Abstract

This paper integrates fertility choice and exhaustible resource dynamics in a tractable model of endogenous technological change. The analysis shows that, under the right conditions, the interdependence of population, resources and technology produces a transition from unsustainable resource-based growth to sustainable knowledge-based growth that consists of three phases: (1) an initial phase where agents build up the economy by exploiting exhaustible natural resources to support population growth; (2) an intermediate phase where agents turn on the Schumpeterian engine of endogenous innovation in response to population-led market expansion; (3) a terminal phase where growth becomes fully driven by knowledge accumulation and no longer requires growth of physical inputs. The last phase is crucial: not only economic growth no longer requires growth of physical inputs, but technological change also compensates for the exhaustion of the natural resource stock.

Keywords: Endogenous Growth, Market Structure, Natural Resources.

JEL Classification Numbers: E10, L16, O31, O40
1 Introduction

One of the liveliest debates of our times concerns the sustainability of living standards in a world of limited, possibly vanishing, natural resources. To contribute to the debate, this paper integrates fertility choice and exhaustible resource dynamics in a tractable model of endogenous technological change. It then shows that under the right conditions the interdependence of population, resources and technology produces a transition from unsustainable resource-based growth to sustainable knowledge-based growth that consists of three phases:

1. an initial phase where agents build up the economy by exploiting exhaustible natural resources to support population growth;
2. an intermediate phase where agents turn on the Schumpeterian engine of endogenous innovation in response to population-led market expansion;
3. a terminal phase where growth becomes fully driven by knowledge accumulation and no longer requires growth of physical inputs.

The last phase is crucial: not only economic growth no longer requires growth of physical inputs, but technological change also compensates for the exhaustion of the natural resource stock.

The paper thus proposes a theory of the de-coupling of the growth of living standards from the physical resource base that allows one to investigate analytically issues that to date have been quite challenging. In particular, the theory provides a clear characterization of the conditions under which the economy possesses a steady state with constant exponential growth of consumption per capita despite its dependence on an essential natural resource that runs out due to exhaustion.\(^1\) The characterization provides insights about possible interventions that can ensure sustainability in case the economy fails to meet those conditions.

The paper contributes to a large literature that has accomplished much but that still faces open questions. The analytical framework used to study the relation between resource scarcity and economic growth emphasizes the role of exhaustible natural resources in generating diminishing returns to other physical inputs that worsen over time as natural resources run out.\(^2\) In the last 15-20 years researchers have extended the scope of the analysis, initially limited to the neoclassical model of capital accumulation, to incorporate insights from the theory of endogenous innovation (see, e.g., Smulders 2005 for a review). The need to do so emerges clearly from Stiglitz’s (1974a,b) classic treatment of the scarcity question, which concluded that technological change is the key

\(^{1}\)This operational definition of sustainability is important for the purposes of this paper because it avoids the vast number of issues that plague much of the current analyses. See Pezzey and Toman (2002, 2005) for discussions of the problems that the literature has encountered in defining the concept of “sustainability” in general, that is, without referring to the behavior of a specific variable in a specific model.

\(^{2}\)The foundations of the framework where laid in the 70s by Solow (1974), Stiglitz (1974a,b) and Dasgupta and Heal (1974, 1979). There is now a vast literature elaborating the original insights provided by these contributions. For excellent reviews, see Simpson, Toman and Ayres (2005) and Brock and Taylor (2005).
force capable of compensating for resource exhaustion. It is thus clear that understanding where it comes from, at what cost, and what possible institutional changes should be implemented to provide the right incentives for it to happen, must be a key component of the analysis.

A similar understanding has gradually emerged concerning demographic forces: it is now widely recognized that population dynamics must be a key endogenous component of analyses that project the model forward over long time horizons to explore sustainability (see Bloom and Canning 2001 for further discussion). It is thus important to understand the incentives and constraints that drive reproductive decisions. Doing so requires investigating the complex interactions between traditional Malthusian forces — population expansion puts pressure on the natural environment — and modern Schumpeterian forces — population expansion creates the larger market that ignites and sustains endogenous innovation.

