which such deliberate – or “strategic” – thinking becomes important, and strategic considerations can become considerably more complex than those we encountered in Chapter 23.

Before we can proceed to a more general analysis of strategic behavior, we therefore have to develop some new tools. Known collectively as game theory, these tools find their roots in the pioneering work of John Nash (1928-) in the 1940’s and 1950’s and have become integrated into a variety of social sciences over the following decades.² For economic situations in which strategic thinking matters, the game theory approach models the most salient features of such situations as a “game” in which fictional “players” face incentives that are similar to those faced by the real-world actors in the underlying economic setting. In 1994, this approach received the full recognition of the economics community when John Nash and two succeeding game theorists, John Harsanyi (1920-2000) and Reinhard Selten (1930-), were awarded the Nobel Prize in Economics. Nash’s compelling life story has since been immortalized in the movie “A Beautiful Mind” (which takes some artistic liberties with game theory as explored further in end-of-chapter exercise 24.1).

While game theory thus opens the door to incorporating strategic thinking into economic models, the models still follow the same path that we have seen in our development of competitive markets: First, a model is defined; second, we analyze how individuals “do the best they can” within the context of the model; and finally, we investigate how an “equilibrium” emerges – an equilibrium in which we discover the economic environment that arises when everyone is doing the best he can given what everyone else is doing. The only difference from our competitive models is that there

¹This chapter introduces a new set of tools and does not directly build on any previous material.
²You will also find these same tools having found their way into evolutionary biology where scientists have modeled biological evolution as if it were guided by the strategic behavior of genes.
Chapter 24. Strategic Thinking and Game Theory

is now an incentive for individuals to strategically consider how their own behavior impacts the equilibrium, a consideration that is absent when individuals are too small to have such an impact. Our goal in this chapter is then to begin to appreciate how one can model equilibria that emerge from such strategic thinking in a systematic way – leaving many of the applications to exercises and later chapters.

Before we begin, however, we point out two basic distinctions between different types of games — distinctions that give rise to four types of games. In some settings, it is reasonable to assume that all economic agents (that are modeled as “players”) have complete information. By complete information we mean that all players know the economic benefits that all the other players will receive as the game unfolds in different ways. In other situations, economic agents do not have such complete information — i.e. they do not fully know how other players fare as the game unfolds in different ways and therefore cannot as easily put themselves in their opponents' shoes. Such games are then characterized by incomplete information. In an auction in which you and I bid for a $100 bill, for instance, both of us can be pretty sure how much the other values the prize. But in a game where you and I bid on a painting, we can’t be sure how much the painting is valued by the other unless we know each other really well.

The second important distinction between games is whether all players in a game have to decide on the actions they will take at the same time or whether some players take actions before others do. We will call a game in which all players move at the same time a simultaneous move game, while we will call a game in which players move in sequence a sequential move game. In the latter, some players therefore know at least a bit about how the other player is playing the game when the time comes to make a move. Simultaneous move games are also sometimes referred to as “static” while sequential move games are often called “dynamic.” The game “Rock, Paper, Scissors” played by my children on long car trips, for instance, is a simultaneous move game, but the game of Chess is a sequential move game.\(^3\)

Combining these two distinctions, we have four basic types of games: (1) complete information, simultaneous move games; (2) complete information, sequential move games; (3) incomplete information, simultaneous move games; and (4) incomplete information, sequential move games. These games become increasingly complex to analyze as one proceeds from (1) to (4), and we will focus in Section A solely on (1) and (2) — games of complete information. In Section B, we then expand our discussion to games of incomplete information – games of type (3) and (4).\(^4\) In addition, many games have both sequential and simultaneous stages, as we will see in our treatment of repeated simultaneous move games – games in which players meet repeatedly and, at each meeting, play a simultaneous move game. Such repeated interactions will have important implications for what kinds of behavior we can expect to observe in the equilibrium. Similarly we will see in Section B that some games have some players that have complete information and other players that have incomplete information. In such games, less informed individuals may attempt to gain information about the more informed players through their own strategic choices. We have in fact already encountered examples in our Chapter 22 treatment of asymmetric information (where, for instance, insurance companies have less information than clients) and in our Chapter 23 treatment of second-degree price discriminating monopolists (who had less information about what type of consumer

\(^3\)If the game “Rock, Paper, Scissors” is unfamiliar to you, it is described in end-of-chapter exercise 24.7A(a).

they were dealing with than the consumers themselves).

24A  Game Theory under Complete Information

In this section, we will introduce the basics of game theory under complete information. In Section 24A.1, we define what we mean by a complete information game theory model – specifying in particular the players, the actions available to each player and the payoffs they can receive depending on how the game is played. In Section 24A.2 we then expand our notion of an “equilibrium” to one that incorporates the strategic element that has been absent from our definition of a competitive equilibrium. This will require us to specify what we mean by a “strategy” in order to describe an outcome in which everyone’s equilibrium strategy is a “best response” to everyone else’s equilibrium strategy. In the process, we will give some examples of games in which the strategic element does not result in any efficiency problems and other examples in which strategic behavior leads to inefficient outcomes (or, in our previous language, to violations of the first welfare theorem). We then focus in Section 24A.3 on a particular game of the latter type – the Prisoner’s Dilemma. In this game, all players agree they would be better off if they cooperated with each other, but their individual incentives are such that they will not choose to cooperate in equilibrium. This game is one that has many real-world applications and has therefore become a “work-horse” of sorts for social scientists interested in problems involving voluntary cooperation. We will also use the example of the Prisoners’ Dilemma to illustrate how to think about repeated simultaneous move games – games in which players interact more than once and each time play the same (simultaneous move) game, and we will show that the repeated nature of certain strategic interactions can fundamentally alter the type of equilibrium we might observe. Finally we will introduce in Section 24A.4 the notion of a “mixed strategy” in which players decide on probabilities with which they will take particular actions rather than arriving at a plan that involves settling on actions with probability 1. This last section is somewhat optional as we will make limited use of it in the remainder of the book, but it nevertheless represents an important way in which game theorists model strategic behavior, particularly in models where there does not exist a “pure strategy” equilibrium.

24A.1  Players, Actions, Sequence and Payoffs

We begin then by defining the basic structure of complete information games. This structure is given by specifying who the players are, what actions they can take, in what sequence they move and what their payoffs are depending on the combination of moves made by the different players. In the rest of the chapter, we will sometimes also refer to players as “agents” or “actors”.

24A.1.1  Players and Actions

Each of $N$ different players in a given game is often permitted to take one of $M$ possible actions. We will denote the set of possible actions for player $n$ as a set $A^n = \{a^n_1, a^n_2, ..., a^n_M\}$. Often, the actions that different players of the game can take are the same for all players, in which case we can dispense with the superscript notation and simply denote the (common) set of possible actions for all players by the same set $A = \{a_1, a_2, ..., a_M\}$. Sometimes, as we will see in end-of-chapter exercises and upcoming chapters, the set of possible actions will instead be continuous. For instance, it might be that a player $n$ can choose any number on the interval $[0,1]$ as an action, in which case we simply denote the set of possible actions for player $n$ as $A^n = [0,1]$. 
Chapter 24. Strategic Thinking and Game Theory

Consider, for instance, a simple game in which two individuals in a small town are the only ones that drive cars. They might choose to drive on the left side of the road or on the right side of the road. In this case, the two players have the same common set of actions $A=\{\text{Left}, \text{Right}\}$. Alternatively, we might have a game involving a single consumer and a single producer, where the producer can set a high price or a low price for her product, and the consumer can decide to buy the product or not buy it. In that case, the set of actions available to the producer would be $A^p=\{\text{High}, \text{Low}\}$ whereas the set of actions available to the consumer would be $A^c=\{\text{Buy}, \text{Don’t Buy}\}$. Or an employer might offer either a high wage or a low wage to a worker, and the worker has the option of accepting or rejecting the offer — resulting in $A^e=\{\text{High Wage}, \text{Low Wage}\}$ and $A^w=\{\text{Accept, Reject}\}$.

Exercise 24A.1 For which of these examples might it be more appropriate to assume that the set of possible actions is continuous?

24A.1.2 Sequence of Actions

As already mentioned at the beginning of this chapter, a further feature of a game involves the sequencing of moves by the different players. In some cases, we might model an economic situation as one where all players have to decide what action to take simultaneously while in other cases we might model a situation where some players will make sequential moves, with the actions of players who move early observable to the players that decide on their actions later on. The first is a simultaneous move game while the second is a sequential move game. For instance, as two gasoline station owners on opposite sides of a street come to work in the morning, they might face a simultaneous choice of what gasoline price to post as rush hour traffic is about to start. Alternatively, one gasoline station owner might show up a half hour later to work, in which case she might be able to observe what her competitor has posted prior to deciding what she will post. Players in a game are therefore defined not only by the the set of actions they have available to choose from but also by whether or not they are able to observe the other players’ moves prior to determining their own.

24A.1.3 The Payoff Matrix for a Simultaneous Move Game

Once we have defined the set of possible actions and the sequence of moves for the relevant players in a game, we have to settle on what the consequences of different combinations of actions will be for each player. These “consequences” are referred to as payoffs, and the payoff for player $n$ may depend on both her own action as well as the action(s) taken by others.

Exercise 24A.2 Suppose that, for every player $n$ in a game, the payoffs for player $n$ depend on player $n$’s action as well as the sum of all the other player’s actions — but no single other player has, by himself, a perceptible influence on player $n$’s payoff. Would such a game characterize a setting in which strategic thinking was important?

Payoffs for 2-player, simultaneous move games in which both players have a discrete number of possible actions they can take are typically represented in a payoff matrix such as that depicted in Table 24.1. In the game that is depicted, each player has two possible actions, with the actions for player 1 appearing in the first column as $a_1^1$ and $a_1^2$ and actions for player 2 appearing in the first row as $a_2^1$ and $a_2^2$. The payoffs for player 1 then appear as either utility values or dollars in the matrix, with $u^1(a_1^1, a_2^1)$ denoting the utility (or dollar) payoff player 1 receives when both she and player 2 take action $a_1$, $u^1(a_1^2, a_2^2)$ denoting her payoff when she plays action $a_1$ but her opponent...
24A. Game Theory under Complete Information

Table 24.1: Payoffs in a 2-Player Simultaneous Move Game

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
<th>a₁</th>
<th>a₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₁</td>
<td>u¹(a₁, a₁), u²(a₁, a₁)</td>
<td>u¹(a₁, a₁), u²(a₁, a₁)</td>
<td>u¹(a₁, a₁), u²(a₁, a₁)</td>
</tr>
<tr>
<td>a₂</td>
<td>u¹(a₂, a₁), u²(a₂, a₁)</td>
<td>u¹(a₂, a₁), u²(a₂, a₁)</td>
<td>u¹(a₂, a₁), u²(a₂, a₁)</td>
</tr>
</tbody>
</table>

Table 24.2: Driving on the Left or Right Side of the Road

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>10, 10</td>
<td>0, 0</td>
</tr>
<tr>
<td>Right</td>
<td>0, 0</td>
<td>10, 10</td>
</tr>
</tbody>
</table>

Suppose, for instance, that we again considered the simple game in which two individuals in a small town had to decide on which side of the road they should drive. In the end, neither individual cares much about which side of the road is ultimately chosen so long as cars don’t crash into each other when the two individuals choose different actions. The payoffs from this game might then be represented in a payoff matrix such as the one depicted in Table 24.2 in which both individuals receive a payoff of 10 when they pick the same action but a payoff of 0 when they pick different actions.

As we will see in later applications (within end-of-chapter exercises as well as in upcoming chapters), payoffs in games where players have a continues set of possible actions – such as $A = [0,1]$ – are instead represented in payoff functions that specify a player n’s payoffs for any combination of actions taken by all the players. In a two player game we would then find player n’s payoff as a function $u^n(a^1, a^2)$ where $u^n$ is a function that assigns a payoff value for n to any combination of player 1 and player 2 actions, both of which are drawn from the interval [0,1] (when $A = [0,1]$ for both players).

24A.1.4 Game Trees for Sequential Move Games

Sequential move games are often represented in game trees that clearly specify the sequence of moves prior to indicating the payoff each player receives as different actions are taken. Graph 24.1 presents an example of such a game tree for the case were two players each have two possible actions to choose from, with player 1 moving before player 2. For player 2, two possible “information nodes” – or just nodes – emerge depending on which action player 1 has taken. If player 1 chooses action
Chapter 24. Strategic Thinking and Game Theory

Graph 24.1: Example of a 2-Player Sequential Move Game

\( a_1 \), player 2 has sufficient information to know that she is making her decision at the left node, whereas if player 1 chooses action \( a_2 \), player 2 knows she is making her decision at the right node in the game tree. At the end of the game tree, the payoffs that result from each possible sequence of actions are indicated as utility values for each player.

Graph 24.2: Driving on the Left or Right Side of the Road with Sequential Moves

Consider, for instance, the same game as we did in Table 24.2 in which each player has a choice of driving on either the right or the left. But instead of assuming that the players choose simultaneously on which side of the road to drive, player 1 gets on the road first and player 2 gets to observe player 1’s choice prior to making her own choice. Graph 24.2 then displays the game
tree for this sequential move game. The payoffs at the bottom of the game tree are the same as those we see in the payoff matrix in Table 24.2, with both players receiving a payoff of 10 if they choose the same side of the road and a payoff of 0 when they crash into each other because they chose different sides of the road.

While game trees indeed represent a very convenient way for us to present the structure of sequential move games in which each player picks from a discrete (and finite) number of possible actions, we will see shortly that it is possible to also represent such games in payoff matrices once we have defined how “strategies” differ from “actions” in sequential form games. It is also possible to represent a simultaneous move game (in which players pick from a discrete (finite) number of possible actions) in a sequential game tree – as long as we indicate that player 2 does not know which of his nodes he is playing from when it becomes his turn to move. (This is explored further in Section B of the chapter and in some of the end-of-chapter exercises where we introduce a way to model players being unsure about which node in a game tree they have reached. For now, however, we will assume throughout Section A that players in sequential move games can identify precisely what node they are playing from when it becomes their turn to make a move.)

24A.2 “Doing the Best We Can” and the Emergence of an Equilibrium

An equilibrium emerges in our game when all players of a game are doing the best they can given how all other players are playing the game. Notice the italicized phrase is subtly different than the phrase “given what all other players are doing in the game.” The difference is more than semantic – with the former referring to the entire plan that other players are following as they play the game and the latter referring to the observable actions that other players are taking as the game unfolds. As we will see, this is the difference between “strategies” and “actions” – and it is a a difference that will become particularly important in sequential move games where early players will need to know what later players are planning to do at each of their decision nodes in the game tree in order to know which action early on in the game has them “doing the best they can”. We will therefore first define strategies as plans of action for each player, and we will then say that an equilibrium has been reached when each player is playing a strategy that is the best response to the strategies played by the other player(s).

24A.2.1 Strategies

Strategies are most straightforwardly defined in simultaneous move games in which all players have to choose a plan of action at the same time. Each player in such a game can either settle on a particular action to take, or she can decide to play particular actions with some probability. A strategy that involves picking a particular action with probability 1 is called a pure strategy while a strategy that places probabilities of less than 1 on more than one action is called a mixed strategy. In most of of the chapter, we will focus only on pure strategies, but we will conclude Section A with an optional discussion of mixed strategies and their role in the development of game theory models. In fact, all strategies can be viewed as mixed strategies, with pure strategies simply special cases that assign probability 0 to all but one action.

In sequential move games, strategies are a little more complicated because some players will already know what other players are doing when they decide on their own actions. Thus, a complete plan of action for a player other than the one who moves first involves a plan for what to do at each possible node at which a player might find herself in the game tree. A pure strategy for player 2 in the game depicted in Graph 24.2, for instance, involves a plan for what to do in case player 1
has chosen the action *Left* and what to do if player 1 has chosen the action *Right*. Pure strategies in simultaneous move games therefore involve simply picking one action, while pure strategies in sequential move games involve picking one action at each node in the game tree. (Just as in simultaneous move games, a mixed strategy in a sequential setting involves playing different pure strategies with probabilities that sum to 1 – but we will limit our discussion of mixed strategies to simultaneous move games.)

When we restrict ourselves to considering pure strategies, player 2 in the game in Graph 24.2 then has *four possible strategies even though she only has two possible actions available*. These strategies are:

| Strategy 1: | Always play *Left* |
| Strategy 2: | Always play *Right* |
| Strategy 3: | Play *Left* if player 1 plays *Left* and play *Right* if player 1 plays *Right* |
| Strategy 4: | Play *Right* if player 1 plays *Left* and play *Left* if player 1 plays *Right*. |

We can denote these four strategies as (*Left, Left*), (*Right, Right*), (*Left, Right*) and (*Right, Left*), with the first action in each pair indicating the plan of action if player 2 ends up on the left node in the game tree and the second action in each pair indicating the plan of action if player 2 finds herself on the right node in the game tree.

**Exercise 24A.3** True or False: *In simultaneous move games, the number of pure strategies available to a player is necessarily equal to the number of actions a player has available.*

Once we recognize that players who move later in the sequence within a sequential move game have more pure strategies than actions available to them, we can see how we can represent the structure of such games in payoff matrices rather than game trees. All we have to do is list the payoffs that each player will receive for each combination of pure strategies. For the game in which players choose the right or left side of the road sequentially, this implies that player 1 has only 2 pure strategies (equal to the actions she is able to take) while player 2 has four pure strategies. The sequential move game represented in Graph 24.2 can then also be represented in the payoff matrix in Table 24.3.5

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><em>Left</em></td>
</tr>
<tr>
<td><em>Left</em></td>
<td>10, 10</td>
</tr>
<tr>
<td><em>Right</em></td>
<td>0, 0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><em>Right</em></td>
</tr>
<tr>
<td><em>Left</em></td>
<td>10, 10</td>
</tr>
<tr>
<td><em>Right</em></td>
<td>0, 0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Left</em></td>
<td>10, 10</td>
</tr>
<tr>
<td><em>Right</em></td>
<td>10, 10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Left</em></td>
<td><em>Left</em></td>
</tr>
<tr>
<td><em>Left</em></td>
<td><em>Right</em></td>
</tr>
</tbody>
</table>

Table 24.3: A Sequential Move Game Represented in a Payoff Matrix

**Exercise 24A.4** Verify that the payoffs listed in Table 24.3 are consistent with those given in the game tree of Graph 24.2.

5Representing a game in a payoff matrix is often referred to as the game’s *normal form*, whereas representing the game in a game tree is often referred to as the game’s *extensive form*. 
24A.2.2 Pure Strategy Nash Equilibrium in Simultaneous Move Games

John Nash was the first to formalize the notion of an equilibrium in games, and what we explore next has therefore come to be called a Nash equilibrium. The definition of such an equilibrium is best given in terms of “best responses”, where a best response for player \( n \) to a set of strategies played by other players is simply a strategy that will result in the highest possible payoff for player \( n \) given the strategies played by others. A Nash equilibrium is reached whenever each player in the game is playing a best response strategy relative to the strategies played by all other players – i.e. whenever everyone’s plan is the best possible plan given the plans that all the others have adopted.

In some cases, we will see that it is very clear what Nash equilibrium will emerge as individual players try to do the best they can given how others are playing the game. Sometimes, a single equilibrium will emerge, while other times multiple different equilibria are possible. Depending on the structure of the game, we will find instances when only pure strategies are employed in equilibrium, but many games also have mixed strategy equilibria. In fact, in games where there are no pure strategy equilibria, there generally exists a mixed strategy equilibrium. And in games in which there are multiple pure strategy equilibria, there generally also exist mixed strategy equilibria (as we will see in Section 24A.4).

Let’s begin by considering again the game represented in the payoff matrix in Table 24.2. Suppose you are player 1 and I am player 2, and suppose you contemplate what pure strategy to play. If I choose to drive on the left side of the road, you know that you will get a payoff of 10 if you also choose the left side but will receive a payoff of 0 if you choose the right side. Your best response to my strategy of playing \( \text{Left} \) is therefore to play \( \text{Left} \) as well. Similarly, if I choose the right side of the road, your best response is to also choose \( \text{Right} \). It is clear in this example that you will do the best you can if you mimic what I do. I of course face exactly the same incentives.

We can then look at each of the four possible outcomes and check to see if the outcome could be a Nash equilibrium supported by strategies that are best responses. The two outcomes that result in 0 payoff for each player cannot possibly be an equilibrium outcome because, if we find ourselves crashing into each other as we are choosing different sides of the road, there is a way for you to improve your fortunes by changing what you do. The two outcomes that result in payoffs of 10, on the other hand, can be equilibrium outcomes. Whenever one of us chooses \( \text{Left} \), the other’s best response is to also choose \( \text{Left} \), and whenever one of us chooses \( \text{Right} \), the best response of the other is to also choose \( \text{Right} \). Put differently, if we end up in the upper left corner of the payoff matrix, neither one of us has an incentive to change what we are doing — implying that we have reached an equilibrium. The same holds for the lower right corner of the payoff matrix.

In this example, it is unclear whether both of us driving on the right side or both of us driving on the left side will emerge as an equilibrium. In the real world, conventions arise and are often formalized in laws that insure everyone knows which equilibrium is to be expected. As you know, in some societies the convention of driving on the left side of the road has become the equilibrium, while in other societies the convention of driving on the right side has emerged. Games like this are sometimes called coordination games because the key for the players is to coordinate their actions to get to one of the possible pure strategy equilibria.

Exercise 24A.5 Are the two pure strategy Nash equilibria we have identified efficient?

It might appear at this point that an equilibrium will necessarily entail both sides achieving the maximum possible payoffs. If this were always the case, the first welfare theorem would still hold in

\[ \text{In the original investigation by Nash on the existence of Nash equilibria, it was in fact proven that such equilibria generally exist so long as the equilibrium concept includes mixed strategies.} \]
Chapter 24. Strategic Thinking and Game Theory

the sense that decentralized decision making by individuals is resulting in efficient outcomes. But this is not necessarily the case. Suppose we changed the payoff matrix in Table 24.2 by assuming that each of us has an innate preference for driving on the left side of the road and thus we only receive a payoff of 5 each if we end up driving on the right side. In this case, both of us driving on the right side of the road is still an equilibrium of the game — if one of us chooses to play Right, it remains a best response for the other to also choose Right. Games with multiple equilibria might therefore have some equilibria that are better for everyone than others. In such cases, a role for non-market institutions emerges to try to get individuals to switch from the sub-optimal equilibrium to the more efficient one.

24A.2.3 Dominant Strategy Equilibria in Simultaneous Move Games

Even in games where there is a single pure strategy Nash equilibrium, however, there is no guarantee that the Nash equilibrium will achieve the maximum possible payoffs for the players. Consider the games defined by the payoff matrices in Tables 24.4 and 24.5. In the first game, a clear optimal strategy for each player is to always play the action Up because regardless of what the other player does, each individual player is better off playing Up rather than Down. This is an example of a game with a clear dominant strategy — a strategy where a player always has the incentive to play a single action regardless of what the opponent does. Even if you think your opponent will play the action Down, it is best for you to play Up because that will give you a payoff of 7 rather than 5. Since both players face the same incentives, a single pure strategy equilibrium emerges in which both players play Up and thus receive a payoff of 10. The game in Table 24.4 therefore unambiguously leads to an equilibrium in which both players receive the highest possible payoff — i.e. the Nash equilibrium is efficient and is particularly compelling since it is both the only equilibrium and it involves each player playing a strategy that is the best for that player regardless of what the other player does.

