1 Appendix[1]

Derivation of Wages from a Rubinstein Bargaining Game

Following the outlines of the proof in Binmore, Shaked and Sutton (1989) (from hereon referred to as BSS) and Binmore, Rubinstein, and Wolinsky (1986), we define m_f and M_f as the infimum and supremum payoffs for the firm, respectively, and m_w and M_w as the infimum and supremum payoffs for the worker, respectively. Match revenue is Y_{ij} and outside options are 0 and R_i for firms and workers, respectively.

In a Rubinstein bargaining game in which the firm moves first (in the absence of a minimum wage), the following inequalities hold:

$$\begin{aligned} m_f &\geq Y_{ij} - \max\{\tau_w M_w, R_i\} \\ Y_{ij} - M_f &\geq \max\{\tau_w m_w, R_i\} \\ m_w &\geq Y_{ij} - \tau_f M_f \\ Y_{ij} - M_w &\geq \tau_f m_f \end{aligned}$$

τ_w represents the worker's discount factor, and τ_f represents the firm's.

Inclusion of minimum wage means that the any bargaining offer (whether supremum or infimum) must be capped from below at the minimum wage, therefore, the inequalities change to:

$$\begin{aligned} m_f &\geq Y_{ij} - \max\{\tau_w M_w, \underline{W}, R_i\} \\ Y_{ij} - M_f &\geq \max\{\tau_w m_w, \underline{W}, R_i\} \\ m_w &\geq Y_{ij} - \tau_f M_f \\ Y_{ij} - M_w &\geq \tau_f m_f \end{aligned}$$

We will examine the case where $\underline{W} \geq R_i$ and $\underline{W} < R_i$ separately. First, when $\underline{W} \geq R_i$, we

[1] This is the proof appendix for the paper: The Distributional Impacts of Minimum Wage Increases when Both Labor Supply and Labor Demand are Endogenous

examine 3 regions, defined similarly to BSS:

$\underline{W} \leq \tau_w m_w$ (region 1), $\tau_w m_w < \underline{W} < \tau_w M_w$ (region 2), and $\underline{W} \geq \tau_w M_w$. (region 3)

Focusing on region 1, the inequalities change to:

$$\begin{aligned} m_f &\geq Y_{ij} - \tau_w M_w \\ Y_{ij} - M_f &\geq \tau_w m_w \\ m_w &\geq Y_{ij} - \tau_f M_f \\ Y_{ij} - M_w &\geq \tau_f m_f \end{aligned}$$

It is easy to show that:

$$\frac{(1-\tau_f)Y_{ij}}{1-\tau_f\tau_w} \leq m_w \leq M_w \leq \frac{(1-\tau_f)Y_{ij}}{1-\tau_f\tau_w}$$

Therefore, $M_w = m_w = \frac{(1-\tau_f)Y_{ij}}{1-\tau_f\tau_w}$.

Define $\beta = \frac{1-\tau_f}{1-\tau_f\tau_w}$. Then, $M_w = m_w = \beta Y_{ij}$, implying that $M_f = m_f = (1-\beta)Y_{ij}$.

We next show that region 2 yields a logical contradiction:

$$\begin{aligned} m_f &\geq Y_{ij} - \tau_w M_w \\ Y_{ij} - M_f &\geq \underline{W} > \tau_w m_w \\ m_w &\geq Y - \tau_f M_f \\ Y_{ij} - M_w &\geq \tau_f m_f \end{aligned}$$

which yields $\frac{(1-\tau_f)Y_{ij}}{1-\tau_f\tau_w} < m_w \leq M_w \leq \frac{(1-\tau_f)Y_{ij}}{1-\tau_f\tau_w}$.

For region 3, the inequalities are:

$$\begin{aligned} m_f &\geq Y_{ij} - \underline{W} \\ Y_{ij} - M_f &\geq \underline{W} \\ m_w &\geq Y_{ij} - \tau_f M_f \\ Y_{ij} - M_w &\geq \tau_f m_f \end{aligned}$$

This yields $m_w = M_w = (1-\tau_f)Y_{ij} + \tau_f \underline{W}$ and $m_f = M_f = Y_{ij} - \underline{W}$. Letting τ_f approach one, we have $m_w = M_w = \underline{W}$ and $m_f = M_f = Y_{ij} - \underline{W}$. When $\underline{W} \geq R_i$ and a worker successfully matches, his wage outcome is $\max\{\beta Y_{ij}, \underline{W}\}$.