In this perspective, this paper provides an integrated view that expands the focus from the resource economist’s traditional concern with the asymptotic behavior of the economy under increasing scarcity to the system’s global dynamics, with special emphasis on the phase transitions that mark shifts to qualitatively different behaviors. This broader focus provides a different vision of the dynamic forces at play. In the first phase of the model’s transition, for example, the economy does not invest resources in the generation of technological change and thus it looks like it is just exploiting the natural environment to expand the population. Without further consideration, such a situation looks clearly unsustainable. What the model says, however, is that this phase of population expansion is in fact sowing the seeds of future growth because it creates the critical market size needed to support investment in new technology by profit-driven firms. The full fruition of such initial, seemingly unsustainable, development arrives in the third and final phase because — if the conditions are right — the economy reaches a steady state where the rate of endogenous technological change is sufficiently fast to compensate for resource exhaustion. Moreover, in this phase the rate of endogenous technological change is divorced from population dynamics so that sustainability is possible even if population ceases growing (or even shrinks).

Given its emphasis on the interaction between population and an exhaustible natural resource, the paper is related to the literature on the rise and fall of civilizations, although most of that literature considers models of renewable resources (see Taylor 2009 for a review). Such models generate rich dynamics, with possible environmental crises that can result in human extinction, and in some examples have been calibrated to replicate the collapse of Easter Island and similar historical episodes (e.g., Brander and Taylor 1998). By and large, however, this literature ignores endogenous technological change and thus provides a very different perspective on sustainable growth from that developed here.

A notable recent contribution is Bretschger (2013) who considers poor substitution (complementarity) between labor and an exhaustible resource in a Romer-style model of endogenous growth that exhibits the strong scale effect. To my knowledge that is the first attempt at integrating in a single model the dynamics of population, exhaustible resources and technology. The analysis developed in this paper builds on the insights developed there and extends the framework to a more
comprehensive model of endogenous technological change capable of producing the rich transition described above. Another important difference is that Bretschger (2013) allows for a backstop technology triggered by a sufficiently high resource price. This paper, instead, sets up the harshest possible environment in which economic activity takes place and thereby sets the highest possible bar for technology to clear to deliver sustainability.

2 The model

The economy is closed. All variables are functions of (continuous) time but to simplify the notation the time argument is omitted unless necessary to avoid confusion. To keep things simple, there is no physical capital.\(^3\)

2.1 Final producers

A competitive representative firm produces a final good \(Y\) that can be consumed, used to produce intermediate goods, invested in the improvement of the quality of existing intermediate goods, or invested in the creation of new intermediate goods. The final good is the numeraire so its price is \(P_Y = 1\). The production technology is

\[
Y = \int_0^N X_i^\theta \left( Z_i^\alpha Z^1-\alpha \frac{L^\gamma R^{1-\gamma}}{N^{1-\sigma}} \right)^{1-\theta} di, \quad 0 < \theta, \alpha, \gamma, \sigma < 1
\]

where \(N\) is the mass of non-durable intermediate goods and \(L\) and \(R\) are, respectively, services of labor and an exhaustible natural resource. Quality is the good’s ability to raise the productivity of the other factors: the contribution of good \(i\) depends on its own quality, \(Z_i\), and on average quality \(Z = \int_0^N (Z_j/N) dj\). The technology features social returns to variety of degree \(\sigma\) and social returns to quality of degree \(1\).\(^4\)

The first-order conditions for the profit maximization problem of the final producer yield that each intermediate producer faces the demand curve

\[
X_i = \left( \frac{\theta}{P_i} \right)^{-\frac{1}{1-\theta}} Z_i^\alpha Z^1-\alpha \frac{L^\gamma R^{1-\gamma}}{N^{1-\sigma}}, \quad \text{(2)}
\]

where \(P_i\) is the price of good \(i\). Let \(w\) denote the wage and \(p\) denote the resource price. The first-order conditions then yield that the final producer pays total compensation

\[
\int_0^N P_i X_i di = \theta Y, \quad wL = \gamma (1-\theta) Y \quad \text{and} \quad pR = (1-\gamma) (1-\theta) Y
\]

to intermediate goods, labor and resource suppliers, respectively.