**Exercise 24A.6** True or False: If a simultaneous move game gives rise to a dominant strategy for a player, then that strategy is a best response for any strategy played by the other players.

![Table 24.4: A Game with a Single Efficient Pure Strategy Nash Equilibrium](image)

Now consider the game in Table 24.5 and suppose that you and I are playing this game. If I play Up, you will receive a payoff of 10 by also choosing Up and a payoff of 15 if you choose Down. Your best response to me playing Up is therefore to play Down. If, on the other hand, I choose to play Down, you will receive a payoff of 0 if you play Up and a payoff of 5 if you play Down. Thus, playing Down is also your best response to me playing Down. Put differently, playing Down is a dominant strategy for you because it is your best response to any strategy I play. Since I face the same incentives, we will both end up playing Down, resulting in the equilibrium outcome
represented by the payoffs (5,5) in the lower right corner of the payoff matrix. Thus, even though we would both prefer the payoffs (10,10) in the upper right corner of the matrix, the incentives in the game are such that we will end up in the lower right corner with payoffs (5,5). The unique Nash equilibrium of this game is therefore inefficient – and it is just as compelling an equilibrium as the one we found in Table 24.4 in that it is the only pure strategy equilibrium and it involves only dominant strategies.

Table 24.5: A Game with a Single Inefficient Pure Strategy Nash Equilibrium

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>10,10</td>
<td>0,15</td>
</tr>
<tr>
<td>Down</td>
<td>15,0</td>
<td>5,5</td>
</tr>
</tbody>
</table>

In Section 24A.3 below, we will discuss this game – known as the “Prisoners’ Dilemma” – in much more detail because it will represent an important game that can be used to analyze many economic situations in the real world. For now, however, it should be clear that we will be unable to come up with something analogous to the First Welfare Theorem we derived for competitive economies when individual players have an incentive to be strategic in their decision making. Put differently, we will not be able to say in general that equilibria which rely on decentralized decision making by individuals are always efficient in economic circumstances that can be modeled by game theory. Sometimes they are, and sometimes they are not.

Exercise 24A.7 Suppose that player 2 has payoffs as in Table 24.4 while player 1 has payoffs as in Table 24.5. Write out this payoff matrix. Is there a dominant strategy equilibrium? Is there a unique Nash equilibrium? If so, is it efficient?

Exercise 24A.8 Suppose both player’s payoffs are as in Table 24.5 except that player 1’s payoff when both players play Up is 20. Is there a dominant strategy equilibrium? Is there a unique Nash Equilibrium? If so, is it efficient?

Exercise 24A.9 Suppose payoffs are as in exercise 24A.8 except that player 2’s payoff from playing Down is 10 less than before (regardless of what player 1 does). Is there a dominant strategy equilibrium? Is there a unique Nash Equilibrium? If so, is it efficient?

24A.2.4 Nash Equilibrium in Sequential Move Games

The notion of a Nash equilibrium can then be straightforwardly applied in sequential move games if we represent the structure of such games within a payoff matrix in which we specify the set of payoffs for each combination of strategies. In Table 24.3, for instance, we depicted the structure of the game in which two players sequentially choose on which side of the road to drive.

Exercise 24A.10 Can you find which strategies in the game depicted in Table 24.3 constitute a Nash equilibrium? (Hint: You should be able to find four combinations of strategies that constitute Nash equilibria.)

A slightly more interesting version of this game arises when we assume again that the players have an innate preference for driving on the left side of the road, resulting in payoffs of 10 if they
both choose Left, payoffs of 5 when they both choose Right and payoffs of 0 when they choose different sides of the road. In the case when the players move simultaneously, we discovered that two pure strategy equilibria emerge: one in which both players drive on the left side of the road and one in which both players drive on the right side of the road — and thus we discovered a simultaneous move game in which one of the equilibria was inefficient. When player 2 makes her choice after player 1 moves, the payoff matrix (analogous to the one we derived in Section 24A.2.1), is given in Table 24.6.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>10, 10</td>
</tr>
<tr>
<td>Right</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Table 24.6: The Sequential Right/Left Game with Left Preferred by All

There are now several Nash equilibria in this game — with the accompanying equilibrium outcomes shaded in Table 24.6. One of these equilibria involves player 1 playing Right and player 2 playing (Right, Right). Given that player 2 always plays Right, it is a best response for player 1 to play Right, and given that player 1 plays Right, player 2’s (Right, Right) strategy is a best response. Thus, the (inefficient) outcome of both players driving on the right side of the road continues to be possible in a Nash equilibrium in the sequential move game.

**Exercise 24A.11** Is it also a Nash equilibrium for player 1 to play Right and player 2 to play (Left, Right)? If not, why was it a Nash equilibrium before when players were indifferent between coordinating on the left or the right side of the road?

In the case where player 1 gets to decide first which side of the road to pick, however, this equilibrium seems very counterintuitive. The only reason this is a Nash equilibrium is that player 2 is in effect threatening to drive on the right side of the road regardless of what player 1 chooses to do. But this threat is fundamentally non-credible because player 1 knows that player 2 is better off driving on the left side of the road once she sees that player 1 has chosen to drive on the left. For this reason, game theorists have developed a more refined notion of Nash equilibrium for sequential move games — a refinement that eliminates the possibility that non-credible threats are taken seriously in equilibrium. This refinement is known as subgame perfection.

**24A.2.5 Subgame Perfect Equilibria in Sequential Move Games**

It is reasonable to assume that players who move early in a sequential move game will look down the game tree and determine what strategies by players that follow are credible — and that only credible strategies can emerge in an equilibrium. This implies that player 1 will look at each node in the game tree of the sequential move game to determine what is optimal for player 2. Player 1 can then infer something about what player 2 plans to do once player 2 has observed the action of player 1.

Consider the game tree in Graph 24.3 that depicts the game we represented in the payoff matrix in Table 24.6. Player 1 can now view each of the 2 nodes that player 2 could face as a separate
subgame in which player 2 is the only player. If the left node is reached (as a result of player 1 playing Left), it is optimal for player 2 to also play Left, which we indicate in the graph by highlighting this action. Thus, player 1 can infer that she will receive a payoff of 10 if she moves Left. If the right node is reached (as a result of player 1 playing Right), on the other hand, player 1 knows it will be optimal for player 2 to play Right, leading to a payoff of 5 for player 1. We again indicate this in the graph by highlighting that action. Thus, in choosing between Left and Right, player 1 knows that she is choosing between a payoff of 10 and a payoff of 5 and will therefore choose to play Left. The only rational response for player 2 is to also play Left, which leads to a unique equilibrium in which both players drive on the left side of the road.

While the outcome in which both players drive on the right side can therefore arise from a Nash equilibrium in which player 2 plays the strategy (Right, Right), this outcome cannot emerge as an equilibrium in which player 1 does not pay attention to non-credible threats. The elimination of Nash equilibria that are supported by non-credible threats then results in subgame perfect equilibria.

Exercise 24A.12 True or False: In sequential move games, all pure strategy subgame perfect equilibria are pure strategy Nash equilibria but not all pure strategy Nash equilibria are subgame perfect.

Now suppose that one of our players is from the U.S. and the other is from the U.K. – and the players therefore do not share the same preferences over which side of the road to choose. In particular, player 1 now receives a payoff of 10 if both end up driving on the left and 5 when both end up driving on the right side, and player 2 receives a payoff of 5 when both end up driving on the left and 10 when both end up driving on the right. The only feature of the game tree in Graph 24.3 that then changes is that the 10 and the 5 on the last line in the graph reverse positions. But player 1 still knows that player 2 will choose the left side of the road if player 2 has to make a decision.

---

The notion of subgame perfection is due to Reinhard Selten who was awarded the Nobel Prize together with John Nash. As we will note in Section 24A.3.2, subgame perfect equilibria can equivalently be defined as Nash equilibria under which the equilibrium strategies represent Nash equilibria for every subgame of the actual sequential game.
Chapter 24. Strategic Thinking and Game Theory

from the left node in the game tree and the right side of the road if she has to make a decision on the right node. Thus, player 1 again knows that she will earn a payoff of 10 from choosing Left and a payoff of 5 from choosing Right — which again results in a unique subgame perfect equilibrium in which both players end up driving on the left side of the road. While player 2 might threaten to always drive on the right side in order to get to the Nash equilibrium in which both players drive on the right, player 1 would be rational not to pay any attention to such a non-credible threat. As a result, player 1 enjoys a first mover advantage because, by moving first, she gets the outcome most favorable to her.

Exercise 24A.13 What are the Nash equilibria and the subgame perfect equilibria if player 2 rather than player 1 gets to move first in this version of the game?

As we will see in later chapters, however, it is not the case that a first mover in a game will always get her way. Suppose, for example, we consider a firm that currently has a monopoly in a particular market but worries about a potential second firm entering the market and competing. To keep the game simple, let’s suppose that the existing firm can set a Low or a High price for the product and that the potential firm can choose to Enter or Not Enter after observing the price set by the existing firm. Suppose further that they payoffs (or profits) in this game are as depicted in Graph 24.4.

![Graph 24.4: Facing Potential Competition](image)

Graph 24.4: Facing Potential Competition

If the potential firm does not enter, it receives a profit of 0, but if it enters, it earns a positive profit when the current price is high and a negative profit when the current price is low. The existing firm, on the other hand, earns the highest profit under a high price and no competition and the lowest profit if it announces a high price and the competitor enters (and undercut that price in order to steal customers). The existing firm then looks down the game tree at each node faced by its potential competitor and determines what the competitor will do at each node. When price is set low by the existing firm, the competitor will not enter (because she would make a profit of -10 by entering) but when price is set high, she will enter. In choosing between Low and High, the existing firm is therefore choosing between a payoff of 20 and a payoff of -10 and will choose...
the low price in order to keep the potential firm from entering. This results in the subgame perfect equilibrium in which the existing firm sets a low price and the potential firm does not enter. Notice that in this case, the subgame perfect equilibrium does not result in the most preferred outcome for the first mover, and it is supported by a credible threat that the potential firm will enter if the price is set high by the existing firm.

**Exercise 24A.14** Suppose the game had a third stage in which the existing firm gets a chance to re-evaluate its price in the event that a new firm has entered the market. This would imply that the game tree in Graph 24.4 continues as depicted in Graph 24.5. What is the subgame perfect equilibrium in this case?

![Graph 24.5: An Extension of the Game in Graph 24.4](image)

Finally, we can note from the sequential move game in Graph 24.4 that, just as we found in simultaneous move games, there is no guarantee that equilibria in game theory are efficient — i.e. there is no general first welfare theorem. The efficient outcome (from the perspective of the two players) is the outcome that maximizes the sum of the profits (or payoffs). In our example, that occurs when the existing firm earns a profit of 30 and faces no competition from potential entrants. But, at least as the game is specified in Graph 24.4, this is not a subgame perfect equilibrium. Rather, the subgame perfect equilibrium results in a profit of 20 for the existing firm and a profit of zero for the potential entrant. From the perspective of the two firms, a move to the outcome in which the existing firm gets to set a high price and the potential firm does not enter makes one player better off without making the other worse off, but it is not an outcome that can be sustained as an equilibrium in the game without some non-market institution altering the incentives of the game.

**Exercise 24A.15** In our example in Graph 24.4, we say that the subgame perfect equilibrium is not efficient from the perspective of the two players. Could it be efficient from the perspective of “society”?
24A.2.6 Solving for (Pure Strategy) Nash and Subgame Perfect Equilibria

While we have already solved for the equilibria in several games, it might be useful to briefly review the method by which we solve for these. In the case of Nash equilibria in which 2 players have a finite number of actions to choose from, we start with the payoff matrix – whether this represents a simultaneous move game or a sequential move game. Let’s refer to the player whose strategies appear in the rows of the matrix as the “row player” and the player whose strategies appear in the columns of the matrix as the “column player”. To solve for pure strategy Nash equilibria, we can then simply start with the first strategy of the row player and ask which strategy (or strategies) the column player would play as a best response. For each of these best response strategies by the column player, we then ask whether the first row strategy is a best response by the row player. When we find a case where the first row strategy is a best response to one of the column player’s best responses, we have identified a Nash equilibrium. Doing this for each row, we end up finding all the pure strategy Nash equilibria.

When the set of possible actions for players in a simultaneous move game are not finite – i.e. when the set \( A \) is a continuum like the line segment \([0,1]\) from which the player can choose any point, we cannot use payoff matrices as just described (because such matrices would have to specify the payoffs from an infinite number of combinations of actions). We will encounter some examples of this in some of the end-of-chapter exercises – and we will develop the method for solving such games explicitly in the next chapter. For now, we just note that the logic of a strategy and an equilibrium remains exactly the same – all we will do is define “best-response functions” that must then intersect in an equilibrium. This is similar to how we solve games with discrete numbers of possible actions for mixed strategy equilibria in Section 24A.4.

In the case of subgame perfect (Nash) equilibria to sequential games in which players have a finite number of actions to choose from, we have to start with the game tree rather than the payoff matrix of the game. In particular, we start at the bottom of the game tree and ask which action is optimal at each node of the last player. These actions are the only actions that could be planned in a credible strategy for that player – and we then assume that these are in fact the actions that would be played at the respective nodes. We then move to the second-to-last player and ask which action (at each of the player’s nodes) is optimal given that the player assumes the final player will play rationally at each of his nodes in the next stage. This then allows us to identify the optimal actions for the second-to-last player – which can be taken as given by the third-to-last player. In this way, we can solve the game backwards to the top – and derive the full set of subgame perfect equilibrium strategies. It is important to keep in mind, however, that the equilibrium is defined by best response strategies – and not just by the path along which the game unfolds in equilibrium. Put differently, the players’ plans “off the equilibrium path” are often crucial to keeping other players “on the equilibrium path”. We will again encounter games in which some players have a continuum of possible actions they can choose from – and we will see in end-of-chapter exercises as well as upcoming chapters that the basic logic for solving such games will again mirror that for games with a finite set of possible actions.

24A.3 The Prisoners’ Dilemma

In Table 24.6, we illustrated a simultaneous move game in which each player has a dominant strategy — and in which the resulting Nash equilibrium is inefficient. This type of game is often referred to as the “Prisoners’ Dilemma,” and it occupies a particularly important place in microeconomics because it so starkly illustrates how strategic behavior can lead to outcomes that can be improved
24A. Game Theory under Complete Information

upon through some type of non-market institution.

The name “Prisoners’ Dilemma” has its origins in the 1950’s when Albert Tucker (1905-1995), a mathematician and dissertation advisor to the young John Nash, attempted to find an accessible way of illustrating the basic incentives of the game with a “story” that made sense to psychology undergraduates at Stanford. The story goes something like this: A prosecutor knows that two individuals he has in custody have committed armed robbery but he does not have enough evidence to convict them on anything other than a relatively minor charge of illegal possession of firearms. So he puts them in separate rooms and tells each of them that they can choose to confess or deny the armed robbery. If one confesses and the other does not, then he will let the confessor out on parole while using his testimony to go for the maximum sentence of 20 years in prison for the one that remains silent. If they both confess, they will each get a plea agreement that will put them in jail for 5 years. If neither confesses, all the prosecutor can do is press the illegal firearms convictions and get them 1 year prison sentences each.

Exercise 24A.16 Why is this outcome inefficient from the perspective of the two players? Could it be efficient from the perspective of “society”?

As we will see in upcoming chapters, many economic circumstances have similar incentives. We may all wish to live in a society in which we smile and are courteous to one another. But smiling and being courteous requires effort — and so regardless of whether others smile and show courtesy, it might be a dominant strategy to individually behave like an ass. We may all want to live in a world in which we look out for our neighbors and provide them with help when they are in need — but helping others requires effort and it might just be a dominant strategy to not bother and just hope others will take care of it. Once you have internalized the incentive structure of the Prisoners’ Dilemma game, you’ll see these incentives all around you. We want to live in a

---

Table 24.7: The Prisoners’ Dilemma (with years in prison as payoffs)

<table>
<thead>
<tr>
<th></th>
<th>Deny</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deny</td>
<td>1,1</td>
<td>20,0</td>
</tr>
<tr>
<td>Confess</td>
<td>0,20</td>
<td>5,5</td>
</tr>
</tbody>
</table>

8The underlying game was already known at the time and played a large role in the Rand Corporation’s investigation of game theory as part of its federally sponsored project to research incentives in global nuclear strategy.
world in which we cooperate with one another for the common good but in which it is often in our self interest to not cooperate and hope everyone else will. The fact that individuals inadvertently cooperate in competitive markets and maximize overall social surplus (as illustrated by the first welfare theorem) simply does not mean they cooperate purposefully when put in situations where they have an incentive to behave strategically.

Once you understand the incentives in Prisoners’ Dilemma games, observing a lack of cooperation in the world is not surprising. What is surprising is how much cooperation we actually do observe in the real world despite the predictions of the Prisoners’ Dilemma. While it may not happen to the extent to which we would hope, we see neighbors helping one another, individuals holding open doors for strangers, charities successfully raising money to combat hunger and disease and soldiers dying in battle to save another’s life. We also see prisoners denying crimes when faced with the incentives in Table 24.7 and firms colluding to set prices even when it appears that they would individually benefit by producing more than their collusive agreement permits (as we will discuss in detail in our treatment of cartels in Chapter 25). In some sense, once we understand the Prisoners’ Dilemma, the question becomes not “why don’t we observe people cooperating more with one another” but rather “why do we see any cooperation in many situations at all”?

### 24A.3.1 Repeated Prisoners’ Dilemma Games and the “Unraveling of Cooperation”

You might think that one possible explanation for cooperation in the real world is that, at least in some circumstances, players run into each other repeatedly and therefore develop a cooperative relationship. It turns out, however, that repeated interaction in circumstances that can be described by the Prisoners’ Dilemma is not enough for game theory to predict cooperation.

Suppose you and I face the payoffs (in, say, dollar terms) in Table 24.8 every time we meet.

<table>
<thead>
<tr>
<th></th>
<th>Cooperate</th>
<th>Don’t Cooperate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>100, 100</td>
<td>0, 200</td>
</tr>
<tr>
<td>Don’t Cooperate</td>
<td>200, 0</td>
<td>10, 10</td>
</tr>
</tbody>
</table>

Table 24.8: Another Prisoners’ Dilemma (with Payoffs in $’s)

**Exercise 24A.17 Why is this a Prisoners’ Dilemma Game?**

Now suppose you and I know that we will run into each other 100 times, and each time we will face the incentives in Table 24.8. This means we are now playing a sequential move game in the sense that we encounter each other (after the first time) knowing what we did in previous encounters, but in each encounter we play a simultaneous move game. We can then apply the logic of subgame perfection to see what would happen. Subgame perfection requires that we start at the very bottom of the game tree that, in this case, consists of 100 different simultaneous move games. We can then ask: What would we expect will happen when we encounter each other for the 100th (and last) time?
Since we will know that we will not encounter each other again, it will be exactly as if we simply played the game one time — with each one of us facing a dominant strategy of not cooperating in that last encounter. When we meet each other the 99th time, it is therefore not credible for either one of us to promise or threaten any action other than not cooperating in the 100th round. Put differently, we will both know in the 99th round that we will not cooperate in the 100th round. But then there is no particular reason to cooperate in the 99th round — once again, regardless of what you do in the 99th round, I will do better by not cooperating. So we both realize when we play the 98th round that we will not cooperate in the 99th or 100th rounds — which, by the same logic, implies we won’t cooperate in the 98th round or in any round before that. The prediction from subgame perfection is that we will not cooperate in the Prisoner’s Dilemma even if we know we will interact repeatedly \( n \) different times. This holds true regardless how large \( n \) is (assuming it is finite).

Notice what is going on in this argument for why cooperation will not arise even under repeated interactions: We might think that if I know we will run into each other 100 times, I could say to you “why don’t we cooperate since we will run into each other repeatedly and we both know we’ll be better off by cooperating.” You would presumably see that what I said is true. I might even try a carrot-and-stick approach by telling you that I will cooperate so long as you cooperate but if I see you not cooperating, I will punish you and never cooperate again. The problem is that my promise to cooperate is not credible — because as you look down the game tree, you know I will not cooperate in the 100th round which means that there is no incentive to cooperate in the 99th round which means there is no incentive to cooperate in the 98th round and so on.

**Exercise 24A.18** Does the same logic hold for any repeated simultaneous game in which the simultaneous game has a single pure strategy Nash equilibrium? Put differently, does subgame perfection require that players in such games always simply repeat the simultaneous game Nash equilibrium?

### 24A.3.2 Infinitely Repeated Games, Trigger Strategies and Cooperation

The reason why cooperation unravels in the repeated Prisoners’ Dilemma is that both of us can look toward the last time we interact and work backwards to realize that there is no credible (i.e. subgame perfect) way of sustaining any cooperation. But what if there was no “last time”? What if we keep running into each other without end? Or more realistically, what if we are never sure whether we’ll run into each other again but each time we run into each other we know there is a good chance we’ll see each other again under similar circumstances?

**Exercise 24A.19** True or False: In an infinitely repeated Prisoners’ Dilemma game, every subgame of the sequential game is identical to the original game.

Before answering this question, we need to briefly address what the concept of “subgame perfection” means in the case of a game that has no end. So far, we have simply thought of subgame perfection as eliminating non-credible strategies by solving the game “from the bottom up”, but now there is no “bottom”! The basic idea of subgame perfection can, however, be expressed a little differently and in a way that then allows us to apply it to infinitely repeated games: When we solve the game backwards in a finite sequential game, we are actually making sure that the Nash equilibrium is such that each subgame of the whole game — i.e. each game that begins at one of the nodes in the game tree — is also in equilibrium. Put differently, we are requiring that the subgames that are “off the equilibrium path” and are never reached still involve strategies that are best responses to each other in the hypothetical case that such subgames were reached. We
can then restate the concept of a subgame perfect equilibrium by defining it as follows: A Nash equilibrium in a sequential move game of complete information is subgame perfect if all subgames of the sequential game – whether they are reached in equilibrium or not – also involve Nash equilibrium strategies.

Now let’s return to our question: What could be a subgame perfect equilibrium in a repeated Prisoners’ Dilemma game in which there is no definitive end to our interactions? Robert Axelrod (1943-), a political scientist, has written a famous series of papers in which precisely this question was analyzed theoretically and experimentally. Consider the case in which you and I meet repeatedly, and each time we meet we know that we will meet again with probability $\gamma$. At the beginning of our interactions, we decide on our strategies. Remember that a “strategy” for me is a complete plan for what I will do each time we run into each other — a plan in which I can make my actions dependent on how we interacted in the past. Axelrod distinguished between two kinds of such plans or strategies we might adopt: those that are “nice” and those that are “not nice”. “Nice” strategies are those in which an individual will not stop cooperating first while “not nice” strategies are those in which an individual is the first to stop cooperating.

**Exercise 24A.20** True or False: If two players play “nice” strategies in the repeated Prisoners’ Dilemma, they will always cooperate with one another every time they meet.

Suppose, for instance, I play a strategy in which I plan to cooperate the first time I see you and then plan to continue to cooperate every time I see you as long as all our previous interactions have been characterized by both of us cooperating — but if at some point we do not cooperate, I will punish you by never cooperating again. One act of non-cooperation, according to this strategy, will “trigger” my non-cooperation at every meeting thereafter — which is why this type of a strategy is sometimes called a trigger strategy.