Now, repeating the exercise with $\underline{W} < R_i$, we see that for regions 1 and 2, results are identical (since we just replace $\underline{W}$ with R_i), and region 3 changes to $m_w = M_w = R_i$ and $m_f = M_f = Y_{ij} - R_i$. Therefore, when $\underline{W} < R_i$ and a worker successfully matches, his wage outcome is $\max\{\beta Y_{ij}, R_i\}$.

Combining these two results, when a worker successfully matches ($Y_{ij} > \underline{W}$), the unique subgame perfect equilibrium outcome of the bargaining game is a wage offer of $\max\{\beta Y_{ij}, \underline{W}, R_i\}$ which is accepted. QED

Proof of Proposition 1

Note that conditional on any $N \in [0, \overline{N}]$, as $J \to \infty$, $q \to 0$. There then exists a $\overline{J}$ such that for all N if $J' > \overline{J}$, profits are negative. Since the partial derivative of π is negative with respect to J,

$$\frac{\partial \pi}{\partial J} = -\frac{q\alpha\left(E\max\{Y_{ij}, W_{ij}\} - C_2\right)}{J} < 0$$

We know that for each value of N there is at most one value of J such that $\pi = 0$.

Similarly, define V as the search value. Taking the partial derivative with respect to N yields:

$$\frac{\partial V}{\partial N} = -\frac{p(1-\alpha)\left(E\max\{W - R_i, 0\}\right)}{N} + \frac{p\partial E\max\{W - R_i, 0\}}{\partial N} - \frac{\partial K_i}{N} < 0$$

as the second two terms must be negative when ordering the individuals according to V_i. We know that for each J there is at most one value of N such that $V = 0$.

We can then define the following mappings:

$$f_1 = \begin{cases} \pi(J,N) & \text{for } J \in (0,\overline{J}], N \in [0,\overline{N}] \\ \max\{\pi(0,N),0\} & \text{for } J = 0, N \in [0,\overline{N}] \end{cases}$$

$$f_2 = \begin{cases} \min\{V(J,\overline{N}),0\} & \text{for } J \in [0,\overline{J}], N = \overline{N} \\ V(J,N) & \text{for } J \in [0,\overline{J}], N \in (0,\overline{N}) \\ \max\{V(J,0),0\} & \text{for } J \in [0,\overline{J}], N = 0 \end{cases}$$

Then for each value of N, there exists a unique value of $J \in [0,\overline{J}]$ that satisfy $f_1 = 0$. Further, since π is continuous in N, this unique value is a continuous function of N. Similarly, for each J, there is a unique $N \in [0,\overline{N}]$ satisfying f_2 which is continuous in J. We can then use functions to define a continuous vector valued function mapping from $[0,\overline{J}] \times [0,\overline{N}]$ into itself. Then by Brouwer's fixed point theorem there exists a doublet $\{J^*,N^*\}$ where $f_1 = 0$ and $f_2 = 0$. QED.

Proof of Lemma 1

We show, given NR, if a worker searches, he accepts all matches. Assume $R_i > \underline{W}$.Then, individuals search when

$$pPr(Y_{ij} \geq R_i)[E(W|Y_{ij} \geq R_i) - R_i] > K_i$$

To derive the lower limit on K_i to make the condition above hold, set $p = 1$. The K_i that satisfies this condition is $\underline{K}$ for all searching workers, and yields the expression in NR. QED

Proof of Proposition 2

Consider the equilibrium before the minimum wage increase. The expected surplus for the firm conditional on matching is $E(\max\{Y_{ij} - W_ij, 0\}|\underline{W_1})$ and the probability of a firm matching is given by q_1. Note that the expected surplus for the firm conditional on matching is weakly decreasing in the minimum wage. The firm's expected zero profit condition is:

$$q_1(E(Y) - E_1(W) - C_2) - C_1 = 0$$

The firm's probability of matching must increase when the expected surplus conditional on matching fall in order for the zero profit condition to still bind. Note further that the probabilities of firms and workers matching is given by:

$$q = A\left(\frac{N}{J}\right)^{1-\alpha} \qquad p = A\left(\frac{J}{N}\right)^{\alpha}$$

The expression for the firm implies that $\frac{N}{J}$ must increase for the zero profit condition to bind. But if this fraction increases then p must fall. QED.