\(^3\)This paper builds on Peretto (2015). To avoid repetition, the interested reader is referred to that paper for details of the model’s production structure not discussed here in full.

\(^4\)See Peretto (2015) for an interpretation of \(\sigma\) in terms of economies of scope and congestion effects in the use of intermediate goods, labor and natural resources.
2.2 Intermediate producers

The typical intermediate firm operates a technology that requires one unit of final output per unit of intermediate good and a fixed operating cost $\phi Z_i^\alpha Z^{1-\alpha}$, also in units of final output. The firm can increase quality according to the technology

$$\dot{Z}_i = I_i,$$

(3)

where $I_i$ is R&D in units of final output. Using (2), the firm’s gross profit (i.e., before R&D) is

$$\Pi_i = \left[ (P_i - 1) \left( \frac{\theta}{P_i} \right)^{\frac{1}{1-\sigma}} \frac{L^\gamma R^{1-\gamma}}{N^{1-\sigma}} - \phi \right] Z_i^\alpha Z^{1-\alpha}. $$

(4)

The firm chooses the time path of its price, $P_i(t)$, and R&D, $I_i(t)$, to maximize

$$V_i(0) = \int_0^\infty e^{-\int_0^t r(s) ds} \left[ \Pi_i(t) - I_i(t) \right] dt$$

(5)

subject to (3) and (4), where $r$ is the interest rate and 0 is the point in time when the firm makes decisions. The firm takes average quality, $Z$, in (4) as given. The characterization of the firm’s decision yields a symmetric equilibrium where

$$r = \frac{\alpha \Pi}{Z},$$

(6)

is the return to quality innovation (see the Appendix for the derivation) and $\alpha$ is now intuitively interpreted as the elasticity of the firm’s gross profit with respect to its own quality.

The process of firm formation is as follows. At time $t$, an agent who wants to create a new firm must sink $\beta X(t)$ units of final output. Because of this sunk cost, the new firm cannot supply an existing good in Bertrand competition with the incumbent monopolist but must introduce a new good that expands product variety. New firms enter at the average quality level, and therefore at average size (this simplifying assumption preserves symmetry of equilibrium at all times), and finance entry by issuing equity. Entry is positive if the value of the firm is equal to its setup cost, i.e., if the free-entry condition $V_i = \beta X$ holds. Taking logs and time derivatives of the free-entry condition and of the value of the firm in (5), and imposing symmetry, yields the return to variety innovation

$$r = \frac{\Pi - I}{\beta X} + \frac{\dot{X}}{X},$$

(7)

2.3 Households

The economy is populated by a continuum of measure one of identical households that supply labor services and purchase financial assets in competitive labor and asset markets. The typical household has preferences:

$$U(0) = \int_0^\infty e^{-\rho t} \left[ \mu \log (C_M(t) M(t)^\eta) + (1 - \mu) \log (C_B(t) B(t)^\eta) \right] dt, \quad \rho > 0, \ 0 < \mu, \eta < 1$$
where 0 is the point in time when the household makes decisions, $\rho$ is the discount rate, $C_M$ is consumption per adult, $M$ is the mass of adults, $C_B$ is consumption per child, $B$ is the mass of children, $\mu$ measures the preference for adult utility versus children utility, $\eta$ regulates the trade-off between the consumption per adult (child) and the mass of adults (children). The mass of adults evolves according to

$$\dot{M} = B - \delta M, \quad M_0 > 0, \quad \delta > 0,$$

where $\delta$ is the exogenous death rate.

In this structure, the decision maker cares about utility of adults and utility of children with weights $\mu$ and $1 - \mu$. Adults and children derive utility from their individual consumption and from the mass of adults and the mass of children. Childhood lasts for one instant and then the child becomes a productive adult. Children consume but do not work.