**Exercise 24A.21** Explain why this type of trigger strategy is “nice”.

What is your best response to this strategy? One possible best response might well be for you to play the same strategy — resulting in us always cooperating. This is because the cost of being punished with non-cooperation from now on is too high to justify the gain from not cooperating one time while I am still cooperating. Whether it’s worth it to you to cheat me at our current encounter by not cooperating (despite knowing that I will never cooperate again thereafter) then depends on two things: the probability $\gamma$ that we will meet again and the degree to which you discount the future. If $\gamma$ is sufficiently high and you do not discount the future too much, you will value future cooperation more than the one-time payoff you could get by cheating me at our present meeting.

**Exercise 24A.22** Would you playing “Cooperate Always” also potentially be a best response for you to my trigger strategy? Would my trigger strategy then be a best response to your “Cooperate Always” strategy?

If you playing the trigger strategy is a best response to me playing this strategy, then it is of course also a best response for me to play this strategy if you play it. And when both of us play this strategy, we will always cooperate with one another. It is certainly possible, then, to have Nash equilibria in which cooperation is sustained in repeated relationships that are characterized by Prisoners’ Dilemma incentives if those relationships have no clear end. But is such a Nash equilibrium subgame perfect? Given our restated definition of subgame perfection as involving only strategies that are Nash equilibrium strategies to every subgame, we have to ask whether the Nash equilibrium strategies we have proposed are also Nash equilibrium strategies in every subgame of the infinitely repeated game. Every such subgame is, of course, once again an infinitely repeated game.

---

*Chapter 24. Strategic Thinking and Game Theory*
identical to the original game – but subgames have different “histories” of previous interactions between us that led up to them. Thus, unlike the first time we meet, I know something about how you are playing the game every time we meet thereafter – and you know something about how I play the game.

When we reach a particular subgame, there are then two possible histories that have brought us there: either we have gotten there by always cooperating, or we have gotten there by not cooperating at some point. Suppose first that we had always cooperated previously. Then, given that we are playing our trigger strategies, we are starting this subgame in exactly the same way as we started the first time we interacted: we both cooperate and plan to continue cooperating unless one of us deviates at some point. If the proposed trigger strategy played by both of us was a Nash equilibrium to the original game, it must therefore be a Nash equilibrium to this subgame. This leaves us to consider the (“off-the-equilibrium path”) case where cooperation broke down at some point in a previous meeting. In this case, our trigger strategies for the next subgame are both to “Never Cooperate”. Given that you will never cooperate, it is a best response for me to never cooperate and the other way around. Thus, we are best-responding to each other in this kind of a subgame – and we have therefore shown that both of us playing the proposed trigger strategy represents a Nash equilibrium in every subgame of our infinitely repeated game. These strategies are therefore subgame perfect.

Put differently, the “threats” required to sustain our cooperation are credible in our example. In fact, as we demonstrate in the appendix, anything between no cooperation and full cooperation can be part of a subgame perfect equilibrium through similar trigger strategies in an infinitely repeated Prisoners’ Dilemma. Thus, when Prisoners’ Dilemma games are repeated infinitely, many possible subgame perfect equilibria emerge even though there is only a single subgame perfect equilibrium when such games are repeated a large but finite number of times.

**Exercise 24A.23** Why can’t the same type of “trigger strategy” sustain cooperation in a repeated Prisoners’ Dilemma that has a definitive end?

**Exercise 24A.24** If you model the decision about whether to be friendly to someone you run into as part of a Prisoners’ Dilemma, why might you expect people in small towns to be friendlier than people in big cities?

### 24A.3.3 The “Evolution” of Cooperation and the Emergence of “Tit-for-Tat”

Axelrod, however, was interested in more than just demonstrating that cooperation could in principle emerge in repeated relationships — he wanted to know what kinds of strategies individuals might use to in fact sustain such cooperation. The answer is far from obvious. Once relationships have no clear end (and we cannot use the concept of subgame perfection to solve the game backwards because each subgame is identical to the whole game), many different strategies, some sustaining cooperation and others not, can be part of a subgame perfect Nash equilibrium. So which will people actually choose?

To answer this question, Axelrod did several very clever experiments. First, he asked the world’s most eminent game theorists to submit strategies that they think might do well in repeated Prisoners’ Dilemmas that have no definitive end. He placed no limit on how complex these strategies could be and included them all in a computer simulation in which different strategies encountered

---

9If you are interested to learn more about these, you may want to read Axelrod, R. *The Evolution of Cooperation*, New York: Basic Books, 1984.
each other randomly. The strategy that consistently outperformed all others was remarkably simple and has become known as the “tit-for-tat” strategy.

Under the *tit-for-tat strategy*, a player begins at a first encounter with someone by cooperating and from then on mimics what the opposing player did at the last meeting. Thus, if the other player also cooperates, then the tit-for-tat player will cooperate next time. If the other player does not cooperate, the tit-for-tat player punishes him at the next meeting by not cooperating and will continue to not cooperate at each successive meeting unless the other player shows good will by cooperating at some point. If so, the tit-for-tat player will begin cooperating again. The strategy reminds me of what my mother told me when I was a child and she sent me to the playground to play with other kids. “Play nice with the other kids,” she would say, “but if someone hits you, you hit them back until they start being nice again.”

Axelrod also took the same strategies submitted by game theorists and did another simulation in which strategies “reproduced” if they achieved high average payoffs and decreased in the population if they received relatively low payoffs. As the computer simulation continued, unsuccessful strategies would therefore die out while successful strategies would increase in number. Eventually, he found, only one strategy survived this evolutionary process and was left standing: you guessed it — tit-for-tat. Eventually Axelrod showed that strategies that were “evolutionarily stable” had to have properties similar to the tit-for-tat strategies.\(^\text{10}\) Put differently, strategies that would do well in evolutionary settings had to (1) attempt cooperation and sustain it if it is reciprocated (i.e. the strategies have to be “nice”), (2) punish non-cooperation but (3) leave the door open for forgiving non-cooperation if a player signals that she is ready to cooperate again.

### 24A.3.4 Sustaining Cooperation (in Prisoners’ Dilemmas) through Institutions

As we have seen, it is possible for cooperation in Prisoners’ Dilemma games to emerge if the same players meet repeatedly without any definitive end of the repetitions. Even in such settings, however, equilibria without cooperation are also possible, and in settings other than that, cooperation unravels under subgame perfection. As we will see throughout the remainder of this text, there are, however, other ways in which market and non-market institutions might emerge to help sustain cooperation when the incentives in each interaction are themselves insufficient.

One possibility is for the individuals in a Prisoners’ Dilemma to write a contract that imposes sufficient penalties for not cooperating. If there is a way to enforce the penalties, such a contract in essence changes the payoffs in the matrix to eliminate the “Dilemma”. The prisoners in our game depicted in Table 24.6, for instance might be part of a “mafia” or a “gang” that has the rule that those who cooperate with prosecutors will be severely punished. In joining the “mafia”, individuals implicitly sign a contract that imposes penalties for not cooperating with the goals of the mafia (i.e. cooperating with prosecutors). Getting out of jail early looses some of its appeal if the prisoner knows he will be killed in some particularly gruesome way as soon as he is out.

But not all institutions that solve Prisoners’ Dilemma problems are as sinister as the mafia. Religious institutions might, for instance, persuade individuals that there are eternal benefits from cooperating — thus changing the way in which we evaluate the payoffs in a Prisoners’ Dilemma because we get “utility” from the act of cooperating. Private fund raisers have developed ways of “personalizing” our participation in large efforts to help the poor — and thus making us view the payoffs from helping others differently. For instance, you may have seen how organizations

---

\(^\text{10}\)The concept of evolutionary stability has precise meaning in a subfield of game theory known as evolutionary game theory (which is beyond the scope of this chapter).
that help poor children in developing countries offer the opportunity for individuals to “sponsor” particular children whose pictures and stories are shared with the donors. There is no particular reason to believe that the children whose pictures are sent to sponsors would not have been helped had the particular sponsor not decided to contribute to the organization, but the use of pictures personalizes the contribution in a way that appears to move people to give more.

And in some cases government policy can alter the payoffs in Prisoners’ Dilemma games – sometimes achieving positive and sometimes, as we will see, achieving less desirable outcomes. If individuals face Prisoners’ Dilemma incentives in their decision to give to charitable organizations, tax breaks for charitable contributions (or other forms of more explicit government subsidies for giving to charitable causes) might change behavior in the direction of greater efficiency. At the same time, if large corporations in concentrated industries face Prisoners’ Dilemma incentives when trying to collude on setting high prices, they might also look to government to act as the enforcer of their collusion. We will discuss this at greater length in our discussion of oligopolies in Chapter 25. For now I simply want to convince you that government policies and civil society institutions often look for ways to alter payoffs in situations in which Prisoners’ Dilemma incentives arise.

Exercise 24A.25 True or False: Whenever individuals find themselves in a Prisoners’ Dilemma game, there is profit to be made if someone can determine a way to commit players to change behavior.

Exercise 24A.26 How might your answer to the previous exercise help explain why we see more cooperation in real world Prisoners’ Dilemma games than we expect from the incentives contained in the game?

24A.3.5 Sustaining Cooperation (in Prisoners’ Dilemmas) through “Reputations”

Another way in which cooperation might emerge is if there is a way for individuals to credibly establish a reputation for cooperating. This is, however, far from trivial and requires the introduction of uncertainty on the part of one player with respect to the type of player she is facing in a (finitely) repeated setting. In other words, it requires the modeling of repeated interactions as sequential games of incomplete information, a topic we take up in Section B. We will therefore return to the role of reputations in finitely repeated Prisoners’ Dilemmas in Section 24B.3.

24A.4 Mixed Strategies

The distinction between “strategies” and “actions” has been most apparent for the case of sequential games where a plan for the game is different (for at least some players) than just picking an action. In simultaneous move games, however, pure strategies have involved simply picking an action – but this is not true for mixed strategies which we now explore.

Consider the following game: You and I are both asked to put a penny on the table. If our pennies “match” in the sense that they both have the same side of the penny showing, I end up getting your penny. If, on the other hand, the pennies do not “match” (in the sense that one shows “heads” and the other “tails”), you get my penny. This simple game, known as matching pennies, is illustrated in Table 24.9.

You should be able to convince yourself fairly quickly that there is no pure strategy Nash equilibrium to this game – my best response to any move of yours is to match it while your best response to any move of mine is to contradict it. In such a game, there is no way to predict for sure what will happen because the very structure of the game prohibits such predictability. A common way to think of this formally is then through the use of “mixed strategies.”
A mixed strategy for a player is simply a probability distribution over the pure strategies. (Even though we will only explore mixed strategies for simultaneous move games, the same definition holds for sequential move games.) For instance, I have 2 pure strategies in the matching pennies game: Heads and Tails. A mixed strategy is a set of two probabilities \((\rho, 1-\rho)\) such that \(0 \leq \rho \leq 1\). If I decide to play the mixed strategy \((0.5,0.5)\), it simply means that I will play Heads with probability 0.5 and Tails with probability 0.5. More generally, if a player has \(n\) different pure strategies available to her, a mixed strategy is a list of \(n\) probabilities \((\rho_1, \rho_2, ..., \rho_n)\) (with \(\rho_i \geq 0\) for all \(i=1,2,...,n\) and the sum of all \(\rho_i\)'s equal to 1).

**Exercise 24A.27** Note that it is always possible to write a pure strategy in the form of a mixed strategy with one probability set to 1 and the others set to zero. How would you write my pure strategy of Heads in the form of a mixed strategy?

### 24A.4.1 Best Responses to Mixed Strategies

Now suppose that I have some belief about the probability \(\lambda\) with which you will play Heads and I am trying to determine how best to respond by setting my own probability \(\rho\) of playing Heads. My goal is to match your penny – so if I think \(\lambda > 0.5\), I will do best by simply playing Heads all the time – i.e. by setting \(\rho = 1\). Similarly, if I think \(\lambda < 0.5\), I should just play Tails, which implies setting \(\rho = 0\). But if I think you are setting \(\lambda = 0.5\), I could always play Heads (i.e. \(\rho = 1\)) or always play Tails (i.e. \(\rho = 0\)), and my expected payoff would be exactly the same in either case.

**Exercise 24A.28** What would be my expected payoff if I play Heads all the time when you play the mixed strategy that places probability 0.5 of Heads?

Furthermore, I could play any mixed strategy in between and also get the same payoff. To see this, note that if you end up playing Heads (which you will do with probability 0.5), I will get your penny with probability \(\rho\) and will lose my penny with probability \((1-\rho)\). In expectation, I will therefore get \(\rho - (1-\rho) = 2\rho - 1\) in the event that you put down Heads. If, on the other hand, you put down Tails (which you will do half the time), I will win a penny with probability \((1-\rho)\) and lose a penny with probability \(\rho\). In expectation, I will therefore get \((1-\rho) - \rho = 1 - 2\rho\). Each of these expectations is equally likely, which means my expected payoff from playing the mixed strategy that places probability \(\rho\) on Heads when I believe you are playing a mixed strategy that places probability 0.5 on Heads is \(0.5(2\rho - 1) + 0.5(1 - 2\rho) = 0\) – exactly the same expected payoff as if I chose to simply always play Heads or always Tails (when you play Heads with probability 0.5).
24A. Game Theory under Complete Information

In panel (a) of Graph 24.6, we then graph my best response mixed strategy to all possible mixed strategies you might be playing. On the horizontal axis, we plot $\lambda$ – the probability you assign to *Heads*, while on the vertical axis we plot $\rho$. For any $\lambda < 0.5$, my best response is $\rho = 0$, and for any $\lambda > 0.5$, my best response is $\rho = 1$. Finally, for $\lambda = 0.5$, my best response can set $\rho$ anywhere between 0 and 1.

**Graph 24.6: Mixed Strategy Nash Equilibrium for Matching Pennies**

Panel (b) does the same from your perspective, illustrating your best response in terms of setting $\lambda$ to any possible $\rho$ that I might set. Finally, we put the two panels together in panel (c) of the graph and note that our best responses intersect at $\lambda = \rho = 0.5$.

### 24A.4.2 Mixed Strategy Nash Equilibrium

Recall that a Nash equilibrium requires each player to play a strategy that is a best response to the strategy played by the opposing player. This is no different for the case of mixed strategies – the only way we are in a Nash equilibrium is if you are “best responding” to my $\rho$ when you set $\lambda$ just as I am “best responding” to your $\lambda$ when I set my $\rho$. Put differently, the only time we are at a Nash equilibrium is if our best responses in Graph 24.6c intersect. In our “matching pennies” game, there is then only a single Nash Equilibrium – one in which both you and I play mixed strategies in which we place probability 0.5 on each of our two possible pure strategies.

The matching pennies game is a natural game to use to motivate the notions of mixed strategies and mixed strategy equilibrium because the game does not give rise to any pure strategy equilibria. But even in games with pure strategy equilibria, there may exist separate mixed strategy equilibria. Consider, for instance, our *Left/Right* game pictured in Table 24.2. In Graph 24.7, we again plot out the best responses for me and you to different mixed strategies by the other. It turns out that my best response function in panel (a) looks exactly like the one we plotted for me in the matching pennies game. This is because I am trying to match your action in both games. But *your* best response in panel (b) differs across the two games because you are trying to contradict my action in the matching pennies game while trying to match it in the *Left/Right* game. As a result, when we put the two best response functions together in panel (c), they now intersect 3 times: at $\rho = \lambda = 0$, at $\rho = \lambda = 0.5$ and at $\rho = \lambda = 1$. 

© Nechyba 2011
Chapter 24. Strategic Thinking and Game Theory

Graph 24.7: Nash Equilibria for Left/Right Game from Table 24.2

Notice that two of the intersections of the best response functions involve both of us playing one of our pure strategies with probability 1. These are simply the pure strategy Nash equilibria we identified earlier. In addition, however, we have now discovered a third Nash equilibrium in mixed strategies, one in which both of us play each of our two possible pure strategies with probability 0.5.

Exercise 24A.29 Is the mixed strategy equilibrium more or less efficient than the pure strategy equilibria in the Left/Right game?

Because of the particular payoff values we have chosen so far, the two mixed strategy equilibria that we have found both involve each player placing equal weight on each of her pure strategies. But one can easily identify games where a mixed strategy equilibrium involves other weights. For instance, in the version of the Left/Right game in which the payoffs for both players choosing Right are 5 rather than 10, you should be able to convince yourself that the mixed strategy equilibrium involves $\rho = \lambda = 1/3$. We can also think of settings in which the two players will place different probabilities on their pure strategies, such as, for instance, when the payoff from both choosing left is 10 for player 1 but 5 for player 2 and the payoff from both choosing right is 5 for player 1 and 10 for player 2. (This game is sometimes referred to as the “Battle of the Sexes” game for reasons explained in exercise 24.8.)

Exercise 24A.30 * Determine the mixed strategy Nash equilibrium for the game described in the previous sentence.

24A.4.3 A Quick Note on the Existence of Nash Equilibria

John Nash proved in 1950 that all well-defined games have at least one Nash equilibrium. The proof makes use of fixed point theorems that are beyond the scope of this text, but the intuition for it is simple: In graphs plotting best response functions to mixed strategies, each player’s best response function must cross the 45 degree line at some point – and this insures that the two players’ best response functions must cross at least once (though not necessarily on the 45 degree line). When
they cross, we have a Nash equilibrium. As we have already seen in the matching pennies game, not all games have pure strategy equilibria. Similarly, you should be able to convince yourself that the Prisoners’ Dilemma is a game in which there does not exist a mixed strategy equilibrium and we are therefore left with only the single pure strategy equilibrium. As a general rule, you can remember the following: If there are no pure strategy equilibria in a game you are asked to analyze, there is sure to be a mixed strategy equilibrium. If there is a single pure strategy equilibrium, you won’t find a mixed strategy equilibrium to the same game. But if there are two pure strategy equilibria, there will also be at least one mixed strategy equilibrium.

**Exercise 24A.31** Plot the best response functions to mixed strategies for the Prisoners’ Dilemma game and illustrate that there exists only a single, pure strategy equilibrium.

### 24A.4.4 How Should we Interpret Mixed Strategies

It is often a little difficult for students to figure out what to make of the concept of a “mixed strategy.” Taken literally, it means that players just randomize over pure strategies in some fashion. But there is another interpretation that many game theorists think makes more sense. In particular, it can be shown that if we change a game of complete information (in which all the players know everyone’s payoffs) to a very similar game with just a little bit of incomplete information (in which there is some uncertainty on the part of some players about the payoffs of other players), a mixed strategy equilibrium in the complete information game can be interpreted as a pure strategy Nash equilibrium in the incomplete information game. Put differently, the “mixing” might arise from a little uncertainty about other players’ payoffs. In Section B below, we now turn toward games of incomplete information, and in end-of-chapter exercise 24.4 you can explore how mixed strategies in games of complete information are in fact related to pure strategy equilibria in similar games with incomplete information. Exercise 24.7 also provides some real world examples where you might find the idea of mixed strategy equilibria somewhat persuasive (as it is in the matching pennies game).

### 24B Game Theory under Incomplete Information

As noted in the introduction to this chapter, we can distinguish between **complete information** games in which the payoffs of all the players are known to all players and **incomplete information** games in which some players do not know the payoffs of other players. So far, we have dealt only with complete information games. But there are economically important situations in which players don’t in fact have such complete information. Think, for example, of a sealed-bid auction in which you and I are bidding on a painting. I know what the painting is worth to me, but I have no idea what it is worth to you. I therefore know only my own payoff from winning the auction. Or think of two firms (in an industry that is not perfectly competitive) competing without knowing quite what costs the other is facing. Each firm will know its own profit under different output prices – but not the other firm’s. We now turn to such games of incomplete information and will distinguish once again between simultaneous move games and sequential games. Games of incomplete information are also often called **Bayesian games**.

#### 24B.1 Simultaneous Bayesian Games

When we first introduced games of complete information, we began by specifying the set of $N$ players, their possible actions and the payoffs each player receives from different combinations of
actions. In particular, we assumed that a player $n$ could take an action from a set of possible actions denoted $A^n$. Player $n$’s payoff was then given by a function $u^n : \mathbb{R}^N \rightarrow \mathbb{R}$ that specifies a payoff value $u^n(a^1, a^2, \ldots, a^N)$ for all possible combinations of actions that the $N$ players might take. In games of incomplete information, we similarly need to specify the set of $N$ players and their possible actions $A^n$, but the payoffs are now no longer common knowledge. We therefore have to introduce beliefs on the part of players about other players’ payoffs.

24B.1.1 “Types” and Beliefs

This is typically accomplished by assuming that players could be one of several (or many) types, and that player $n$’s payoff depends on her type $t$ as well as on the set of actions $(a^1, a^2, \ldots, a^N)$ taken by everyone in the game. If a player $n$ could be one of $T$ different types, she now has $T$ different possible payoff functions $(u^n_1, u^n_2, \ldots, u^n_T)$, with $u^n_t : \mathbb{R}^N \rightarrow \mathbb{R}$ giving the payoff $u^n_t(a^1, a^2, \ldots, a^N)$ when $n$ is type $t$. We will assume that each player knows her own payoff function (which is equivalent to saying that each player knows her own type) before she has to make a move in the game, but at least some players in the game only have beliefs about what type other players are. The set of types, as we will see in the examples below, could be a finite number of possible types (as in Section 24B.1.5) or a continuum of types (as in Section 24B.1.6).

To be more precise, beliefs are simply probability distributions that players have over the set of possible types that other players might be. Suppose there are 2 players, me and you, and that each of us could be one of 3 types. If I know my own type, then there are three possible scenarios I am facing – you could be type 1, 2 or 3. My beliefs about the game can then be characterized by the probability distribution $(\rho_1, \rho_2, \rho_3)$, where $0 \leq \rho_i \leq 1$ for all $i$ and $\sum \rho_i = 1$. This means that I believe you are a type 1 player with probability $\rho_1$, a type 2 player with probability $\rho_2$ and a type 3 player with probability $\rho_3$. If there are 3 players and 3 possible types, then I face 6 possible scenarios (assuming I know my own type) with beliefs given by the probability distribution $(\rho_{11}, \rho_{12}, \rho_{13}, \rho_{21}, \rho_{22}, \rho_{23})$ where $\rho_{ij}$ is the probability that the first player is of type $i$ and the second player is of type $j$. And if an opposing player can take on types from a continuous interval such as $T = [0, 1]$, we will see in the example of Section 24B.1.6 that the probability distribution is given in terms of a function $\rho : T \rightarrow \mathbb{R}$, with $\rho(t)$ equal to the probability that the player is a type less than or equal to $t$.

Exercise 24B.1 If there are $N$ players and $T$ possible types, how many probabilities constitute my beliefs about the other players in the game?

Note that this structure of beliefs as probability distributions makes it possible for some player $n$’s payoffs to be known with certainty by everyone – the other players’ beliefs would simply assign probability zero to player $n$ being of a different type. We therefore do not require that everyone is equally uncertain about what type everyone else in the game is but, even if one player is uncertain about another one’s type, we will call this a Bayesian game (of incomplete information).