Proof of Proposition 3

Differentiating the matching function with respect to the minimum wage yields:

$$\frac{dx}{d\underline{W}} = \alpha q \frac{dJ}{d\underline{W}} + (1-\alpha) p \frac{dN}{d\underline{W}}$$

Rewrite as:

$$\begin{aligned}
\frac{dx}{d\underline{W}} &= \alpha \frac{x}{J}\frac{dJ}{d\underline{W}} + (1-\alpha)\frac{x}{N}\frac{dN}{d\underline{W}} \\
&= x\left(\alpha \frac{\frac{dJ}{J}}{d\underline{W}} + (1-\alpha)\frac{\frac{dN}{N}}{d\underline{W}}\right) \\
&= \frac{x}{\underline{W}}\left(\alpha \frac{\frac{dJ}{J}}{\frac{d\underline{W}}{\underline{W}}} + (1-\alpha)\frac{\frac{dN}{N}}{\frac{d\underline{W}}{\underline{W}}}\right) \\
&= \frac{x}{\underline{W}}\left(\alpha\varepsilon_{LD} + (1-\alpha)\varepsilon_{LS}\right)
\end{aligned}$$

Therefore, for the employment effect to be positive ($\frac{dx}{d\underline{W}} > 0$), it must be that $(\alpha\varepsilon_{LD} + (1-\alpha)\varepsilon_{LS}) > 0$, where ε_{LD} is the elasticity of labor demand and ε_{LS} is the elasticity of labor supply. QED

Proof of Proposition 4

In order for all workers to benefit from an increase in the minimum wage it is sufficient to show that the workers with the lowest reservation values, zero, are made better off by the increase. The value of search for these workers can be written as:

$$V = A\left(\frac{N}{J}\right)^{-\alpha} E(W) - K_i$$

Note that the zero profit condition for firms can be written as:

$$A\left(\frac{N}{J}\right)^{1-\alpha} (E(Y) - E(W) - C_2) - C_1 = 0$$

and that both of these conditions depend on N and J only through the ratio N/J. Further, the zero profit condition for the firm is an identity. Differentiating profits with respect to an increase in the minimum wage yields:

$$A\left(\frac{N}{J}\right)^{1-\alpha} \left((1-\alpha)(E(Y) - E(W) - C_2)\left(\frac{N}{J}\right)^{-1} \frac{\left(d\frac{N}{J}\right)}{d\underline{W}} - \frac{dE(W)}{d\underline{W}}\right) = 0$$

Solving for $d(N/J)/d\underline{W}$ yields:

$$\frac{d\left(\frac{N}{J}\right)}{d\underline{W}} = \frac{N}{(1-\alpha)(E(Y) - E(W) - C_2)J} \frac{dE(W)}{d\underline{W}}$$

We now have all components necessary to sign $dV/d\underline{W}$ for those with a reservation value of zero. Differentiating V with respect to $\underline{W}$ yields:

$$E(W)A(-\alpha)\left(\frac{N}{J}\right)^{-\alpha-1} \frac{d\left(\frac{N}{J}\right)}{d\underline{W}} + A\left(\frac{N}{J}\right)^{-\alpha} \frac{dE(W)}{d\underline{W}}$$

substituting in for $d(N/J)/d\underline{W}$ and rewriting yields:

$$\frac{pdE(W)}{d\underline{W}}\left[1 - \frac{\alpha E(W)}{(1-\alpha)(E(Y) - E(W) - C_2)}\right]$$

Since $dE(W)/d\underline{W} > 0$, we have the result. QED