The household owns an initial stock $S_0$ of an exhaustible resource and thus faces the constraints

$$S_0 \geq \int_0^{\infty} R(t) \, dt, \quad R \geq 0, \quad S_0 > 0, \quad \dot{S} = -R,$$

where $R$ is the flow of the resource that the household sells for price $p$. Each adult is endowed with one unit of labor that he supplies entirely in the labor market. Since children do not work, the household faces the flow budget constraint

$$\dot{A} = rA + wM + pR - C_M M - C_B B, \quad A_0 \geq 0,$$

where $A$ is assets holding, $r$ is the rate of return on assets and $w$ is the wage.

### 3 The economy’s general equilibrium

This section characterizes first the behavior of the household. It then imposes general equilibrium conditions and characterizes how market interactions determine the dynamics of resource supply and use. Finally, it characterizes how these dynamics drive the evolution of the economy.

#### 3.1 Household behavior

The current value Hamiltonian for the typical household is

$$CVH = \mu \log C_M + \mu \eta \log M + (1 - \mu \log C_B + (1 - \mu) \eta \log B$$

$$+ \lambda_A \left[ rA + wM + pR - (C_M M + C_B B) \right] + \lambda_M (B - \delta M) - \lambda_S R,$$

where the $\lambda$s denote the shadow value of respectively, financial assets, household size, and the resource stock. The first order conditions for the control variables $C_M, C_B, B, R$ are:

$$\frac{\mu}{C_M M} = \lambda_A = \frac{1 - \mu}{C_B B}, \quad \frac{(1 - \mu) \eta}{B} + \lambda_M = \lambda_A C_B; \quad \lambda_{AP} = \lambda_S.$$
The conditions for the state variables \(A, M, S\) are:

\[
\begin{align*}
    r + \frac{\dot{\lambda}_A}{\lambda_A} &= \rho; \\
    \frac{\eta + \lambda_A (wM - C_M M - C_B B)}{\lambda_M M} + \left( \frac{\dot{\lambda}_M}{\lambda_M} + \frac{\dot{M}}{M} \right) &= \rho; \\
    \frac{\dot{\lambda}_S}{\lambda_S} &= \rho.
\end{align*}
\]

Associated to these are the transversality conditions that the value of each state variable times its shadow value converges to zero as \(t \to \infty\).

Let \(C = C_M M + C_B B\) be aggregate consumption. The conditions for \(C_M\) and \(C_B\) yield \(C = C_M M + C_B B = 1/\lambda_A\). Thus, let the ratio of consumption to final output be \(c \equiv C/Y\), births per adult be \(b \equiv B/M\) and the shadow value of adult population be \(h \equiv \lambda_M M\). The empirical counterpart of \(b\) is the crude fertility rate (often called CBR). The empirical counterpart of \(c\) is not the traditional one minus the saving rate because \(Y\) is not GDP; see below for the formal mapping between \(c\) and GDP. The result \(C = 1/\lambda_A\) and the first-order condition for financial wealth \(A\) yield the Euler equation for saving

\[
r = \rho + \frac{\dot{C}}{C} = \rho + \frac{\dot{c}}{c} + \frac{\dot{Y}}{Y}.
\]

The result \(C = 1/\lambda_A\) and the conditions for fertility \(B\), financial wealth \(A\) and adult population size \(M\), yield the fertility rule

\[
    h = \frac{(1 - \mu)(1 - \eta)}{b}
\]

and, recalling that \(wM = \gamma(1 - \theta)Y\), the asset-pricing-like equation

\[
    \dot{h} = \rho h - \eta - \frac{wM - C}{C} = \rho h - \eta - \frac{\gamma(1 - \theta) - c}{(1 - \mu)c}.
\]

that characterizes the evolution of the shadow value of adult population. Using (10), this can be rewritten

\[
    \frac{\dot{b}}{b} = \left[ \frac{\gamma(1 - \theta)}{c(1 - \eta)} - 1 \right] \frac{b}{1 - \mu} - \rho.
\]

Finally, the result \(C = 1/\lambda_A\), the extraction flow \(R\), the resource stock \(S\) and the Euler equation (9) yield the Hotelling rule

\[
    \frac{\dot{p}}{p} = \lambda_S \Rightarrow \frac{\dot{p}}{p} = \rho + \frac{\dot{C}}{C} = r.
\]