24B.1.2 The Role of “Nature”

For reasons that will become clearer shortly, it has become common to introduce into Bayesian games a non-strategic fictional player called “Nature” (who has no payoffs) that moves prior to any other move. Thus, even simultaneous move Bayesian games have a sequential structure in the sense that Nature goes first and then everyone else moves at the same time. The only role played by Nature is that she assigns a type to each player, with knowledge of one’s own type becoming
private information for each individual. In some games, Nature might also share some information about other players' types with some of the players, perhaps leaving some players more informed than others. Only if all information about player types were shared with everyone in the game would the game cease to be one of incomplete information. In this sense, we can think of games of complete information as a special case of games of incomplete information. The crucial assumption we will make throughout is that all players know the probability distribution Nature uses to assign types to players, and each player is assigned her type independently of others. Put differently, all players in the game begin (prior to Nature moving) with the same initial beliefs about types.

24B.1.3 Strategies

Recall from our discussion of complete information games that a strategy is a complete plan of action prior to the beginning of the game. In the case of simultaneous move games with complete information, this implied that a pure strategy for player \( n \) involves picking an action from the set \( A^n \), but in the sequential move case, it meant something more than simply choosing an action for those players that moved later in the game. Specifically, in a sequential game, a strategy involved specifying an action for each possible prior history of the game. In the two player case, this meant that player 2’s strategy involved a plan for what to do for each possible action that player 1 might have taken in the first stage of the game – even if player 1 never chooses a particular action in equilibrium.

This is relevant for our discussion of simultaneous move Bayesian games because we have embedded the simultaneous moves that the players make into a sequential structure in which the fictional player Nature moves first. Since the game begins with Nature’s move – and since a strategy is a complete plan for how to play the game prior to the beginning of the game, a strategy now involves each player settling on what action she will take depending on which type Nature assigns to her. Put differently, by introducing the fictional player Nature as the first player in the game, we implicitly require that every actual player determines a plan for how to play the game before finding out what type of player he is.

At first glance, this may seem silly. After all, the player “Nature” is just a fiction – so why can’t we just assume that each player will simply decide on a a plan of action once she finds out her type? But think of it this way: Suppose you and I are in a simultaneous Bayesian game and I know what type Nature has assigned to me. Now I want to figure out what my best course of action is. In order to do that, I have to think about what your strategy will be, and your strategy will depend on what you think I will do. Since only I know my true type, you will have to use your beliefs to infer what I will do, which means you will need to think about what I would do depending on what type I am and then appropriately weight each of the possibilities by the probability your beliefs assign to me being a particular type. Thus, you have to be thinking about what I would do for each possible type that I could in fact have been assigned. And that in turn means that I need to think about what I would have done had I been assigned another type because this goes into your thinking about what you will do in the game.

A strategy in a simultaneous Bayesian game is therefore a plan of action for each possible type that a player might be assigned by Nature. If a player’s type is drawn from the set of possible types \( T \) and this player can choose from actions in the set \( A \), her strategy is then a function \( s: T \rightarrow A \) – i.e. a function that assigns to every possible type in \( T \) an action from \( A \). Such a strategy might have a player choosing the same action regardless of what type she was assigned, or it might have the player choose a different action for each type she might be assigned. We will later refer to the
first type of strategy as a “separating strategy” and the second as a “pooling strategy”. Regardless, however, it is important to remember that we will no more be able to find an equilibrium in a Bayesian game without fully specified strategies than we would be able to find an equilibrium in a sequential complete information game without specifying full strategies. Put differently, plans for what to do “off the equilibrium path” can, in either case, affect the nature of the equilibrium.

**Exercise 24B.2** In what sense does the distinction between Nash and subgame perfect equilibrium illustrate how “off the equilibrium path” plans – i.e. plans that are never executed in equilibrium – can be important?

### 24B.1.4 Bayesian Nash Equilibrium

Once we have fully understood the set-up of a simultaneous move Bayesian game and its implications for what a strategy is for each player, the definition for a Nash equilibrium is then exactly the same as it has always been, with one twist at the end: A (Bayesian) Nash equilibrium in a simultaneous move game of incomplete information occurs when each player’s strategy is a best response to every other player’s strategy given the player’s beliefs that are consistent with how the game is being played.

The “twist at the end” – the part that extends the concept of a Nash equilibrium to incomplete information games, is important in simultaneous games for the following reason: As we have already noted, we assume that everyone knows the probabilities with which the player Nature assigns types to players in the stage of the game that precedes the simultaneous move game. Unless new information is revealed in the course of the game – which does not happen when the rest of the game is a simultaneous move game – each person’s beliefs are therefore just the probabilities with which types are assigned. (This will change in a sequential game of incomplete information where information may be revealed in the actions taken by players that move early in the game). In simultaneous move Bayesian games, having beliefs be “consistent with how the game is being played” therefore means that equilibrium beliefs have to be consistent with how the player Nature plays the game.

**Exercise 24B.3** Do you agree or disagree with the following statement: “Both complete and incomplete information simultaneous move games can be modeled as games in which Nature moves first, but Nature plays only pure strategies in complete information games while it plays mixed strategies in incomplete information games.”

### 24B.1.5 A Simple Example

Suppose, for instance, that we consider the two (complete information) games from exercises 24A.8 and 24A.9 which are depicted at the bottom of Graph 24.8. These games differ only in terms of player 2’s payoffs – with payoffs for playing $R$ being 10 more in the first game than in the second. In both games, player 2 has a dominant strategy, but player 1’s best response will depend on player 2’s strategy. In particular, player 1’s best response to $L$ is $U$ (giving a payoff of 20 instead of 15), and his best response to $R$ is $D$ (giving payoff 5 instead of 0). Since player 1’s payoffs are the same in both games, these best responses to strategies played by player 2 are the same in both games.

**Exercise 24B.4** What is player 2’s dominant strategy in each of the two games?

Now suppose that there is a probability $\rho$ that player 2 will be of type I (with payoffs as in the first game) and a probability $(1 - \rho)$ that player 2 will be of type II (with payoffs as in the second
game). Player 2 knows what type he is before the game starts, but player 1 does not know what type he is facing in player 2. This is then a simultaneous move Bayesian game in which player 2 could be one of two possible types. To model this, we introduce a third player – “Nature” – that moves before the simultaneous game begins – assigning type I to player 2 with probability $\rho$ and type II with probability $(1 - \rho)$. If $\rho = 1$, the game is a complete information game in which player 1 plays a player 2 of type I – i.e. the two players simply play the game captured by the payoff matrix in the bottom left of Graph 24.8. If $\rho = 0$, the game is similarly a complete information game – but this time player 1 plays a player 2 of type II; i.e. the two players play the game captured by the payoff matrix on the bottom right of Graph 24.8. In the first case, player 1 would play $D$ in equilibrium – and in the second case, he would play $U$. But what will he play if $0 < \rho < 1$?

**Exercise 24B.5** How can we be sure that player 1 will play $D$ in equilibrium in the left-hand side game but $U$ in the right-hand side game?

Before answering this question, we need to show how we can illustrate – using either a payoff matrix or a game tree – the kind of game we have just introduced. Note first that our 2-player Bayesian game actually has 3 players once we introduce the fictional player Nature – and this makes it difficult to depict such a game in a payoff matrix. Second, note that this third player adds a sequential structure to the simultaneous game – which suggests that the resulting game might best be illustrated in a game tree (or “extensive form”). Such a tree would begin with Nature moving first – as is done in the game tree in Graph 24.8. If player 1 then moves second, we have to furthermore find a way to indicate that player 1 does not know the outcome of Nature’s move when it is his turn to play – because Nature only reveals player 2’s type to player 2. We do this by pulling both of player 1’s nodes in the game tree – the left-hand node that results from Nature assigning type I to player 2 and the right-hand node that results from Nature assigning type II to
player 2 – into a single information set. This is depicted in Graph 24.8 with the magenta oval that contains both of these nodes – and it indicates that player 1 is uncertain about which of his two possible nodes he is playing from when it comes time to make his move.

**Exercise 24B.6** Since all players know the probabilities with which types are assigned, how would you characterize player 1’s beliefs about which node he is playing from once the game reaches his information set?

Next, note that the two players play the (complete information) game depicted in the payoff matrix on the lower left of the graph if they are playing from player 1’s left-hand side node, and they play the (complete information) game depicted in the payoff matrix on the lower right of the graph if they are playing from player 1’s right-hand side node. In order for us to depict the Bayesian game (that includes Nature’s move) in a game tree, we therefore have to now find a way to depict the complete information games from these payoff matrices in game tree format. In Section A, when we showed how a sequential game can be depicted in a payoff matrix, we hinted at the fact that it was possible to represent a simultaneous move (complete information) game in a game tree — but we postponed illustrating this because there was no particular need to do so at the time and because we were still missing a key ingredient – the concept of an information set – which we just introduced.

Given this new tool, all we have to do is to make sure that the information sets over the nodes following Nature’s move are such that no new information is conveyed through the actions of any player – because all players following Nature’s move are playing simultaneously. Thus, player 2 does not know whether player 1 moved Up or Down – which means both actions by player 1 must end in the same information set for player 2. Put differently, we cannot allow player 2 to infer anything from the fact that player 1 has taken a particular action, because player 2 is acting at the same time as player 1 even though the game tree shows her making a decision farther down the game tree. But player 2 does know whether Nature assigned type I or type II – and thus whether she is playing the left-hand side or the right-hand side game. As a result, the information sets for player 2 do not cross from one side of the tree into the other.

**Exercise 24B.7** True or False: If we depict a simultaneous move (complete information) game in a game tree, each player only has one information set.

**Exercise 24B.8** How would you depict the complete information game from either of the payoff matrices in the graph if you had player 2 rather than player 1 at the top of the game tree?

Notice that the game tree in Graph 24.8 now fully captures all aspects of a simultaneous move Bayesian game – the actions that each player has available, the types that players might be assigned by Nature, the beliefs captured by the probability $\rho$ and the payoffs for each player and type. Reading the game tree from the top down, we see that Nature begins by moving left with probability $\rho$ and right with probability $(1 - \rho)$. We then see from player 1’s information set that player 1 cannot tell what Nature did when the time comes for him to choose between the actions Up and Down. We can furthermore note from player 2’s two information sets that player 2 can never tell whether player 1 has decided to go Up or Down but she can tell whether Nature moved left or right. Player 2 therefore has more information than player 1.

**Exercise 24B.9** You could also draw the game tree in Graph 24.8 with Player 2 going first and player 1 going second. What do the information sets look like if you depict the game in this way?
While (pure) strategies in each of the games at the bottom of Graph 24.8 are simply actions, strategies in the Bayesian game depicted in the graph are now more complicated for player 2 because they have to represent complete plans of action prior to the beginning of the game – prior to Nature’s move. Put differently, player 2’s strategy must specify an action for each of her information sets; i.e. for the case where Nature assigns her type I and for the case where Nature assigns her type II. Player 1, on the other hand, has only a single information set in the game, which implies that a pure strategy for player 1 is simply an action for that one information set.

**Exercise 24B.10** True or False: Player 2 has 4 possible strategies while player 1 has 2 possible strategies.

Now, since each of the two simultaneous move games at the bottom of Graph 24.8 has a dominant strategy for player 2, we know that player 2 will play \( R \) if she is assigned type I and \( L \) if she is assigned type II. Player 1 knows this and knows that she is at the first node in her information set with probability \( \rho \) and at the second node with probability \((1 - \rho)\). This implies that her expected payoff from playing \( U \) is \( 0\rho + 20(1 - \rho) \) while her expected payoff from playing \( D \) is \( 5\rho + 15(1 - \rho) \). The former is larger than the latter so long as \( \rho < 0.5 \). Thus, if \( \rho < 0.5 \), player 1 will play \( U \) and if \( \rho > 0.5 \) she will play \( D \). (If \( \rho = 0.5 \), she is indifferent between her two possible actions and could play either.)

**Exercise 24B.11** How would the outcome be different if the two games at the bottom of Graph 24.8 were the games in Table 24.5 and exercise 24A.7 (with player 2’s actions labeled \( L \) and \( R \) instead of \( U \) and \( D \)?)

### 24B.1.6 Another Example: Sealed Bid Auctions

One of the most common applications for simultaneous games of incomplete information is in the area of auctions. In a *sealed bid auction*, for instance, different players bid on the same item at the same time by submitting sealed bids, with none of the players knowing exactly what the item is worth to the other players. Consider such an auction in which the player that bids the most ends up getting the item and has to pay the price that she bid. This type of auction is called a *first-price sealed bid auction* (which you can compare to a *second-price sealed bid auction* described in end-of-chapter exercise 24.10.)

Suppose, for instance, that you and I are bidding on a painting. I know that the painting is worth at most \( t^i \) to me, and you know that it is worth at most \( t^j \) to you. But I do not know how much the painting is worth to you and you do not know how much it is worth to me. Suppose that all we know is that, for any potential bidder \( n \), the private value \( t^n \) is drawn randomly (and independently) from the uniform distribution on the interval \([0,1]\).\(^{11}\) Thus, the set of possible types is \( T = [0,1] \), and the probability that Nature assigns to a player a type \( t \) less than \( \overline{t} \) (for any \( 0 \leq \overline{t} \leq 1 \)) is simply \( \overline{t} \).

**Exercise 24B.12** What is the probability that Nature assigns a type greater than \( \overline{t} \) to a player?

Each player \( n \) has to choose an action \( a^n \) that is just her bid for the painting. If a player wins the auction, her payoff is her consumer surplus \((t^n - a^n)\). If a player loses the auction, on the other hand, she does not get the painting and does not have to pay anything – leaving her with payoff of 0. Finally, we will assume that, when both players bid the same amount, the auctioneer will flip a coin – which gives each player a 50% chance of winning the auction and thus an expected payoff

\(^{11}\)Assuming that individual valuations are drawn independently means that you cannot infer something about my valuation of the painting from knowing your valuation. Assuming that the distribution is uniform simply means that each value on the interval \([0,1]\) is equally likely to be drawn.
of \((t^n - a^n)/2\). (We will, however, be able to ignore the possibility of ties in our example because they happen with probability zero.)

**Exercise 24B.13** What is the set of possible actions \(A\) for this game?

A strategy for each of the bidders in this auction has to once again be a complete plan of action for every possible type that a player might be assigned. A type in this game is determined by the valuation \(t^n\) that a player was assigned by Nature – which could lie anywhere on the continuum between 0 and 1. Thus, a strategy must be a function \(s^n : [0, 1] \rightarrow \mathbb{R}\) that specifies a bid for each possible value that a player might place on the painting. It is possible to formally demonstrate that such strategies in this setting will, in equilibrium, take on a linear form; i.e. \(s^n(t^n) = \alpha_n + \beta_n t^n\).12

Suppose, then, that you play the strategy \(s_j(t^j) = \alpha_j + \beta_j t^j\). My best response to this strategy is to maximize my expected payoff which is (ignoring the possibility of a tie)

\[
\max_{a^i} (t^i - a^i) \text{Prob} \{ a^i > \alpha_j + \beta_j t^j \}. \quad (24.1)
\]

Simply by rearranging the terms in the inequality above, we can write the probability term as

\[
\text{Prob} \{ a^i > \alpha_j + \beta_j t^j \} = \text{Prob} \left\{ t^j < \frac{a^i - \alpha_j}{\beta_j} \right\} \quad (24.2)
\]

But recall that, given the underlying uniform probability distribution on the interval \([0, 1]\) with which Nature assigns types, the probability that \(t^j < \frac{a^i - \alpha_j}{\beta_j}\) is simply \(\frac{a^i - \alpha_j}{\beta_j}\), which implies

\[
\text{Prob} \left\{ t^j < \frac{a^i - \alpha_j}{\beta_j} \right\} = \frac{a^i - \alpha_j}{\beta_j} \quad (24.3)
\]

We can then rewrite equation (24.1) as

\[
\max_{a^i} (t^i - a^i) \frac{a^i - \alpha_j}{\beta_j} \quad (24.4)
\]

which solves to

\[
a^i = \frac{t^i + \alpha_j}{2}. \quad (24.5)
\]

**Exercise 24B.14** Verify that this is correct.

Thus, my best response to you playing \(s^j(t^j) = \alpha_j + \beta_j t^j\) is \(s^i(t^i) = \alpha_i + \beta_i t^i\) where \(\alpha_i = \alpha_j/2\) and \(\beta_i = 1/2\). If I play \(s^i(t^i) = \alpha_i + \beta_i t^i\), then the exact same steps imply that your best response is \(s^j(t^j) = \alpha_j + \beta_j t^j\) where \(\alpha_j = \alpha_i/2\) and \(\beta_j = 1/2\). But \(\alpha_i = \alpha_j/2\) and \(\alpha_j = \alpha_i/2\) can both hold only if \(\alpha_i = \alpha_j = 0\) – which implies that our equilibrium strategies are

\[
s^i(t^i) = \frac{t^i}{2} \quad \text{and} \quad s^j(t^j) = \frac{t^j}{2}. \quad (24.6)
\]

In other words, in equilibrium we will each bid half of the value that we attach to the painting.

\[12\] Demonstrating this involves the use of differential equations and is thus beyond the scope of this text. The mathematically inclined reader is referred to Gibbons’ text.
Exercise 24B.15 Suppose that both bidders know how much each of them value the painting; i.e. suppose the game was one of complete information. What would be the Nash equilibrium bidding behavior then? How does it differ from the incomplete information game?

This is, of course, a very simple auction setting, and there exist many different types of auctions and different economically relevant beliefs that might be introduced in different settings. In fact, over the past two decades, an extensive literature on auctions has developed (and an entire course could now be taught simply about auctions), all based on game theoretic modeling of the underlying incentives. This literature has guided the design of large auctions – such as auctions for rights to harvest timber on federal land or for rights to broadcast on particular frequencies. Many of these auctions, however, have a sequential structure that goes beyond the simultaneous Bayesian games we have defined so far.

24B.2 Sequential Bayesian Signaling Games

While we can think of economically interesting applications of simultaneous games of incomplete information, the set of potential applications of sequential games of incomplete information is much richer. Such games have the feature that some players have private information and, through their actions in the early part of the game, they can reveal some, all or none of that information to the other players. In our chapter on asymmetric information, we already dealt with situations of this kind – situations where buyers had less information than sellers (as in the used car market) or workers had more information (about their productivity) than potential employers or insurance clients had more information (about their risk type) than the insurance company. These instances of asymmetric information are precisely the kinds of economic situations that can be represented in sequential games of incomplete information – games in which the more informed party can signal something about herself or in which the less informed party can set up incentives so as to extract information.

Just as we needed to extend the concept of Nash equilibrium to that of subgame perfect Nash equilibrium in the sequential complete information case, we need to extend the concept of a Bayesian Nash equilibrium to that of a (subgame) perfect Bayesian Nash equilibrium in the sequential incomplete information case. And we need to do so for exactly the same reason as before – to eliminate implausible Nash equilibria that rely on non-credible behavior off the equilibrium path. To do so, however, we will again need to make beliefs, and not just strategies, part of the equilibrium. More precisely, we will need to specify what beliefs players hold on and off the equilibrium path in order to be sure the equilibrium strategies are in fact part of an equilibrium, and we need to make sure that players update their beliefs (from those they hold at the beginning of the game) if new information is revealed by the actions taken early on the game. We will return to these issues more formally after first illustrating them in concrete settings where we will simply use the logic of subgame perfection to find sensible equilibria in sequential Bayesian games. By the “logic of subgame perfection”, we will simply mean attacking the sequential game from the “bottom up” as we did in the complete information games of Section A.

24B.2.1 Simple Signaling when Beliefs Don’t Matter

We will use one of the most common families of games of incomplete information to fix ideas. This family of games is known as signaling games – games in which a person first finds out (from Nature) what type she is, then sends a “signal” to the other player before that other player takes an action
that impacts both players. Thus, the signaling player initially has private information that she might choose to reveal before the other player makes a move.

The simplest such setting is one in which one player (whom we will call the *sender*) might be one of two possible types and can send one of two possible signals; and in which the other player (whom we will call the *receiver*) then has to choose among two actions. Consider, for example, a sequential version of the simultaneous game we introduced in Graph 24.8. In that game, player 2 was one of two possible types – with her payoff depending on which type she was. To turn this game into a signaling game, player 2 would first find out her type, would then be able to play the actions $L$ or $R$ before player 1, *after observing player 2’s signal*, gets a chance to undertake her action of either $U$ or $D$. Thus, player 2 becomes the sender who signals through her choice of $L$ or $R$, and player 1 becomes the receiver. A convenient way to represent the new structure of this game is then given in Graph 24.9.

![Graph 24.9: Turning the Simultaneous Game in Graph 24.8 into a Signaling Game](image)

Unlike the game trees we have looked at so far, this tree begins in the center with Nature revealing the sender’s type – assigning type I with probability $\rho$ and type II with probability $(1 - \rho)$. After finding out her type, the sender can then play $L$ or $R$ (going either left or right in
the graph). The receiver only observes the sender’s actions – not her type. Thus, the receiver’s two nodes on the left (following $L$ by the sender) are in the same information set, as are the receiver’s two nodes on the right (following $R$ by the sender). Note that the person we called “player 2” in the simultaneous version of the game gets the private information and thus moves first in the signaling game. As you compare payoffs in Graph 24.9 to those in Graph 24.8, keep in mind that the first payoff at each terminal point in the sequential game should therefore correspond to player 2’s payoff in the simultaneous game.

**Exercise 24B.16** Check that the payoffs listed in Graph 24.9 correspond to the payoffs in Graph 24.8.

First, note that the receiver’s (subgame perfect) strategy is particularly easy to figure out in this game because, once the receiver observes which action the sender has taken, she knows exactly what she wants to do even if she is uncertain about which of the nodes in her information set she has reached. To be more precise, if the sender plays $L$, the receiver’s best response is $U$ regardless of what type the sender is, and if the sender plays $R$, the receiver’s best response is $D$ (again regardless of the sender’s type). This is indicated in the graph through the bold lines at each node for the receiver. The receiver’s (subgame perfect) strategy therefore must be $(U, D)$, where the first action indicates her plan if the sender plays $L$ and the second indicates her plan if the sender plays $R$. Since this strategy is optimal for the receive regardless of what type the sender is, beliefs do not play an important role in this game.

Next, let’s consider the possible strategies that the sender could employ and let’s recall that a strategy in a Bayesian game is a complete plan of action prior to the beginning of Nature’s move. Thus, the sender has to have a plan for what to do depending on what type she turns out to be. She therefore has four possible pure strategies: $(L, L)$, $(R, R)$, $(L, R)$ and $(R, L)$, where the first action in each pair corresponds to her plan if she turns out to be type I and the second action corresponds to her plan if she turns out to be type II. If she chooses one of the two latter strategies, she will implicitly reveal her type to the receiver because she is taking a different action depending on which type she is. This is therefore called a **separating strategy** because it involves separate observable actions depending on which type is assigned to the sender. The first two strategies, on the other hand, provide no information to the receiver beyond what the receiver already knows – i.e. the probabilities that nature assigns one type rather than the other. Such a strategy is called a **pooling strategy** because the different types of sender end up looking as if they came from the same pool.\(^\text{13}\)

We can then begin to look at each strategy for the sender and see if it could plausibly be part of an equilibrium. Suppose the sender plays $(L, L)$. We have already determined that the receiver’s optimal strategy is $(U, D)$ regardless of whether the sender reveals any information through her strategy – and so $(U, D)$ is a best response to $(L, L)$. Now all we have to do is check whether $(L, L)$ is also a best response for the sender to the receiver’s $(U, D)$. Note that both sender types would do worse by switching to $R$ given that the receive would respond by playing $D$ – with sender type I getting 5 rather than 10 and sender type II getting -5 rather than 10. Thus, $(L, L)$ for the sender and $(U, D)$ for the receiver are part of a (subgame perfect) equilibrium. You should also be able to convince yourself that none of the other possible pure strategies for the sender could be a (subgame perfect) equilibrium because in each case at least one of the types of sender would have an incentive to deviate given that the receiver is playing $(U, D)$.

**Exercise 24B.17** Determine for each of the three remaining sender pure strategies why the strategy cannot be part of a (subgame perfect) equilibrium.

---

\(^\text{13}\)When there are more than two types, we might get hybrid strategies in which some types pool and some separate.
Exercise 24B.18 Suppose the minus 5 payoff in the lower right corner of the game tree were 0 instead. Would we still get the same subgame perfect equilibrium? Could \((R,R)\) be part of a Nash equilibrium that is not subgame perfect?

Exercise 24B.19 Suppose that we changed the -5 payoff in Graph 24.9 to 20. Demonstrate that this would imply that only the separating strategy \((L,R)\) can survive in equilibrium.

Since the equilibrium we have identified involves both sender types playing the same “signal” \(L\), the receiver gets no information about the sender’s type from observing the sender’s action – and therefore the receiver cannot update her beliefs from those she held at the beginning of the game when she knew that Nature would assign type I to the sender with probability \(p\) and type II with probability \((1 - p)\). These are, then, the equilibrium beliefs for the receiver. But in this game, the receiver’s beliefs play not role because her response to either action on the part of the sender is clear cut and independent of her beliefs. This is not generally true in signaling games, and when it is not true, beliefs take on a much more critical role.

24B.2.2 Signaling Games where Beliefs Matter

Now suppose we change the game in Graph 24.9 slightly by changing the payoff for the receiver in the upper right of the graph (where type I sender plays \(R\) and the receiver plays \(U\)) from 0 to 10. This is depicted in Graph 24.10, and as a result of this change, the receiver’s optimal action when she observes \(R\) from the sender is no longer the same irrespective of her beliefs about which node within her information set she occupies when choosing the action. To be more precise, the optimal receiver action after the sender plays \(R\) is \(U\) if the sender is of type I and \(D\) if the sender is of type II (as indicated again through the bold lines in the graph). If the receiver observes \(L\) from the sender, she will still unambiguously play \(U\).

Recall that we extended the concept of a Nash equilibrium to a subgame perfect Nash equilibrium by insisting that a Nash equilibrium in a sequential game also consist of a Nash equilibrium in each subgame of the sequential game. Subgames were defined as beginning at a particular node that had been reached in the game tree. The problem we now face is that such subgames may not be readily available in games of the type depicted in Graphs 24.9 and 24.10. When the receiver gets to move after receiving a signal from the sender, she does not find herself at a particular node – rather she finds herself at an information set that contains two nodes, with some belief about which of two nodes she might actually be playing from. Those beliefs now become important for determining what the best response for the receiver should be if she observes \(R\).

Exercise 24B.20 In the previous section, we talked about subgame perfect strategies in ways that we cannot do here. What is different?

Suppose her belief after observing \(R\) is that the sender is of type I with probability \(\delta\) and of type II with probability \((1 - \delta)\). Her expected payoff from playing \(U\) is then \(10\delta + 0(1 - \delta)\) while her expected payoff from playing \(D\) is \(5\delta + 5(1 - \delta) = 5\). The latter is greater than the former if \(\delta < 0.5\), which implies that the receiver’s best response to observing \(R\) is to play \(D\) only if her belief is that the sender is more likely to be of type II than of type I. The receiver will then play \(U\) if she observes \(L\) regardless of what type she believes the sender to be, but she will play \(U\) after observing \(R\) only if she believes the sender is of type I with probability of at least 0.5. Otherwise, she will play \(D\).

Now we can check to see if the pooling strategy \((L,L)\) for the sender can still be part of an equilibrium. If the sender plays that strategy, we know that the receiver will play \(U\) – resulting in
Graph 24.10: A Small Change to the Previous Game and Beliefs Matter

payoff (10, 20) for the two players (regardless of what type the sender is). Now we can ask whether either of the sender types could do better by playing $R$, and the answer depends on what the receiver would do if she ever saw a signal $R$. If $(L, L)$ is indeed part of an equilibrium, the receiver will in fact never see the signal $R$ – but a full plan of action still requires her to have a plan in case she does see $R$, and we need to know what that plan is in order to be able to answer whether either of the sender types could do better by sending $R$ rather than $L$. If the receiver were to plan $U$ following a signal $R$, then a type I sender would indeed be better off sending $R$ rather than $L$, which in turn would imply that $(L, L)$ cannot be part of an equilibrium. And we just concluded in the previous paragraph that the receiver will play $U$ after observing $R$ only if $\delta > 0.5$. In order for the pooling strategy $(L, L)$ to be an equilibrium strategy, the receiver must therefore believe that the sender is more likely to be type II if a signal $R$ is observed. Put differently, the receiver’s beliefs have to be appropriately specified as part of the pooling equilibrium. And we see in this example that beliefs “off the equilibrium path” can be critical for sustaining an equilibrium; that is, in the equilibrium $\{(L, L), (U, D)\}$ where $\delta < 0.5$, it matters what the receiver believes in the event that $R$ is observed even though $R$ is not observed in equilibrium.
Exercise 24B.21 Is there any way for \((R, R)\) to be an equilibrium sender strategy? (Your answer should be no – can you explain why?)

We can also ask whether there is a separating equilibrium in this case; i.e. an equilibrium that involves the sender playing either \((L, R)\) or \((R, L)\). Consider first the strategy \((L, R)\). Under this strategy, the receiver knows with certainty which type the sender is because different sender types play different actions observable to the receive. As a result, the receive will update her beliefs; i.e. \(\gamma\), the probability that the sender is type I if \(L\) is observed, is 1 and \(\delta\), the probability that the sender is of type I if \(R\) is observed, is 0. That means that the receiver will play \(U\) after observing \(L\) and \(D\) after observing \(R\). Given this response by the receiver, a type I sender cannot do better by changing her signal to \(R\) because that would reduce her payoff from 10 to 5. But a type II sender can get a higher payoff by switching from the signal \(R\) to \(L\) given the receiver’s response. Thus, \((L, R)\) cannot be part of an equilibrium.

Exercise 24B.22 How much higher a payoff would a type II sender get by switching her signal in this way?

Next consider the other separating strategy – \((R, L)\). If the sender plays this strategy, the receiver will know that the sender is of type I if she observes \(R\) (i.e. \(\delta = 1\)), and she will know that the sender is of type II if she observes \(L\) (i.e. \(\gamma = 0\)). Either way, her best response is to play \(U\). For this to be an equilibrium, we now have to again make sure that neither of the two sender types could do better (given that the receiver will always play \(U\)). If type I switched, her payoff would fall from 15 to 10, and if type II switched, her payoff would fall from 10 to 5. Thus, neither type can benefit from deviating from the strategy \((R, L)\), which means we have found a separating equilibrium \(\{(R, L), (U, U)\}\) with equilibrium beliefs \(\delta = 1\) and \(\gamma = 0\). (The initial probability \(\rho\) with which Nature assigned types no longer matters because all information is revealed in the separating strategy played by the sender.)

Exercise 24B.23 * For the game in Graph 24.10, we have therefore found both a separating and a pooling equilibrium, but for the pooling equilibrium we needed to place a restriction on out-of-equilibrium beliefs. Do you find these restrictions “reasonable” in this example?\(^{14}\)

24B.2.3 Signaling Games where Beliefs and Nature’s Probabilities Matter

In the previous example, we have seen that out-of-equilibrium beliefs on the part of the receiver might be critical in sustaining a pooling equilibrium. This was because the optimal action differed across the two nodes in the information set that is not reached in equilibrium. Beliefs along the pooling equilibrium path have not yet played a crucial role because so far we have had examples in which the optimal action from each node in the information set that is reached in the pooling equilibrium is the same.

Now suppose we change the game in Graph 24.10 a little more by changing the receiver’s payoff from playing \(U\) when she faces a type I sender who plays \(L\) from 20 to 10. This new game is depicted in Graph 24.11, with the optimal receiver actions from each node again highlighted. Note that now we have a game in which the receiver’s optimal action differs across the nodes in each of her two information sets.

First, we can begin with the receiver and ask which way she will play from each of her information sets. If she observes \(L\) and thus plays from her left information set, her payoff from \(U\) is \(10\gamma + \)

\(^{14}\)This is far from a trivial question and it has concerned game theorists a great deal. After all, what does it mean for beliefs related to events that do not happen in equilibrium to be reasonable? An approach to this, known as the “Intuitive Criterion” has been derived. You can read more about this in Section 4.4 of Gibbons.
Graph 24.11: Beliefs Matter Even More

20(1 − γ) = 20 − 10γ while her payoff from D is 15γ + 15(1 − γ) = 15. The former is larger than the latter so long as γ < 0.5, which means the receiver will play U following L if she believes the probability that the sender is of type I is less than 0.5 and D if she believes that probability is greater than 0.5. Similarly, from what we did in the previous section we know that, the receiver will play U following R if δ > 0.5 and D if δ < 0.5.

Next we can begin again with the pooling strategy (L, L) and see whether it can still be part of an equilibrium. The receiver’s response would (as we just argued) depend on her belief γ, but if the two sender types both always play L, the receiver’s belief about the probability that she is facing each type after observing L should be unaltered from what it was at the beginning of the game. Since we assume that all players know the probability ρ with which Nature assigns types, this means that, under the sender strategy (L, L), γ = ρ. Since we determined that the receiver will play U from her left information set if γ < 0.5, this means that we know she will play U under the pooling strategy (L, L) so long as ρ < 0.5 and D so long as ρ > 0.5. But if the receiver were to play D, type I senders can make themselves better off by playing R since their payoff would be greater than 0 regardless of what the receiver planned in that event. So (L, L) cannot be a pooling equilibrium if ρ > 0.5, only if ρ < 0.5.
Exercise 24B.24 What has to be true about $\delta$ in order for $(L, L)$ to be an equilibrium pooling strategy when $\rho < 0.5$?

We can also check again if the second pooling strategy – $(R, R)$ – could be part of an equilibrium. If $(R, R)$ is played, the sender again reveals nothing about herself, which means that the receiver should not change her beliefs about what sender type she is facing if she observes $R$. Thus, $\delta = \rho$. Since the receiver will play $U$ from her right information set if $\delta > 0.5$ and $D$ if $\delta < 0.5$, we then know she will play $U$ if $\rho > 0.5$ and $D$ if $\rho < 0.5$. But if the receiver were to play $D$, type II senders can do better by deviating and playing $L$ since both possible payoffs for her would then be larger than -5. So $(R, R)$ cannot be part of an equilibrium if $\rho < 0.5$.

To insure that $(R, R)$ can be an equilibrium pooling strategy with the receiver playing $U$ after seeing $R$ when $\rho > 0.5$, we now need to make sure that type II senders can’t do better by deviating. Since such senders would get a payoff of 5 under the proposed equilibrium, this means they can’t think that the receiver would play $U$ if she observed $L$ (since that would result in a payoff for type II players of 10). We concluded before that the receiver would in fact play $D$ if she believed $\gamma > 0.5$. Thus, $(R, R)$ combined with $(D, U)$ are pooling equilibrium strategies so long as $\rho > 0.5$ and $\gamma > 0.5$. Since $L$ is never played in this equilibrium, any belief $\gamma$ is an out-of-equilibrium belief – and thus could take on any form including $\gamma > 0.5$. But still, despite the fact that $L$ is not played in this pooling equilibrium, it can be an equilibrium only if the receiver thinks an $L$ signal (that is never sent) is most likely indicative of a type I sender. We therefore have a pooling equilibrium $\{(R, R), (D, U)\}$ with beliefs $\delta = \rho > 0.5$ and $\gamma > 0.5$.

Finally, consider the separating equilibrium strategy $(R, L)$. If the sender plays this strategy, the receiver will best respond by playing $(U, U)$, which results in payoffs of 15 and 20 for type I and II senders respectively. Neither sender type can do better by deviating, which means we have found a separating equilibrium $\{(R, L), (U, U)\}$ with $\delta = 1$ and $\gamma = 0$ – an equilibrium where the source reveals his type and the receiver therefore knows with certainty which type the sender is by the time she has to choose an action.

Exercise 24B.25 Could the separating strategy $(L, R)$ be part of an equilibrium in this case?

Exercise 24B.26 * Suppose that, in the game in Graph 24.10, we had changed the receiver’s payoff from playing $U$ when facing a sender of type II who plays left from 20 to 5 instead. Could there be a separating equilibrium in that game? Is there a pure strategy equilibrium for all values of $\rho$?

24B.2.4 (Subgame) Perfect Bayesian Nash Equilibria in Signaling (and other) Games

So far, we have talked through several different signaling games, illustrating the possibility of separating and pooling equilibria and demonstrating the role that beliefs play in supporting such equilibria. Given the intuition we have developed, we can now be a little more precise about what we mean by an equilibrium in a sequential game of incomplete information such as a signaling game.

Recall that a game of incomplete information (or a Bayesian game) has the following components: (1) actions for each player; (2) types for each player; (3) beliefs about other players’ types; and (4) payoffs that depend on which types are actually in the game and what actions they take. Furthermore, recall that we have assumed throughout that all players know the probabilities with which Nature assigns types to individuals – and that these probabilities therefore form everyone’s initial beliefs. In simultaneous move games, those initial beliefs are the same throughout the game since no new information about other players’ types is revealed before an action has to be taken. But in sequential move games, individuals will update their beliefs if actions by others reveal new information.
We have seen such updating of beliefs in the signal game when we considered separating strategies by the sender. In that case, the sender fully revealed her type through the signals she sent, allowing the receiver to update her beliefs. In the case where the sender did not reveal additional information (because of the use of a pooling strategy), no updating had to be done once the receiver reached her information set – leaving her with the same beliefs she had at the beginning of the game. Off the equilibrium path, we did not restrict the receiver’s beliefs because it is not clear how one forms beliefs in circumstances that happen with zero probability. (We did hint in one of the exercises, however, that game theorists have developed reasonable restrictions (that are beyond the scope of this text) on such out-of-equilibrium beliefs).

More generally, updating of beliefs in sequential Bayesian games satisfies what is known as Bayes rule. Bayes rule in the context of sequential Bayesian games simply means the following: Suppose that a particular information set \( I \) contains nodes \( N_1, N_2, \ldots, N_k \), with \( P(N_i) \) giving the probability that node \( N_i \) is reached (and the probability of the information set \( I \) being reached therefore equal to \( \sum_{i=1}^{k} P(N_i) \)). Now suppose that, as the game progresses, the information set \( I \) is actually reached. Then the updated probability that \( N_i \) has been reached given that the information set \( I \) has been reached is

\[
P(N_i|I) = \frac{P(N_i)}{P(I)}.
\]

(24.7)

Suppose, for instance, that player 1 in a game moves first and has 3 available actions: \( a_1, a_2 \) and \( a_3 \). Suppose player 1 is playing a mixed strategy that places equal weight of 1/4 on \( a_1 \) and \( a_2 \) and 1/2 on \( a_3 \), and suppose that player 2 can tell whether player 1 has played \( a_1 \) but cannot tell the difference between player 1 having played \( a_2 \) and \( a_3 \). Thus, player 2 has two information sets – \( I_1 = \{a_1\} \) and \( I_2 = \{a_2, a_3\} \). Now suppose that player 2 faces a decision after reaching information set \( I_2 \) – i.e. suppose player 2 knows that player 1 did not play \( a_1 \). Then, according to Bayes’ Rule, player 2 now believes that player 1 has played actions \( a_2 \) and \( a_3 \) with probabilities 1/3 and 2/3 because

\[
P(a_2|I_2) = \frac{P(a_2)}{P(I_2)} = \frac{1/4}{3/4} = \frac{1}{3} \quad \text{and} \quad P(a_3|I_2) = \frac{P(a_3)}{P(I_2)} = \frac{1/2}{3/4} = \frac{2}{3}.
\]

(24.8)

with \( P(a_1|I_2) = 0 \). If, on the other hand, player 2 reaches information set \( I_1 \), then Bayes Rule says the updated probabilities are \( P(a_1|I_1) = \frac{1/4}{1/4} = 1 \) and \( P(a_2|I_1) = P(a_3|I_1) = 0 \).

Note that implicitly we have applied Bayes Rule a number of times as we updated beliefs in our signaling games. Suppose the sender played a pooling strategy \( L \), thus taking the receiver to the information set on the left of our game trees with probability 1. Let’s denote that information set as \( I_L \) which contains two nodes defined by whether the sender was a type I or a type II. To make the upcoming notation a bit easier to read, let’s denote type I as \( T_1 \) and type II as \( T_2 \) The receiver knows that Nature, at the beginning of the game, assigned \( T_1 \) to the sender with probability \( \rho \) and \( T_2 \) with probability \( 1 - \rho \). If the sender then plays a pooling strategy that results in the receiver making decisions from the information set \( I_L \), Bayes Rule implies that the receiver should have beliefs \( P(T_1|I_L) = P(T_1)/P(I_L) = \rho/1 = \rho \) and \( P(T_2|I_L) = P(T_2)/P(I_L) = (1 - \rho)/1 = (1 - \rho) \). Put differently, since the sender’s pooling strategy adds no information, no updating of beliefs occurs. Under a separating strategy where the sender plays \( L \) if type I and \( R \) if type II, Bayes rule implies \( P(T_1|I_L) = P(T_1)/P(I_L) = \rho/\rho = 1, P(T_2|I_L) = P(T_2)/P(I_L) = (1 - \rho)/(1 - \rho) = 1 \) and \( P(T_2|I_L) = 0 = P(T_1|I_R) \).
Exercise 24B.27 If the sender plays a pooling strategy \((L, L)\), why is the receiver’s belief about nodes in the information set \(I_R\) undefined according to Bayes’ Rule?

Earlier, we said a Bayesian Nash equilibrium occurs when each player’s strategy is a best response to every other player’s strategy given the player’s beliefs that are consistent with how the game is being played. We can now extend this formally to say that, in a sequential Bayesian game, a (subgame) perfect Bayesian Nash equilibrium is a Bayesian Nash equilibrium in which all the strategies and beliefs in all subgames (that begin at each information set) also constitute a Bayesian Nash equilibrium for each subgame. This is exactly analogous to the relationship between Nash equilibria and subgame perfect Nash equilibria in a complete information game – where subgame perfection in sequential settings required all subgames to be in equilibrium as well (and thus eliminated Nash equilibria that relied on non-credible strategies down the game tree). The difference in sequential Bayesian games is that at least some subgames now begin with information sets that contain more than a single node – and this in turn requires the specification of beliefs.

All such beliefs have to be “consistent with how the game is played” – which simply meant that all players shared beliefs consistent with Nature’s probabilities in our initial simultaneous move game where no new information could arise for players to update their beliefs. In a sequential setting, however, it means that beliefs have to be updated using Bayes rule wherever it applies (beginning with initial beliefs consistent with the probabilities employed by Nature). And Bayes rule applies at information sets that are reached with positive probability under the equilibrium strategies. At information sets that are reached with probability 0, however, Bayes rule does not apply and beliefs are therefore unrestricted – which is not the same as saying they can remain unspecified. In order to sustain an equilibrium, these “off-the-equilibrium-path” beliefs have to be structured so as to make the equilibrium strategies best responses to one another in all subgames that are not reached.

Exercise 24B.28 Is every Bayesian Nash equilibrium also a (subgame) perfect Bayesian Nash equilibrium? Is every (subgame) perfect Bayesian Nash equilibrium also a Bayesian Nash equilibrium? Explain.

Exercise 24B.29 True or False: When a game tree is such that all information sets are single nodes, then subgame perfect Nash equilibrium is the same as (subgame) perfect Bayesian Nash equilibrium.

24B.3 “Reputations” in Finitely Repeated Prisoners’ Dilemmas

In Section A of this chapter, we placed a lot of emphasis on the Prisoner’s Dilemma because, as we will see in the remainder of the text, it is a game that has particular relevance in many economic settings. We solved the simultaneous Prisoner’s Dilemma and found that there exists a single Nash equilibrium which involves both parties in the game choosing not to cooperate with one another despite the fact that the cooperative outcome is preferred by both to the non-cooperative outcome. We also found that, if two players face each other repeatedly a finite number of times, then the only subgame perfect Nash equilibrium again involves a lack of cooperation in every stage of the repeated game. But we noted that in experimental settings as well as in many real world settings, we see significantly more cooperation than what the model predicts, and we discovered a way to think about repeated Prisoners’ Dilemma games in which the players are uncertain about whether they will meet again each time that they meet or in which players expect to interact an infinite number of times. In such a setting, we argued, it is plausible that cooperation can emerge, and we show in the appendix that anything between no cooperation and full cooperation can in fact emerge
in infinitely repeated Prisoners’ Dilemma games (assuming players do not discount the future too heavily).

This set of results is, in some ways, quite odd. In finitely repeated Prisoners’ Dilemma games, not the slightest bit of cooperation can emerge under subgame perfection, while in the infinitely repeated game (or a game in which individuals are uncertain about whether they will meet again but think it sufficiently likely each time), all levels of cooperation can be sustained under subgame perfection. In some sense, one model seems to predict too little cooperation, the other potentially predicts too much.

We will now introduce a Bayesian element to repeated Prisoners’ Dilemma games in which players are uncertain about what type they face (and not about whether they will interact again). What we will find is that the introduction of uncertainty of a certain kind can result in equilibrium cooperation even in finitely repeated Prisoners’ Dilemma settings. We will see that the introduction of uncertainty on the part of one player about the type of player she is facing opens the possibility for the opposing player to establish a “reputation” for cooperation – a reputation that will cause cooperation to persist for some time even among rational players in finitely repeated Prisoners’ Dilemmas.

### 24B.3.1 Introducing the Possibility of a “Tit-for-Tat” Player

Suppose that Nature moves before the beginning of a finitely repeated Prisoners’ Dilemma game involving me and you (with me being player 1 and you being player 2), and suppose that the payoffs in each stage of this game (after Nature moves) are as in Table 24.10. Nature’s move determines my type, assigning me with probability \( \rho \) the “Tit-for-Tat” type \( t_1 \) and with probability \( (1 – \rho) \) the “rational player” type \( t_2 \). If I am assigned the Tit-for-Tat type, I will play the Tit-for-Tat strategy – “begin by playing \( C \) and then mimic for the rest of the game the last action played by the opposing player in the previous period.” If, on the other hand, I am assigned the “rational player” type, I simply maximize my own utility as we have assumed throughout. As in our signaling games, we assume that I learn my own type but you do not.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Cooperate</th>
<th>Don’t Cooperate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>10, 10</td>
<td>0, 15</td>
</tr>
<tr>
<td>Don’t Cooperate</td>
<td>15, 0</td>
<td>5, 5</td>
</tr>
</tbody>
</table>

Table 24.10: Prisoner’s Dilemma

Note that this is a little different than previous incomplete information introduced into our Bayesian games in which Nature assigned different payoffs to different types. Here Nature is rather assigning me a particular strategy (Tit-for-Tat) with probability \( \rho \) – thus removing choice about the strategy that I adopt in the event that I am assigned this type. One could argue that we are assuming Nature is making me “irrational” with probability \( \rho \) – but irrational in a particular way. One could also model this more in line with our previous models as a change in payoffs for the first
type such that Tit-for-Tat is the optimal strategy.\footnote{For instance we could simply assume that there is a chance that I was raised to believe Tit-for-Tat is the correct moral path in life, that I am deeply committed to this path and that I would suffer greatly if I chose a different path.}

24B.3.2 Considering a Twice-Repeated Prisoners’ Dilemma Game

Suppose first that we know we are going to play the Prisoners’ Dilemma twice and, to keep things as simple as possible, let’s suppose we do not discount the future. From our earlier discussion, we know that a typical rational player will choose $D$ the second time we play. If I end up being a $t_2$ (“rational”) player, I know this when we play the first time and will therefore choose $D$ each time we meet. If I am a $t_1$ type, I have no choice and will play the Tit-for-Tat strategy. But this leaves an open question for you: Should you play $C$ the first time we meet in the hope of me being a Tit-for-Tat type – which would mean you could get the cooperative payoff 10 the first time we meet and then get 15 the second time we meet (by playing $D$ when the Tit-for-Tat type will play $C$)?

Playing $C$ followed by $D$ then gives you a combined payoff across the two periods of 25 if you face a Tit-for-Tat type, but it gives you a payoff of only 5 if you end up facing me as a “rational” $t_2$. Put differently, playing $C$ first followed by $D$ gives you an expected payoff of $25\rho + 5(1 - \rho) = 20\rho + 5$ – while your expected payoff from playing $D$ both periods is $10\rho + 10$.

Exercise 24B.30 Verify the last sentence.

Thus, your expected payoff from playing $C$ followed by $D$ is larger than your expected payoff from playing $D$ always if $\rho > 0.5$. If I am therefore more likely to be a Tit-for-Tat player than a “rational” $t_2$ player, the (subgame) perfect Bayesian Nash equilibrium has you playing $(C, D)$, with me playing $(C, C)$ if I am a Tit-for-Tat player and $(D, D)$ if I am not.

Exercise 24B.31 What are the beliefs that support this as a (subgame) perfect Bayesian equilibrium?

24B.3.3 Considering a Thrice-Repeated Prisoners’ Dilemma Game

Now suppose that we instead know at the beginning that we are going to play the game three times and suppose that $\rho > 0.5$. I learn at the very beginning whether I am a Tit-for-Tat player or not, but you learn it only if I choose to reveal it by violating the Tit-for-Tat strategy when I am a type $t_2$.

Suppose, then, that I learn I am $t_2$ (and thus do not have to play the tit-for-tat strategy). If I play $D$ in the first game, I will have revealed to you that I am not a Tit-for-Tat type, and Bayesian updating of your beliefs will imply that you now place probability 1 on me being a $t_2$ type by the time we begin the second game. Knowing that, it will be best for you to play $D$ in the second and third game. If, on the other hand, I play $C$ in the first game after finding out that I am a $t_2$ type, I am at this point acting as if I was a Tit-for-Tat player by beginning the game with a pooling strategy. Bayes Rule then tells us that you have no information to update your beliefs about what type I am, which means we enter game 2 with the same information that we had at the beginning of the Twice-Repeated game we have just analyzed. This implies that, if you also played $C$ in the first round and thus a Tit-for-Tat type would begin playing game 2 with $C$, the beginning of game 2 is identical to the Twice-Repeated Prisoners’ Dilemma and our previous analysis holds for the rest of the game. Put differently, if both of us cooperate in the first stage, we know that you will play $C$ followed by $D$ in the second and third game while I will play $C$ for the rest of the game if I am a Tit-for-Tat type and $D$ for the rest of the game if I am not. If you observe me playing $D$
in the first stage, however, you will plan to play $D$ for the rest of the game. The question we now want to think about is whether the following strategies are part of a (subgame) perfect Bayesian Nash equilibrium:

**Strategy for Me if I am Type $t_2$:** Play $C$ in the first game and $D$ in the second and third game.

**Strategy for You:** Play $C$ in the first game. If you observe me also playing $C$ in the first game, play $C$ in the second game. Otherwise, play $D$ in the second game. Finally, play $D$ in the last game.

**Exercise 24B.32** Verify that, if we play these strategies, your expected payoff will be $35\rho + 15(1 - \rho) = 20\rho + 15$, and my payoff as a $t_2$ type will be 30.

Suppose, then, that I in fact play this strategy. Can you do better by playing $D$ in the first game? Since we know that you will do best playing $D$ in the third (i.e. the last) game, you will play either $D - D - D$ or $D - C - D$ over the three games if you choose $D$ in the first stage. By playing $D - D - D$, your payoffs will be $15 + 5 + 5 = 25$ if you face a Tit-for-Tat $t_1$ type (who will mimic your $D$'s in the second and third game) — exactly the same as if you faced a “rational” $t_2$ type who plays the suggested equilibrium strategy $C - D - D$. Your expected payoff from playing $D - D - D$ is then 25. By playing $D - C - D$, on the other hand, you will get payoffs $15 + 0 + 15 = 30$ if you face a Tit-for-Tat opponent (who will respond with $C - D - C$) and payoffs $15 + 0 + 5 = 20$ if you face a “rational” $t_2$ opponent — giving expected payoff $20\rho + 15$ (i.e. 20) from playing $D - C - D$. Since we are assuming $\rho > 0.5$ throughout, your expected payoff from $D - D - D$ (i.e. 25) then falls short of your expected payoff from $D - C - D$ (i.e. $20\rho + 15$) — which exceeds your expected payoff from the deviation $D - C - D$ given that $\rho > 0.5$. The suggested equilibrium strategy is therefore a best response to the $t_2$ strategy suggested for me (given that there is a probability $\rho > 0.5$ that I am a Tit-for-Tat player).

Next we can check if I have an incentive to deviate from the proposed strategy. We know from our work on the Twice-Repeated game that, if I do not deviate in the first stage by playing $D$, I cannot benefit from deviating in the second and third stage by playing $C$ (since the game starting in the second stage is identical to the Twice-Repeated game if both of us play $C$ in the first stage). So the only question is if I can benefit by playing $D$ rather than $C$ in the first stage — thereby revealing in the first game that I am a $t_2$. If I do so, I will get a payoff of 15 in the first game followed by payoffs of 5 in the next two stages for a total payoff of 25. But by playing the proposed strategy, my payoff is 30. I therefore cannot benefit from deviating from the proposed strategy, which means the proposed strategy is a best response to your proposed strategy.

We have therefore demonstrated that your suggested strategy is a best response to mine and mine is a best response to yours. In the “Thrice-Repeated” Prisoners’ Dilemma, both of us cooperating in the first game can therefore emerge as part of a (subgame) perfect Bayesian Nash equilibrium so long as the probability of me being a Tit-for-Tat player is sufficiently high. The reason for this is that it is now in my interest as a rational player to try to establish a “reputation” for being a Tit-for-Tat player (or, more generally, for being a cooperative player) in order to get you to cooperate with me for a while.
It should be intuitive that if we are trying to show that a (subgame) perfect Bayesian Nash equilibrium exists for $N$-Times Repeated Prisoners' Dilemma games with players cooperating up to some point in the game, such an equilibrium with early cooperation will also exist for an $(N + 1)$-Times Repeated Prisoners' Dilemma. Thus, by demonstrating that you and I (as a $t_2$ type) might choose to cooperate in the first game of a 3-Times repeated prisoners' dilemma if the probability of me being a Tit-for-Tat player is high enough, we have picked an unlikely game for which to demonstrate our result. As $N$ becomes larger, cooperation in the early part of the game becomes easier to sustain and can emerge for smaller probabilities of me being a Tit-for-Tat player. In fact, for large but finite $N$, this probability can get very close to zero – meaning that we will observe cooperation in Finitely Repeated Prisoners' Dilemma games even if there is only a small chance that one of the players is a Tit-for-Tat player.

It is furthermore the case that, for the payoffs in the game of Table 24.10, there is a perfect-Bayesian Nash equilibrium under which cooperation will persist (between you and me when I am a type $t_2$ player) in all games prior to the second to last game in an $N$-Times Repeated Prisoner’s Dilemma so long as $\rho > 0.5$. Thus, for a sufficiently high probability that one of the players is a Tit-for-Tat player, cooperation in an $N$-Times Repeated Prisoners' Dilemma can persist for long periods – $(N - 2)$ periods to be exact. Such cooperation will of course persist for a shorter period as $\rho$ falls.

If you have thought a bit about this problem, these results for the $N$-Times Repeated Prisoners’ Dilemma may seem intuitive, but it takes a little doing to prove formally. We will therefore forego formal proofs and simply note that we have, using the concept of (subgame) perfect Bayesian Nash equilibrium, arrived at one possible explanation for why we see cooperation in finitely repeated settings when subgame perfection suggests that such cooperation should not occur among rational players. That explanation essentially says that, in environments where there is some uncertainty about the type of opponents that players face, players (like me) may want to establish a reputation for being cooperative in order to sustain cooperation over some period of time.

Conclusion

In this chapter, we have developed a number of tools to help us think about economic situations in which individuals have an incentive to think strategically because their actions can influence the equilibrium that defines the economic environment we face. We had already begun doing this in Chapter 23 for the case of a monopolist – but the game theory tools developed here will now help us to extend our analysis of strategic thinking into a variety of other areas in which individuals are “large” relative to their economic environment.

In some ways, however, what we are doing is not different than what we have been doing all along: we are assuming that individuals seek to do the best they can given what everyone else is “doing” – or, more accurately, what others are “planning to do”. In strategic situations, this implies that individuals have to arrive at complete plans of actions – or strategies – and that doing the best they can given what everyone else is planning is the same as playing “best response strategies” to the strategies played by other players. When all players are “best responding” to all other players in this way, we have reached a Nash equilibrium (or a Bayesian Nash equilibrium in games involving incomplete information). And when such best responses involve players giving no credence to non-credible threats in sequential move games, it means that we have reached a subgame perfect
Nash equilibrium (or a (subgame) perfect Bayesian Nash equilibrium in incomplete information games). (In Bayesian games, we have seen that these equilibrium strategies must be accompanied by equilibrium beliefs that allow players to calculate their expected payoffs from different strategies.)

One of the fundamental insights from this chapter is that such equilibria in game theory models may not result in efficiency. Put differently, the equilibrium that emerges in a game might be such that there are alternative outcomes which all players would in fact prefer but which their rational individual (decentralized) decisions cannot reach without intervention by non-market institutions. Put into language developed earlier, decentralized decision making in strategic settings may violate the First Welfare Theorem because the equilibrium in such settings is not “competitive” but rather involves some form of market power (in the sense that individuals are “large” relative to the economic setting and thus have the power to shape the economic environment).

In the context of markets in which producers supply goods to consumers, such market power may imply that producers are “large” relative to the market, which further implies that producers may have the power to influence prices through their choice of how much to supply to the market (as we have already seen for monopolists). In the next two chapters, we will focus a bit more on this type of market power – but in cases that are less extreme than those for monopolists in Chapter 23. In the process we will see strategic thinking play a large role in choices made by producers, and we will see how such strategic choices may result in dead weight losses (and thus violations of the First Welfare Theorem). Throughout, we will draw on the game theory tools developed in this chapter. Following our treatment of market power on the part of producers, we will then proceed to other cases in which strategic thinking plays important roles, such as in market provision of public goods (Chapter 27) that will cause us to revisit the topic of externalities, and in the choices made by politicians (Chapter 28).

Appendix: Infinitely Repeated Games and The Folk Theorem

Consider the Prisoners’ Dilemma in Table 24.5 which is again depicted here in panel (a) of Graph 24.12 (with the actions relabeled C for “Cooperate” and D for “Don’t Cooperate”). The four possible payoff combinations are then graphed in panel (b). Each of these is of course a possible average (per-period) payoff in the infinitely repeated game if the two players were to always play the actions that lead to those payoffs in the simultaneous game. But by alternating different combinations of actions in different stages of the sequential game, other combinations of average payoffs per game become possible. For instance, if we alternated between both playing C and both playing D, we would alternate between payoffs of 5 and 10, thus getting an average payoff of 7.5 each. If we alternated between both playing C and player 1 playing D while player 2 plays C, player 1 would get an average payoff of 10 while player 2 would get an average payoff of 2.5.

**Exercise 24B.33** Propose a way that average payoffs could be 5 for player 1 and 12.5 for player 2.

You should be able to see that, by combining different ways of playing the game in different periods, any payoff combination in the shaded region in panel (c) of the graph can then arise in the infinitely repeated game as the average payoffs for the two players. The question we would like to turn to now is which of these average payoff combinations could arise in a subgame perfect Nash equilibrium.

The answer is relatively easy to see.\(^{16}\) We will begin by showing that the fully cooperative

\(^{16}\)This was first proposed in for repeated games more generally by Friedman, J. (1971): “A Non-cooperative Equilibrium for Supergames,” *Review of Economic Studies* 38, 1-12.
average payoff outcome (10,10) can emerge under subgame perfection and will then discuss how the same logic can lead to many other average equilibrium payoffs.

Suppose each player in the game plays what we previously called a trigger strategy of the following kind: Play $C$ in the first stage of the infinitely repeated game and continue to do so as long as both players cooperated in all previous stages; otherwise, play $D$. We can first check that these are best responses to one another. Suppose player 1 plays this strategy. If player 2 also plays the same strategy, she will receive a payoff of 10 in every stage of the game. Recall that 

$$1 + \delta + \delta^2 + \ldots = \frac{1}{1 - \delta},$$

which implies that the present discounted value of receiving a payoff of 10 in every period from now on is

$$10 + 10\delta + 10\delta^2 + \ldots = \frac{10}{(1 - \delta)},$$

(24.9)

where $\$1$ one period from now is worth $\delta < 1$. If player 2 decides to deviate from this trigger strategy, she will play $D$ now knowing that this will get her a payoff of 15 this period but then relegate her future payoffs to 5 per period as no more cooperation takes place. Thus, her present discounted value from deviating is

$$15 + 5\delta + 5\delta^2 + \ldots = 15 + 5\delta \left(1 + \delta + \delta^2 + \ldots\right) = 15 + \frac{5\delta}{(1 - \delta)},$$

(24.10)

As long as $\delta > 0.5$, equation (24.9) is greater than (24.10) and deviation from the trigger strategy does not pay in expected value terms. Put differently, as long as the players do not excessively discount the future (to the point where $\$1$ next period is worth less than 50 cents this period), the proposed trigger strategies are best responses to each other and thus constitute Nash equilibrium strategies.

To check whether these strategies are also subgame perfect, we need to check that they represent Nash equilibrium strategies for every subgame of the infinitely repeated game. Every subgame is
identical to the original game (since it, too, is an infinitely repeated Prisoner’s dilemma), and one of two possible histories of the game could have led up to any particular subgame: either all previous meetings between the players have resulted in both players playing $C$, or in at least one previous game at least one of the players played $D$. In the first case, we are still playing the same trigger strategy in the subgame which is identical to the original game for which we already demonstrated these trigger strategies to be a Nash equilibrium. In the second case, we are simply playing the strategy “Always $D$”. Given the other player $i$ plays this strategy, it is a best response for player $j$ to do the same – and so again we have a Nash equilibrium in the subgame. We can therefore conclude that the proposed trigger strategies are subgame perfect – and they result in full cooperation with average per period payoffs of 10 for each player.

The Folk Theorem, however, says more than this – not only is full cooperation possible through the use of the particular trigger strategy we specified, but partial cooperation is also possible. By partial cooperation, we mean sequences of equilibrium actions that result in payoffs for the two players that give more than the non-cooperative average payoff of 5 to each player. This corresponds to the average payoff combinations that lie in the shaded region in Graph 24.13.

![Graph 24.13: The Folk Theorem for the Infinitely Repeated Prisoners’ Dilemma](image_url)

It should not be too difficult to see how any of these payoffs could in fact emerge in a subgame perfect Nash equilibrium of the infinitely repeated game. Pick any payoff combination in the shaded area of Graph 24.13. By definition, these payoffs are greater than what a player could get under non-cooperation. Determine the sequence of actions necessary to insure the average payoff combination you chose and then define a trigger strategy that says “Play this sequence as long as the other player plays her part; otherwise switch forever to $D$.” You should be able to see that, as long as $\delta$ is sufficiently close to 1 – i.e. as long as we do not discount the future too much, it is a subgame perfect equilibrium for both players to play this trigger strategy.

You should also see that similar logic can be extended to games other than the Prisoners’ Dilemma where average payoffs above any simultaneous Nash equilibrium payoffs can arise under subgame perfection through the use of similar trigger strategies. Thus, the Folk Theorem is consid-
erably more general than simply applying to repeated Prisoners’ Dilemmas, and in fact it has been extended in ways that will become relevant when we discuss oligopoly behavior in Chapter 25.

End of Chapter Exercises

24.1 In the Hollywood movie “A Beautiful Mind”, Russel Crowe plays John Nash who developed the Nash Equilibrium concept in his PhD thesis at Princeton University. In one of the early scenes of the movie, Nash finds himself in a bar with three of his fellow (male) mathematics PhD students when a group of five women enters the bar.\(^{17}\) The attention of the PhD students is focused on one of the five women, with each of the four PhD students expressing interest in asking her out. One of Nash’s fellow students reminds the others of Adam Smith’s insight that pursuit of self interest in competition with others results in the socially best outcome, but Nash – in what appears to be a flash of insight – claims “Adam Smith needs revision”.

A: In the movie, John Nash then explains that none of them will end up with the woman they are all attracted to if they all compete for her because they will block each other as they compete – and that furthermore they will not be able to go out with the other women in the group thereafter (because none of them will agree to a date once they know they are at best everyone’s second choice). Instead, he proposes, they should all ignore the woman they are initially attracted to and instead ask the others out – it’s the only way they will get a date. He quickly rushes off to write his thesis – with the movie implying that he had just discovered the concept of Nash Equilibrium.

(a) If each of the PhD students were to play the strategy John Nash suggests – i.e. each one selects a woman other than the one they are all attracted to, could this in fact be a pure strategy Nash Equilibrium?
(b) Is it possible that any pure strategy Nash equilibrium could result in no one pursuing the woman they are all attracted to?
(c) Suppose we simplified the example to one in which it was only Nash and one other student encountering a group of two women. We then have two pure strategies to consider for each PhD student: Pursue woman A or pursue woman B. Suppose that each viewed a date with woman A as yielding a “payoff” of 10 and a date with woman B as yielding a payoff of 5. Each will in fact get a date with the woman that is approached if they approach different women, but neither will get a date if they approach the same woman in which case they both get a payoff of 0. Write down the payoff matrix of this game.
(d) What are the pure strategy Nash Equilibria of this game?
(e) Is there a mixed strategy Nash Equilibrium in this game?
(f) Now suppose there is also a woman C in the group of women – and a date with C is viewed as equivalent to a date with B. Again, each PhD student gets a date if he is the only one approaching a woman, but if both approach the same woman, neither gets a date (and thus both get a payoff of zero). Now, however, the PhD students have 3 pure strategies: A, B and C. Write down the payoff matrix for this game.
(g) What are the pure strategy Nash Equilibria of this game? Does any of them involve woman A leaving without a date?
(h) In the movie, Nash then explains that “Adam Smith said the best result comes from everyone in the group doing what’s best for themselves.” He goes on to say “...incomplete ... incomplete ... because the best result will come from everyone in the group doing what’s best for themselves and the group ... Adam Smith was wrong.” Does the situation described in the movie illustrate any of this?
(i) While these words have little to do with the concept of Nash Equilibrium, in what way does game theory – and in particular games like the Prisoners’ dilemma – challenge the inference one might draw from Adam Smith that self interest achieves the “best” outcome for the group?

B: Consider the 2-player game described in part A(c). (Note: Part (a) and (b) below can be done without having Section B of the Chapter.)
(a) Suppose that the players move sequentially – with player 1 choosing A or B first – and player 2 making his choice after observing player 1’s choice. What is the subgame perfect Nash equilibrium?
(b) Is there a Nash equilibrium in which player 2 goes out with woman A? If so, is there a non-credible threat that is needed to sustain this as an equilibrium?

\(^{17}\)Nash is actually with 4 others, but the rest of the scene unfolds as if there were 4 of them in total.
24B. Game Theory under Incomplete Information

(c) Next, consider again the simultaneous move game from A(c). Draw a game tree for this simultaneous move game – with player 1’s decision on the top. (Hint: Use the appropriate information set for player 2 to keep this game a simultaneous move game). Can you state different beliefs for player 2 (when player 2 gets to his information set) such that the equilibria you derived in A(d) and A(e) arise?

(d) Continue to assume that both players get payoff of 0 if they approach the same woman. As before, player 1 gets a payoff of 10 if he is the only one to approach woman A and a payoff of 5 if he is the only one to approach woman B. But player 2 might be one of two possible types: If he is type 1, he has the same tastes as player 1, but if he is of type 2, he gets a payoff of only 5 if he is the only one to approach woman A and a payoff of 10 if he is the only one to approach women B. Prior to the beginning of the game, Nature assigns type 1 to player 2 with probability \( \delta \) (and thus assigns type 2 to player 2 with probability \( (1 - \delta) \)). Graph the game tree for this game – using information sets to connect nodes where appropriate.

(e) What are the pure strategy equilibria in this game? Does it matter what value \( \delta \) takes?

24.2 Consider a sequential game which is known as the Centipede Game. In this game, each of two players chooses between “Left” and “Right” each time he or she gets a turn. The game does not, however, automatically proceed to the next stage unless players choose to go “Right” rather than “Left”.

A: Player 1 begins – and if he plays “Left”, the game ends with payoff of (1,0) (where here, and throughout this exercise, the first payoff refers to player 1 and the second to player 2). If however, he plays “Right”, the game continues and it’s player 2’s turn. If player 2 then plays “Left”, the game once again ends, this time with payoffs (0,2), but if she plays “Right”, the game continues and player 1 gets another turn. Once again, the game ends if player 1 decides to play “Left” – this time with payoffs of (3,1), but if he plays “Right” the game continues and it’s once again player 2’s turn. Now the game ends regardless of whether player 2 plays “Left” or “Right”, but payoffs are (2,4) if she plays “Left” and (3,3) if she plays “Right”.

(a) Draw out the game tree for this game. What is the subgame perfect Nash Equilibrium of this game.

(b) Write down the 4 by 4 payoff matrix for this game. What are the pure strategy Nash Equilibria in this game? Is the subgame perfect Nash Equilibrium you derived in (a) among these?

(c) Why are the other Nash Equilibria in the game not subgame perfect?

(d) Suppose you changed the (2,4) payoff pair to (2,3). Do we now have more than 1 subgame perfect Nash Equilibrium?

(e) How does your answer to (b) change?

(f) Consider again the original game but suppose I came as an outsider and offered to change the payoff pairs in the final stage from (2,4) and (3,3) to (2,2) and (4,4). How much would each of the two players be willing to pay me to change the game in this way (assuming we know that players always play subgame perfect equilibria)?

B: Consider the original centipede game described in part A. Suppose that, prior to the game being played, Nature moves and assigns a type to player 2, with type 1 assigned with probability \( \rho \) and type 2 with probability \( (1 - \rho) \). Throughout, type 1 is a rational player who understands subgame perfection.

(a) Suppose type 2 is a super-naive player that simply always goes “Right” whenever given a chance. For what values of \( \rho \) will player 1 go “Right” in the first stage?

(b) Suppose instead that type 2 always goes “Right” the first time and “Left” the second time. How does your answer change?

(c) (Note: This (and the next) part requires that you have read Chapter 17.) We have not explicitly mentioned this in the chapter – but game theorists often assume that payoffs are given in utility terms, with utility measured by a function \( u \) that allows gambles to be represented by an expected utility function. Within the context of this exercise, can you see why?

(d) Suppose the payoffs in the centipede game are in dollar terms, not in utility terms. What do your answers to (a) and (b) assume about the level of risk aversion of player 1?

24.3 Consider a simultaneous game in which both players choose between the actions “Cooperate”, denoted by \( C \), and “Defect”, denoted by \( D \).

A: Suppose that the payoffs in the game are as follows: If both players play \( C \), each gets a payoff of 1; if both play \( D \), both players get 0; and if one player plays \( C \) and the other plays \( D \), the cooperating player gets \( \alpha \) while the defecting player gets \( \beta \).

(a) Illustrate the payoff matrix for this game.
Chapter 24. Strategic Thinking and Game Theory

(b) What restrictions on \( \alpha \) and \( \beta \) would you have to impose in order for this game to be a Prisoners’ dilemma?

Assume from now on that these restrictions are in fact met.

**B:** *Now consider a repeated version of this game in which players 1 and 2 meet 2 times. Suppose you were player 1 in this game, and suppose that you knew that player 2 was a “Tit-for-Tat” player – i.e. a player that does not behave strategically but rather is simply programmed to play the Tit-for-Tat strategy.

(a) Assuming you do not discount the future, would you ever cooperate with this player?

(b) Suppose you discount a dollar in period 2 by \( \delta \) where \( 0 < \delta < 1 \). Under what condition will you cooperate in this game?

(c) Suppose instead that the game was repeated 3 rather than 2 times. Would you ever cooperate with this player (assuming again that you don’t discount the future)? (Hint: Use the fact that you should know the best action in period 3 to cut down on the number of possibilities you have to investigate.)

(d) In the repeated game with 3 encounters, what is the intuitive reason why you might play \( D \) in the first stage?

(e) If player 2 is strategic, would he ever play the “Tit-for-Tat” strategy in either of the two repeated games?

(f) Suppose that each time the two players meet, they know they will meet again with probability \( \gamma > 0 \). Explain intuitively why “Tit-for-Tat” can be an equilibrium strategy for both players if \( \gamma \) is relatively large (i.e. close to 1) but not if it is relatively small (i.e. close to 0).

### 24.4 Interpreting Mixed Strategies in the Battle of the Sexes

One of the most famous games treated in early game theory courses is known as the “Battle of the Sexes” – and it bears close resemblance to the game in which you and I choose sides of the street when you are British and I am American. In the “Battle of the Sexes” game, two partners in a newly blossoming romance have different preferences for what to do on a date, but neither can stand the thought of not being with the other. Suppose we are talking about you and your partner. You love opera and your partner loves football.\(^{18}\) Both you and your partner can choose to go to the opera and today’s football game, with each of you getting 0 payoff if you aren’t at the same activity as the other, 10 if you are at your favorite activity with your partner, and 5 if you are at your partner’s favorite activity with him/her.

**A:** In this exercise, we will focus on mixed strategies.

(a) Begin by depicting the game in the form of a payoff matrix.

(b) Let \( \rho \) be the probability you place on going to the opera, and let \( \delta \) be the probability your partner places on going to the opera. For what value of \( \delta \) are you indifferent between showing up at the opera or showing up at the football game?

(c) For what values of \( \rho \) is your partner indifferent between these two actions?

(d) What is the mixed strategy equilibrium to this game?

(e) What are the expected payoffs for you and your partner in this game?

**B:** *In the text, we indicated that mixed strategy equilibria in complete information games can be interpreted as pure strategy equilibria in a related incomplete information game. We will illustrate this here. Suppose that you and your partner know each other’s ordinal preferences over opera and football – but you are not quite sure just how much the other values the most preferred outcome. In particular, your partner knows your payoff from both showing up at the football game is 5, but he thinks your payoff from both showing up at the opera is \( 10 + \alpha \) with some uncertainty about what exactly \( \alpha \) is. Similarly, you know your partner gets a payoff of 5 if both of you show up at the opera, but you think his/her payoff from both showing up at the football game is \( 10 + \beta \), with you unsure of what exact value \( \beta \) takes. We will assume that both \( \alpha \) and \( \beta \) are equally likely to take any value in the interval from 0 to \( x \); i.e. \( \alpha \) and \( \beta \) are drawn randomly from a uniform distribution on \([0, x]\). We have thus turned the initial complete information game into a related incomplete information game in which your type is defined by the randomly drawn value of \( \alpha \) and your partner’s type is defined by the randomly drawn value of \( \beta \), with \([0, x]\) defining the set of possible types for both of you.

(a) Suppose that your strategy in this game is to go to the opera if \( \alpha > \alpha \) (and to go to the football game otherwise), with \( \alpha \) falling in the interval \([0, x]\). Explain why the probability (evaluated in the absence of knowing \( \alpha \)) that you will go to the opera is \( (x - \alpha)/x \). What is the probability you will go to the football game?

---

\(^{18}\)Since this game dates back quite a few decades, you can imagine which of the two players was referred to as the “husband” and which as the “wife” in early incarnation. I will attempt to write this problem without any such gender (or other) bias and apologize to the reader if he/she is not a fan of opera.
(b) Suppose your partner plays a similar strategy: go to the football game if $\beta > b$ and otherwise go to the opera. What is the probability that your partner will go to the football game? What is the probability that he/she will go to the opera?

(c) Given you know the answer to (b), what is your expected payoff from going to the opera for a given $\alpha$? What is your expected payoff from going to the football game?

(d) Given your partner knows the answer to (a), what is your partner’s expected payoff from going to the opera? What about the expected payoff from going to the football game?

(e) Given your answer to (c), for what value of $\alpha$ (in terms of $b$ and $x$) are you indifferent between going to the opera and going to the football game?

(f) Given your answer to (d), for what value of $\beta$ (in terms of $a$ and $x$) is your partner indifferent between going to the opera and going to the football game?

(g) Let $a$ be equal to the value of $\alpha$ you calculated in (e), and let $b$ be equal to the value of $\beta$ you calculated in (f). Then solve the resulting system of two equations for $a$ and $b$ (using the quadratic formula).

(h) Why do these values for $a$ and $b$ make the strategies defined in (a) and (b) pure (Bayesian Nash) equilibrium strategies?

(i) How likely is it in this equilibrium that you will go to the opera? How likely is it that your partner will go to the football game? How do your answers change as $x$ approaches zero – and how does this compare to the probabilities you derived for the mixed strategy equilibrium in part A of the exercise? (Hint: Following the rules of taking limits, you will in this case have to take the derivative of a numerator and a denominator before taking the limit.)

(j) True or False: The mixed strategy equilibrium to the complete information Battle of the Sexes game can be interpreted as a pure strategy Bayesian equilibrium in an incomplete information game that is almost identical to the original complete information game – allowing us to interpret the mixed strategies in the complete information game as arising from uncertainty that players have about the other player.

**24.5 Everyday Application: Splitting the Pot** Suppose two players are asked to split $100 in a way that is agreeable to both.

A: The structure for the game is as follows: Player 1 moves first – and he is asked to simply state some number between zero and 100. This number represents his “offer” to player 2 – the amount player 1 offers for player 2 to keep, with player 1 keeping the rest. For instance, if player 1 says “30”, he is offering player 2 a split of the $100 that gives $70 to player 1 and $30 to player 2. After an offer has been made by player 1, player 2 simply chooses from two possible actions: either “Accept” the offer or “Reject” it. If player 2 accepts, the $100 is split in the way proposed by player 1; if player 2 rejects, neither player gets anything. (A game like this is often referred to as an ultimatum game.)

(a) What are the subgame perfect equilibria in this game assuming that player 1 is restricted to making his “offer” in integer terms – i.e. assuming that player 1 has to state a whole number.

(b) Now suppose that offers can be made to the penny – i.e. offers like $31.24 are acceptable. How does that change the subgame perfect equilibria? What if we assumed dollars could be divided into arbitrarily small quantities (i.e. fractions of pennies)?

(c) It turns out that there are at most two subgame perfect equilibria to this game (and only 1 if dollars are assumed to be fully divisible) – but there is a very large number of Nash equilibria regardless of exactly how player 1 can phrase his offer (and an infinite number when dollars are assumed fully divisible). Can you, for instance, derive Nash equilibrium strategies that result in player 2 walking away with $80? Why is this not subgame perfect?

(d) This game has been played in experimental settings in many cultures – and, while the average amount that is “offered” differs somewhat between cultures, it usually falls between $25 and $50, with players often rejecting offers below that. One possible explanation for this is that individuals across different cultures have somewhat different notions of “fairness” – and that they get utility from “standing up for what’s fair”. Suppose player 2 is willing to pay $30 to stand up to “injustice” of any kind, and anything other than a 50-50 split is considered by player 2 to be unjust. What is now the subgame perfect equilibrium if dollars are viewed as infinitely divisible? What additional subgame perfect equilibrium arises if offers can only be made in integer amounts?

(e) Suppose instead that player 2 is outraged at “unfair” outcomes in direct proportion to how far the outcome is removed from the “fair” outcome, with the utility player 2 gets from rejecting an unfair offer equal to the difference between the amount offered and the “fair” amount. Suppose player 2 believes the “fair”
outcome is splitting the $100 equally. Thus, if the player faces an offer \( x < 50 \), the utility she gets from rejecting the offer is \((50 - x)\). What are the subgame perfect equilibria of this game now under the assumption of infinitely divisible dollars and under the assumption of offers having to be made in integer terms?

**B:** Consider the same game as that outlined in **A** and suppose you are the one that splits the $100 and I am the one who decides to accept or reject. You think there is a pretty good chance that I am the epitome of a rational human being who cares only about walking away with the most I can from the game. But you don’t know me that well – you think there is some chance \( \rho \) that I am a self-righteous moralist who will reject any offer that is worse for me than a 50-50 split. (Assume throughout that dollars can be split into infinitesimal parts.)

(a) Structure this game as an incomplete information game.

(b) There are two types of pure strategy equilibria to this game (depending on what value \( \rho \) takes). What are they?

(c) How would your answer change if I, as a self-righteous moralist (which I am with probability \( \rho \)) reject all offers that leave me with less than $10?

(d) What if it’s only less than $1 that is rejected by self-righteous moralists?

(e) What have we implicitly assumed about risk aversion?

### 24.6 Everyday Application: Another Way to Split the Pot

Suppose again, as in exercise 24.5, that two players have $100 to split between them.

**A:** But now, instead of one player proposing a division and the other accepting or rejecting it, suppose that player 1 divides the $100 into two piles and player 2 then selects his preferred pile.

(a) What is the subgame perfect equilibrium of this game?

(b) Can you think of a Nash equilibrium (with an outcome different than the subgame perfect outcome) that is not subgame perfect?

(c) In exercise 24.5, we considered the possibility of restricting offers to be in integer amounts, to be in pennies, etc. Would our prediction differ here if we made different such assumptions?

(d) Suppose that the pot was $99 and player 1 can only create piles in integer (i.e. full dollar) amounts. Who would you prefer to be: player 1 or 2?

(e) Suppose that player 2 has three possible actions: Pick up the smaller pile, pick up the larger pile, and set all of it on fire. Can you now think of Nash equilibria that are not subgame perfect?

**B:** In exercise 24.5, we next considered an incomplete information game in which you split the $100 and I was a self-righteous moralist with some probability \( \rho \). Assuming that the opposing player is some strange type with some probability can sometimes allow us to reconcile experimental data that differs from game theory predictions.

(a) Why might this be something we introduce into the game from exercise 24.5 but not here?

(b) If we were to introduce the possibility that player 2 plays a strategy other than the “rational” strategy with probability \( \rho \), is there any way that this will result in player 1 getting less than $50 in this game?

### 24.7 Everyday Application: Real World Mixed Strategies

In the text, we discussed the “Matching Pennies” game and illustrated that such a game only has a mixed strategy equilibrium.

**A:** Consider each of the following and explain (unless you are asked to do something different) how you might expect there to be no pure strategy equilibrium – and how a mixed strategy equilibrium might make sense.

(a) A popular children’s game, often played on long road trips, is “Rock, Paper, Scissors”. The game is simple: Two players simultaneously signal through a hand gesture one of three possible actions: Rock, Paper, or Scissors. If the two players signal the same, the game is a tie. Otherwise, Rock beats Scissors, Scissors beats Paper and Paper beats Rock.

(b) One of my students objects: “I understand that Scissors can beat Paper, and I get how Rock can beat Scissors, but there is no way Paper should beat Rock. What ... Paper is supposed to magically wrap around Rock leaving it immobile? Why can’t Paper do this to Scissors? For that matter, why can’t Paper do this to people?... I’ll tell you why: Because Paper can’t beat anybody!”

If Rock really could beat Paper, is there still a mixed strategy Nash Equilibrium?

---

19My student continues (with some editing on my part to make it past the editorial censors): “When I play “Rock, Paper, Scissors”, I always choose Rock. Then, when someone claims to have beaten me with Paper, I can punch them in the face with my already clenched fist and say – oh, sorry – I thought paper would protect you, moron.”
(c) In soccer, penalty kicks often resolve ties. The kicker has to choose which side of the goal to aim for, and, because the ball moves so fast, the goalie has to decide simultaneously which side of the goal to defend.

(d) How is the soccer example similar to a situation encountered by a professional tennis player whose turn it is to serve?

(e) For reasons I cannot explain, teenagers in the 1950’s sometimes played a game called “chicken”. Two teenagers in separate cars drove at high speed in opposite directions on a collision course – and whoever swerved to avoid a crash lost the game. Sometimes, the cars crashed and both teenagers were severely injured (or worse). If we think behavior in these games arose within an equilibrium, could that equilibrium be in pure strategies?

B: If you have done part B of exercise 24.4, appeal to incomplete information games with almost complete information to explain intuitively how the mixed strategy equilibrium in the chicken game of A(e) can be interpreted.

24.8 Everyday Application: Burning Money, Iterated Dominance and the Battle of the Sexes

Consider again the “Battle of the Sexes” game described in exercise 24.4. Recall that you and your partner have to decide whether to show up at the opera or a football game for your date – with both of you getting a payoff of 0 if you show up at different events and therefore aren’t together. If both of you show up at the opera, you get a payoff of 10 and your partner gets a payoff of 5, with these reversed if you both show up at the football game.

A: In this part of the exercise, you will have a chance to test your understanding of some basic building blocks of complete information games whereas in part B we introduce a new concept related to dominant strategies. Neither part requires any material from Section B of the chapter.

(a) Suppose your partner works the night shift and you work during the day – and, as a result, you miss each other in the morning as you leave for work just before your partner gets home. Neither of you is reachable at work – and you come straight from work to your date. Unable to consult one another before your date, each of you simply has to individually decide whether to show up at the opera or at the football game. Depict the resulting game in the form of a payoff matrix.

(b) In what sense is this an example of a “coordination game”?

(c) What are the pure strategy Nash equilibria of the game.

(d) After missing each other on too many dates, you come up with a clever idea: Before leaving for work in the morning, you can choose to burn $5 on your partner’s nightstand – or you can decide not to. Your partner will observe whether or not you burned $5. So we now have a sequential game where you first decide whether or not to burn $5, and you and your partner then simultaneously have to decide where to show up for your date (after knowing whether or not you burned the $5). What are your four strategies in this new game?

(e) What are your partner’s four strategies in this new game (given that your partner may or may not observe the evidence of the burnt money depending on whether or not you chose to burn the money.)

(f) Illustrate the payoff matrix of the new game assuming that the original payoffs were denominated in dollars. What are the pure strategy Nash Equilibria?

B: In the text, we defined a dominant strategy as a strategy under which a player does better no matter what his opponent does than he would under any other strategy he could play. Consider now a weaker version of this: We will say that a strategy B is weakly dominated by a strategy A for a player if the player does at least as well playing A as he would playing B regardless of what the opponent does.

(a) Are there any weakly dominated strategies for you in the payoff matrix you derived in A(f)? Are there any such weakly dominated strategies for your partner?

(b) It seems reasonable that neither of you expects the other to play a weakly dominated strategy. So take your payoff matrix and strike out all weakly dominated strategies. The game you are left with is called a reduced game. Are there any strategies for either you or your partner that are weakly dominated in this reduced game? If so, strike them out and derive an even more reduced game. Keep doing this until you can do it no more – what are you left with in the end?

(c) After repeatedly eliminating weakly dominated strategies, you should have ended up with a single strategy left for each player. Are these strategies an equilibrium in the game from A(f) that you started with?

(d) Selecting among multiple Nash equilibria to a game by repeatedly getting rid of weakly dominated strategies is known as applying the idea of iterative dominance. Consider the initial game from A(a) (before we introduced the possibility of you burning money). Would applying the same idea of iterative dominance narrow the set of Nash equilibria in that game?
24.9 * Everyday and Business Application: Bargaining over a Fixed Amount: Consider a repeated version of the game in exercise 24.5. In this version, we do not give all the proposal power to one person but rather imagine that the players are bargaining by making different proposals to one another until they come to an agreement. In part A of the exercise we analyze a simplified version of such a bargaining game, and in part B we use the insights from part A to think about an infinitely repeated bargaining game. (Note: Part B of the exercise, while conceptually building on part A, does not require any material from Section B of the Chapter.)

A: We begin with a 3-period game in which $100 gets split between the two players. It begins with player 1 stating an amount $x_1$ that proposes she should receive $x_1$ and player 2 should receive $(100 - x_1)$. Player 2 can then accept the offer – in which case the game ends with payoff $x_1$ for player 1 and $(100 - x_1)$ for player 2; or player 2 can reject the offer, with the game moving on to period 2. In period 2, player 2 now has a chance to make an offer $x_2$ which proposes player 1 gets $x_2$ and player 2 gets $(100 - x_2)$. Now player 1 gets a chance to accept the offer – and the proposed payoffs – or to reject it. If the offer is rejected, we move on to period 3 where player 1 simply receives $x$ and player 2 receives $(100 - x)$. Suppose throughout that both players are somewhat impatient – and they value $1$ a period from now at $\delta$ ($\delta < 1$). Also suppose throughout that each player accepts an offer whenever he/she is indifferent between accepting and rejecting the offer.

(a) Given that player 1 knows she will get $x$ in period 3 if the game continues to period 3, what is the lowest offer she will accept in period 2 (taking into account that she discounts the future as described above)?

(b) What payoff will player 2 get in period 2 if he offers the amount you derived in (a)? What is the present discounted value (in period 2) of what he will get in this game if he offers less than that in period 2?

(c) Based on your answer to (b), what can you conclude player 2 will offer in period 2?

(d) When the game begins, player 2 can look ahead and know everything you have thus far concluded. Can you use this information to derive the lowest possible period 1 offer that will be accepted by player 2 in period 1?

(e) Given that player 1 knows she will get $x$ in period 3 if the game continues to period 3, what is the lowest offer she will accept in period 2 (taking into account that she discounts the future as described above)?

(f) What payoff will player 2 get in period 2 if he offers the amount you derived in (a)?

(g) True or False: The more player 1 is guaranteed to get in the third period of the game, the less will be offered to player 2 in the first period (with player 2 always accepting what is offered at the beginning of the game).

B: Now consider an infinitely repeated version of this game; i.e. suppose that in odd-numbered periods – beginning with period 1 – player 1 gets to make an offer that player 2 can accept or reject, and in even-numbered periods the reverse is true.

(a) True or False: The game that begins in period 3 (assuming that period is reached) is identical to the game beginning in period 1.

(b) Suppose that, in the game beginning in period 3, it is part of an equilibrium for player 1 to offer $x$ and player 2 to accept it at the beginning of that game. Given your answer to (a), is it also part of an equilibrium for player 1 to begin by offering $x$ and for player 2 to accept it in the game that begins with period 1?

(c) In part A of the exercise, you should have concluded that – when the game was set to artificially end in period 3 with payoffs $x$ and $(100 - x)$, player 1 ends up offering $x_1 = 100 - \delta(100 - \delta x)$ in period 1, with player 2 accepting. How is our infinitely repeated game similar to what we analyzed in part A when we suppose, in the infinitely repeated game beginning in period 3, the equilibrium has player 1 offer $x$ and player 2 accepting the offer?

(d) Given your answers above, why must it be the case that $x = 100 - \delta(100 - \delta x)$?

(e) Use this insight to derive how much player 1 offers in period 1 of the infinitely repeated game. Will player 2 accept?

(f) Does the first mover have an advantage in this infinitely repeated bargaining game? If so, why do you think this is the case?
24B. Game Theory under Incomplete Information

24.10 Everyday and Business Application: Auctions: Many items are sold not in markets but in auctions where bidders do not know how much others value the object that is up for bid. We will analyze a straightforward setting like this here – which technically means we are analyzing (for much of this exercise) an incomplete information game of the type covered in Section B of the chapter. The underlying logic of the exercise is, however, sufficiently transparent for you to be able to attempt the exercise even if you have not read Section B of the chapter. Consider the following – known as a second-price sealed bid auction. In this kind of auction, all people who are interested in an item \( x \) submit sealed bids (simultaneously). The person whose bid is the highest then gets the item \( x \) at a price equal to the second highest bid.

**A:** Suppose there are \( n \) different bidders who have different marginal willingness to pay for the item \( x \). Player \( i \)'s marginal willingness to pay for \( x \) is denoted \( v_i \). Suppose initially that this is a complete information game – i.e. everyone knows everyone’s marginal willingness to pay for the item that is auctioned.

(a) Is it a Nash equilibrium in this auction for each player \( i \) to bid \( v_i \)?

(b) Suppose individual \( j \) has the highest marginal willingness to pay. Is it a Nash equilibrium for all players other than \( j \) to bid zero and player \( j \) to bid \( v_j \)?

(c) Can you think of another Nash equilibrium to this auction?

(d) Suppose that players are not actually sure about the marginal willingness to pay of all the other players – only about their own. Can you think of why the Nash equilibrium in which all players bid their marginal willingness to pay is now the most compelling Nash equilibrium?

(e) Now consider a sequential first price auction in which an auctioneer keeps increasing the price of \( x \) in small increments and any potential bidder signals the auctioneer whether she is willing to pay that price. (Assume that the signal from bidders to auctioneer is not observable by other bidders.) The auction ends when only a single bidder signals a willingness to pay the price – and the winner then buys the item \( x \) for the price equal to his winning bid. Assuming the increments the auctioneer uses to raise the price during the auction are sufficiently small, approximately what will each player’s final bid be?

(f) In equilibrium, approximately what price will the winner of the sequential auction pay?

(g) True or False: The outcome of the sealed bid second price auction is approximately equivalent to the outcome of the sequential (first price) auction.

**B:** This part provides a real-world example of how an auction of the type analyzed in part A can be used. When I became Department Chair in our economics department at Duke, the chair was annually deciding how to assign graduate students to faculty to provide teaching and research support. Students were paid a stipend by the department but their services were free to the faculty member to whom they were assigned.

(a) Under this system, faculty complained perpetually of a “teaching assistant shortage”. Why do you think this was?

(b) I replaced the system with the following: Aside from some key assignments of graduate students as TAs to large courses, I no longer assigned any students to faculty. Instead, I asked the faculty to submit dollar bids for the right to match with a graduate student. If we had \( N \) graduate students available, I then took the top \( N \) bids, let those faculty know they had qualified for the right to match with a student and then let the matches take place (with students and faculty seeking each other out to create matches). Every faculty member who had a successful bid was then charged (to his/her research account) a price equal to the lowest winning bid which we called the “market price”. (Students were still paid by the department as before – the charges to faculty accounts simply came into the chair discretionary account and were then redistributed in a lump sum way to all faculty.) Given that we have a large number of faculty, should any individual faculty member think that his/her bid would appreciably impact the “market price”?

(c) In my annual e-mail to the faculty at the beginning of the auction for rights to match with students, I included the following line: “For those of you who are not game theorists, please note that it is a dominant strategy for you to simply bid the actual value you place on the right to match with a student.” Do you agree or disagree with this statement? Why?

(d) Would it surprise you to discover that, for the rest of my term as chair, I never again heard complaints that we had a “TA shortage”? Why or why not?

(e) Why do you think I called the lowest winning bid the “market price”? Can you think of several ways in which the allocation of students to faculty might have become more efficient as a result of the implementation of the new way of allocating students?
Chapter 24. Strategic Thinking and Game Theory

24.11 Business Application: *Monopoly and Price Discrimination*: In Chapter 23, we discussed first, second and third degree price discrimination by a monopolist. Such pricing decisions are strategic choices that can be modeled using game theory – which we proceed to do here. Assume throughout that the monopolist can keep consumers who buy at low prices from selling to those how are offered high prices.

**A:** Suppose a monopolist faces two types of consumers – a high demand consumer and a low demand consumer. Suppose further that the monopolist can tell which consumer has low demand and which has high demand; i.e. the consumer types are observable to the monopolist.

(a) Can you model the pricing decisions by the monopolist as a set of sequential games with different consumer types?

(b) Suppose the monopolist can construct any set of two-part tariffs – i.e. a per-unit price plus fixed fee for different packages. What is the subgame perfect equilibrium of your games?

(c) *True or False:* First degree price discrimination emerges in the subgame perfect equilibrium but not in other Nash equilibria of the game.

(d) How is this analysis similar to the game in exercise 24.5?

(e) Next, suppose that the monopolist cannot charge a fixed fee but only a per-unit price – but he can set different per-unit prices for different consumer types. What is the subgame perfect equilibrium of your games now?

**B:** Next, suppose that the monopolist is unable to observe the consumer type but knows that a fraction $\rho$ in the population are low demand types and a fraction $(1-\rho)$ are high demand types. Assume that firms can offer treatments of labor demand earlier in the text, we assumed that firms could observe the marginal revenue product of workers – and thus would hire until wage is equal to marginal revenue product. But suppose a firm cannot observe a worker’s productivity perfectly, and suppose further that the worker himself has some control over his productivity through his choice of whether to exert effort or “shirk” on the job. In part A of the exercise we will consider the subgame perfect equilibrium of a game that models this, and in part B we will see how an extension of this game results in the prediction that firms might combine “above market” wages with the threat to fire the worker if he is not productive. Such wages – known as *efficiency wages* – essentially have firms employing a “carrot-and-stick” approach to workers: Offer them high wages (the carrot), thus making the threat of firing more potent. (Note: It is recommended that you only attempt this problem if you have covered the whole chapter.)

**A:** Suppose the firm begins the game by offering the worker a wage $w$. Once the worker observes the firm’s offer, he decides to accept or decline the offer. If the worker rejects the offer, the game ends and the worker is employed elsewhere at his market wage $w^*$. (a) Suppose the worker’s marginal revenue product is $MRP = w^*$. What is the subgame perfect equilibrium for this game when marginal revenue product is not a function of effort?

(b) Next, suppose the game is a bit more complicated in that the worker’s effort is correlated with the worker’s marginal revenue product. Assuming he accepted the firm’s wage offer, the worker can decide to exert effort $e > 0$ or not. The firm is unable to observe whether the worker is exerting effort – but it does observe how well the firm is doing overall. In particular, suppose the firm’s payoff from employing the worker is $(x - w)$ if the worker exerts effort, but if the worker shirks, the firm’s payoff is $(x - w)$ with probability $\gamma < 1$ and $(w - w)$ with probability $(1 - \gamma)$. For the worker, the payoff is $(w - e)$ if the worker exerts effort and $w$ if he does not. What is the firm’s expected payoff if the worker shirks?

(c) How must $w^*$ be related to $\gamma$ and $x$ in order for it to be efficient for the worker not to be employed by the firm if the worker shirks?

(d) Suppose the worker exerts effort $e$ if hired by the firm. Since $e$ is a cost for the worker, how must $w^*$ be related to $(x - e)$ in order for it to be efficient for non-shirking workers to be hire by the firm?

(e) Suppose $w^*$ is related to $\gamma$, $x$ and $e$ such that it is efficient for workers to be hired by the firm only if they don’t shirk – i.e. if the conditions you derived in (c) and (d) hold. What is the subgame perfect equilibrium? Will the firm be able to hire workers?
The problem in the game defined in part A is that we are not adequately capturing the fact that firms and workers do not typically interact just once if a worker is hired by a firm. Suppose, then, that we instead think of the relationship between worker and firm as one that can potentially be repeated infinitely. Each day the firm begins by offering a wage \( w \) to the worker; the worker accepts or rejects the offer – walking away with a market wage \( w^* \) (and ending the relationship) if he rejects. If he accepts, the worker either exerts effort \( e \) or shirks – and the firm observes whether it ends the day with a payoff of \( (x - w) \) (which it gets for sure if the worker exerts effort but only with probability \( \gamma < 1 \) if the worker shirks) or \( (-w) \) (which can happen only if the worker shirks). Everyone goes home at the end of the day and meets again the next day (knowing how all the previous days turned out).

(a) Consider the following strategy for the firm: Offer \( w = \overline{w} > w^* \) on the first day; then offer \( w = \overline{w} \) again every day so long as all previous days have yielded a payoff of \( (x - \overline{w}) \); otherwise offer \( w = 0 \). Is this an example of a trigger strategy?

(b) Consider the following strategy for the worker: Accept any offer \( w \) so long as \( w \geq w^* \); reject offers otherwise. Furthermore, exert effort \( e \) upon accepting an offer so long as all previous offers (including the current one) have been at least \( \overline{w} \); otherwise shirk. Is this another example of a trigger strategy?

(c) Suppose everyone values a dollar next period at \( \delta < 1 \) this period. Suppose further that \( P_e \) is the present discounted value of all payoffs for the worker assuming that firms always offer \( w = \overline{w} \) and the worker always accepts and exerts effort. Explain why the following must then hold: \( P_e = (\overline{w} - e) + \delta P_e \).

(d) Use this to determine the present discounted value \( P_e \) of the game (as a function of \( \overline{w}, e \) and \( \delta \)) for the worker assuming it is optimal for the worker to exert effort when working for the firm.

(e) Suppose the firm offers \( w = \overline{w} \). Notice that the only way the firm can ever know that the worker shirked is if its payoff on a given day is \( (-\overline{w}) \) rather than \( (x - \overline{w}) \) – and we have assumed that this happens with probability \( 1 - \gamma \) when the worker exerts no effort. Thus, a worker might decide to take a chance and shirk – hoping that the firm will still get payoff of \( (x - \overline{w}) \) (which happens with probability \( \gamma \)). What is the worker’s immediate payoff (today) from doing this?

(f) Suppose that the worker gets unlucky and is caught shirking the first time – and that he therefore will not be employed at a wage other than the market wage \( w^* \) starting on day 2. In that case, what is the present discounted value of the game that begins on day 2? (Note: The infinite sum \( \delta + \delta^2 + \delta^3 + \ldots \) is equal to \( \delta/(1 - \delta) \).)

(g) Suppose that the worker’s expected payoff from always shirking is \( P_s \). If the worker does not get caught the first day he shirks, he starts the second day exactly under the same conditions as he did the first – implying that the payoff from the game beginning on the second day is again \( P_s \). Combining this with your answer to parts (e) and (f), explain why the following equation must hold:

\[
P_s = \overline{w} + \delta \left[ \gamma P_s + (1 - \gamma) \frac{w^*}{1 - \delta} \right]. \tag{24.11}
\]

Derive from this the value of \( P_s \) as a function of \( \delta, \gamma, \overline{w}, e \) and \( w^* \).

(h) In order for the worker’s strategy in (b) to be a best response to the firm’s strategy in (a), it must be that \( P_e \geq P_s \). How much of a premium above the market wage \( w^* \) does this imply the worker requires in order to not shirk? How does this premium change with the cost of effort \( e \)? How does it change with the probability of getting caught shirking? Does this make sense?

(i) What is the highest that \( \overline{w} \) can get in order for the firm to best respond to workers (who play the strategy in (b)) by playing the strategy in (a)? Combining this with your answer to (h), how must \( (x - e) \) be related to \( w^*, \delta, \gamma \) and \( e \) in order for the strategies in (a) and (b) to constitute a Nash equilibrium? Given your answer to (i), will it always be the case that firms hire non-shirking workers whenever it is efficient?

### 24.13 Policy Application: Negotiating with Pirates, Terrorists (and Children):

While we often think of Pirates as a thing of the past, piracy in international waters has been on the rise. Typically, pirates seize a commercial vessel and then demand a monetary ransom to let go of the ship. This is similar to some forms of terrorism where, for instance, terrorists kidnap citizens of a country with which the terrorists have a grievance – and then demand some action by the country in exchange for the hostages.

**A:** Oftentimes, countries have an explicit policy that “we do not negotiate with terrorists” – but still we often discover after the fact that a country (or a company that owns a shipping vessel) paid a ransom or took some other action demanded by terrorists in order to resolve the crisis.
An employer gets profit by expending effort, but it costs type 1 workers to offer; and (1 − δ) if she hires a type 1 worker at wage w. (Employers get zero profit if they do not hire a worker). We then assume that the worker decides in stage 1 how much education to get; then, in stage 2, he approaches two competing employers who decide simultaneously how much of a wage to offer; and finally, in stage 3, he decides which wage offer to accept.

A: Suppose first that worker productivity is directly observable by employers; i.e. firms can tell who is a type 1 and who is a type 2 worker by just looking at them.

(a) Solving this game backwards, what strategy will the worker employ in stage 3 when choosing between wage offers?

(b) Given that firms know what will happen in stage 3, what wage will they offer to each of the two types in the simultaneous move game of stage 2 (assuming that they best respond to one another)? (Hint: Ask yourself if the two employers could offer two different wages to the same worker type, and – if not – how competition between them impacts the wage that they will offer in equilibrium.)

(c) Note that we have assumed that worker productivity is not influenced by the level of education e chosen by a worker in stage 1. Is there any way that the level of e can then have any impact on the wage offers that a worker gets in equilibrium?

(d) Would the wages offered by the two employers be any different if the employers moved in sequence – with employer 2 being able to observe the wage offer from employer 1 before the worker chooses an offer?

(e) What level of e will the two worker types then get in any subgame perfect equilibrium?

(f) True or False: If education does not contribute to worker productivity and firms can directly observe the productivity level of job applicants, workers will not expend effort to get education, at least not for the purpose of getting good wage offers.

B: Now suppose that employers cannot tell the productivity level of workers directly – all they know is the fraction δ of workers that have high productivity and the education level e of job applicants.

(a) Will workers behave any differently in stage 3 than they did in part A of the exercise?

(b) Suppose that there is a separating equilibrium in which type 2 workers get education that differs from the education level type 1 workers get – and thus firms can identify the productivity level of job applicants by observing their education level. What level of education must type 1 workers be getting in such a separating equilibrium?
(c) What wages will the competing firms offer to the two types of workers? State their complete strategies and the beliefs that support these.

(d) Given your answers so far, what values could $\pi$ take in this separating equilibrium? Assuming $\pi$ falls in this range, specify the separating perfect Bayesian Nash equilibrium — including the strategies used by workers and employers as well as the full beliefs necessary to support the equilibrium.

(e) Next, suppose instead that the equilibrium is a pooling equilibrium — i.e. an equilibrium in which all workers get the same level of education $\pi$ and firms therefore cannot infer anything about the productivity of a job applicant. Will the strategy in stage 3 be any different than it has been?

(f) Assuming that every job applicant is type 2 with probability $\delta$ and type 1 with probability $(1 - \delta)$, what wage offers will firms make in stage 2?

(g) What levels of education $\pi$ could in fact occur in such a perfect Bayesian pooling equilibrium? Assuming $\pi$ falls in this range, specify the pooling perfect Bayesian Nash equilibrium — including the strategies used by workers and employers as well as the full beliefs necessary to support the equilibrium.

(h) Could there be an education level $\pi$ that high productivity workers get in a separating equilibrium and that all workers get in a pooling equilibrium?

(i) What happens to the pooling wage relative to the highest possible wage in a separating equilibrium as $\delta$ approaches 1? Does this make sense?

24.15 Everyday, Business and Policy Application: To Fight or Not to Fight: In many situations, we are confronted with the decision of whether to challenge someone who is currently engaged in a particular activity. In personal relationships, for instance, we decide whether it is worthwhile to push our own agenda over that of a partner; in business, potential new firms have to decide whether to challenge an incumbent firm (as discussed in one of the examples in the text); and in elections, politicians have to decide whether to challenge incumbents in higher level electoral competitions.

A: Consider the following game that tries to model the decisions confronting both challenger and incumbent:

The potential challenger moves first — choosing between staying out of the challenge, preparing for the challenge and engaging in it, or entering the challenge without much preparation. We will call these three actions "out," $P$ to fight the challenge ($F$) and $O$ (for "unprepared entry"). The incumbent then has to decide whether to fight the challenge ($F$) or give into the challenge ($G$) if the challenge takes place; otherwise the game simply ends with the decision of the challenger to play $O$.

(a) Suppose that the payoffs are as follows for the five potential combinations of actions, with the first payoff indicating the payoff to the challenger and the second payoff indicating the payoff to the incumbent: $(P, G)$ leads to $(3,3)$; $(P, F)$ leads to $(1,1)$; $(U, G)$ leads to $(4,3)$; $(U, F)$ leads to $(0,2)$; and $O$ leads to $(2,4)$. Graph the full sequential game tree with actions and payoffs.

(b) Illustrate the game using a payoff matrix (and be careful to account for all strategies).

(c) Identify the pure strategy Nash equilibria of the game and indicate which of these is subgame perfect.

(d) Next, suppose that the incumbent only observes whether or not the challenger is engaging in the challenge (or staying out) but does not observe whether the challenger is prepared or not. Can you use the logic of subgame perfection to predict what the equilibrium will be?

(e) Next, suppose that the payoffs for $(P, G)$ changed to $(3,2)$, the payoffs for $(U, G)$ changed to $(4,2)$ and the payoffs for $(U, F)$ changed to $(0,3)$ (with the other two payoff pairs remaining the same). Assuming again that the incumbent fully observes both whether he is being challenged and whether the challenger is prepared, what is the subgame perfect equilibrium?

(f) Can you still use the logic of subgame perfection to arrive at a prediction of what the equilibrium will be if the incumbent cannot tell whether the challenger is prepared or not as you did in part (d)?

B: Consider the game you ended with in part A(f).

(a) Suppose that the incumbent believes that a challenger who issues a challenge is prepared with probability $\delta$ and not prepared with probability $(1 - \delta)$. What is the incumbent’s expected payoff from playing $G$? What is his expected payoff from playing $F$?

(b) For what range of $\delta$ is it a best response for the incumbent to play $G$? For what range is it a best response to play $F$?

(c) What combinations of strategies and (incumbent) beliefs constitute a pure strategy subgame perfect Nash equilibrium? (Be careful: In equilibrium, it should not be the case that the incumbent’s beliefs are inconsistent with the strategy played by the challenger!)
Chapter 24. Strategic Thinking and Game Theory

(d) Next, suppose that the payoffs for \( (P,G) \) changed to \((4,2)\) and the payoffs for \( (U,G) \) changed to \((3,2)\) (with the remaining payoff pairs remaining as they were in A(f)). Do you get the same pure strategy subgame perfect equilibria?

(e) In which equilibrium – the one in part (c) or the one in part (d) – do the equilibrium beliefs of the incumbent seem more plausible?

24.16 * Everyday and Policy Application: Reporting a Crime: Most of us would like to live in a world where crimes are reported and dealt with, but we’d sure prefer to have others bear the burden of reporting a crime. Suppose a crime is witnessed by \( N \) people, and suppose the cost of picking up the phone and reporting a crime is \( c > 0 \).

A: Begin by assuming that everyone places a value \( x > c \) on the crime being reported, and if the crime goes unreported, everyone’s payoff is 0. (Thus, they payoff to me if you report a crime is \( x \), and the payoff to me if I report a crime is \( (x - c) \).)

(a) Each person then has to simultaneously decide whether or not to pick up the phone to report the crime. Is there a pure strategy Nash equilibrium in which no one reports the crime?

(b) Is there a pure strategy Nash equilibrium in which more than one person reports the crime?

(c) There are many pure strategy Nash equilibria in this game. What do all of them have in common?

(d) Next, suppose each person calls with probability \( \delta < 1 \). In order for this to be a mixed strategy equilibrium, what has to be the relationship between the expected payoff from not calling and the expected payoff from calling for each of the players?

(e) What is the payoff from calling when everyone calls with probability \( \delta < 1 \)?

(f) What is the expected payoff from not calling when everyone calls with probability \( \delta \)? (Hint: The probability that one person does not call is \( (1 - \delta) \) – and the probability that \( (N - 1) \) people don’t call is \( (1 - \delta)^{(N-1)} \).)

(g) Using your answers to (d) through (f), derive \( \delta \) as a function of \( c \), \( x \) and \( N \) such that it is a mixed strategy equilibrium for everyone to call with probability \( \delta \). What happens to this probability as \( N \) increases?

(h) What is the probability that a crime will be reported in this mixed strategy equilibrium? (Hint: From your work in part (f), you should be able to conclude that the probability that no one else reports the crime – i.e. \( (1 - \delta)^{(N-1)} \) – is equal to \( c/x \) in the mixed strategy equilibrium. The probability that no one reports a crime is then equal to this times the probability that the last person also does not report the crime.) How does this change as \( N \) increases?

(i) True or False: If the reporting of crimes is governed by such mixed strategy behavior, it is advantageous for few people to observe a crime – whereas if the reporting of crime is governed by pure strategy Nash equilibrium behavior, it does not matter how many people witnessed a crime.

(j) If the cost of reporting the crime differed across individuals (but is always less than \( x \)), would the set of pure Nash equilibria be any different? Without working it out, can you guess how the mixed strategy equilibrium would be affected?

B: Suppose from here on out that everyone values the reporting of crime differently, with person \( n \)’s value of having a crime reported denoted \( x_n \). Assume that everyone still faces the same cost \( c \) of reporting the crime. Everyone knows that \( c \) is the same for everyone, and person \( n \) discovers \( x_n \) prior to having to decide whether to call. But the only thing each individual knows about the \( x \) values for others is that they fall in some interval \([0,b]\), with \( c \) falling inside that interval and with the probability that \( x_n \) is less than \( x \) given by \( P(x) \) for all individuals.

(a) What is \( P(0) \)? What is \( P(b) \)?

(b) From here on out, suppose that \( P(x) = x/b \). Does what you concluded in (a) hold?

(c) Consider now whether there exists a Bayesian Nash equilibrium in which each player \( n \) plays the strategy of reporting the crime if and only if \( x_n \) is greater than or equal to some critical value \( y \). Suppose that everyone other than \( n \) plays this strategy. What is the probability that at least one person other than individual \( n \) reports a crime? (Hint: Given this strategy, the probability that person \( k \) will not report a crime is equal to the probability that \( x_k \) is less than \( y \) – which is equal to \( P(y) \). The probability that \( K \) individuals do NOT report the crime is then \( (P(y))^K \).)

(d) What is the expected payoff of not reporting the crime for individual \( n \) whose value is \( x_n \)? What is the expected payoff of reporting the crime for this individual?

(e) What is the condition for individual \( n \) to optimally not report the crime if \( x_n < y \)? What is the condition for individual \( n \) to optimally report the crime when \( x_n \geq y \)?
(f) For what value of $y$ have we identified a Bayesian Nash equilibrium?

(g) What happens to the equilibrium probability of a crime being reported as $N$ increases?

(h) How is the probability of a crime being reported (in this equilibrium) affected by $c$ and $b$? Does this make sense?

24.17 Policy Application Some Prisoners’ Dilemmas: We mentioned in this chapter that the incentives of the prisoners’ dilemma appear frequently in real world situations.

A: In each of the following, explain how these are prisoners’ dilemmas and suggest a potential solution that might address the incentive problems identified in such games.

(a) When I teach the topic of prisoners’ dilemmas in large classes that also meet in smaller sections once a week, I sometimes offer the following extra credit exercise: Every student is given 10 points. Each student then has to decide how many of these points to donate to a “section account” and convey this to me privately. Each student’s payoff is a number of extra credit points equal to the number of points they did not donate to their section plus twice the average contribution to the section account by students registered in their section. For instance, if a student donates 4 points to his section and the average student in the section donated 3 points, then this student’s payoff would be 12 extra credit points – 6 because the student only donated 4 of his 10 points, and 6 because he gets twice the average donated in his section.

(b) People get in their cars without thinking about the impact they have on other drivers by getting on the road – and at certain predictable times, this results in congestion problems on roads.

(c) Everyone in your neighborhood would love to see some really great neighborhood fireworks on the next national independence day – but somehow no fireworks ever happen in your neighborhood.

(d) People like downloading pirated music for free but would like to have artists continue to produce lots of great music.

(e) Small business owners would like to keep their businesses open during “business hours” and not on evenings and weekends. In some countries, they have successfully lobbied the government to force them to close in the evening and on weekends. (Laws that restrict business activities on Sunday are sometimes called blue laws.)

B: In Chapter 21, we introduced the Coase Theorem, and we mentioned in Section 21A.4.4 the example of beekeeping on apple orchards. Apple trees, it turns out, don’t produce much honey (when frequented by bees), but bees are essential for cross-pollination.

(a) In an area with lots apple orchards, each owner of an orchard has to insure that there are sufficient numbers of bees to visit the trees and do the necessary cross-pollination. But bees cannot easily be kept to just one orchard – which implies that an orchard owner who maintains a bee hive is also providing some cross-pollination services to neighboring orchards. In what sense to orchard owners face a prisoners’ dilemma?

(b) How does the Coase Theorem suggest that orchard owners will deal with this problem?

(c) We mentioned in Chapter 21 that some have documented a “custom of the orchards” – an implicit understanding among orchard owners that each will employ the same number of bee hives per acre as the other owners in the area. How might such a custom be an equilibrium outcome in a repeated game with indefinite end?