### 3.2 The resource extraction path

The natural resource market clears when the flow of the resource supplied by the household equals the final sector demand, i.e., \(pR = (1 - \gamma)(1 - \theta)Y\). Log-differentiating this expression and using the Hotelling rule (11) yields

\[
    \frac{\dot{R}}{R} = \frac{\dot{Y}}{Y} - \frac{\dot{p}}{p} = \frac{\dot{Y}}{Y} - r = - \left( \frac{\dot{c}}{c} + \rho \right).
\]
Integrating this expression and defining the average growth rate of the extraction flow between time 0 and time $t$, i.e.,

$$
\varepsilon (t) \equiv \frac{1}{t} \int_0^t \frac{\dot{c}(s)}{c(s)} ds,
$$

yields

$$
R(t) = R_0 e^{-\varepsilon(t)t}.
$$

Substituting this result into the constraint

$$
S_0 = \int_0^\infty R(t) dt
$$

yields

$$
R_0 = \left[ \int_0^\infty e^{-\varepsilon(t)t} dt \right]^{-1} \cdot S_0,
$$

where the term in brackets is a constant that depends on the fundamentals. Therefore, the resource extraction path is

$$
R(t) = \frac{e^{-\varepsilon(t)t}}{\int_0^\infty e^{-\varepsilon(t)t} dt} \cdot S_0
$$

and the resource stock evolves according to

$$
S(t) = S_0 - \int_0^t R(s) ds = S_0 \cdot \left[ 1 - \frac{\int_0^t e^{-\varepsilon(s)s} ds}{\int_0^\infty e^{-\varepsilon(t)t} dt} \right],
$$

converging to zero as $t \to \infty$.

According to this extraction path, the representative household chooses the initial extraction flow $R_0$ as proportional to the endowment $S_0$ and thereafter the path of the extraction flow is pinned down by the path of the growth rate of the ratio of consumption to final output $c$. In other words, the forward-looking representative household that populates this model realizes that it has at its disposal three forms of wealth — financial, human, and natural — and chooses the extraction path jointly with its consumption-saving path. Consequently, it takes into account that in order to sustain faster consumption growth it needs to extract more aggressively.

### 3.3 GDP and market structure dynamics

Labor market clearing yields $L = M$. Recall that the equilibrium is symmetric because intermediate firms make identical decisions. Using the demand schedule (2) to eliminate $X$, the production function (1) yields

$$
Y = \theta^\frac{2\theta}{1-\theta} \cdot N^\sigma Z M^\gamma R^{1-\gamma},
$$

where $N^\sigma Z$ is Hicks-neutral TFP in the final output sector.

Equations (6)-(7) and the definition of gross profit (4) say that the returns to innovation are functions of the quality-adjusted size of the firm $x_i \equiv X_i/Z_i$, which in symmetric equilibrium reads $x_i = x = X/Z$. Since the final producer pays total compensation $N \cdot P X = \theta Y$ to intermediate producers and intermediate producers set $P = 1/\theta$, one has $NX = \theta^2 Y$. Substituting in the definition of $x$ and using the reduced-form production function (14) yields

$$
x = \frac{X}{Z} = \frac{NX}{NZ} = \frac{\theta^2 Y}{NZ} = \theta^\frac{2\theta}{1-\theta} \cdot \frac{M^\gamma R^{1-\gamma}}{N^{1-\sigma}}.
$$
Now let $G$ denote this economy’s GDP. Subtracting the cost of intermediate production from the value of final production and using (15) yields

$$G = \left[ 1 - \theta^2 \left( 1 + \frac{\phi}{x} \right) \right] \cdot Y.$$  

The term in brackets is increasing in $x$ because the unit cost of production of the typical intermediate firm falls as its scale of operation rises. According to this result, the consumption ratio traditionally defined is

$$\frac{C}{G} = \frac{C}{\left[ 1 - \theta^2 \left( 1 + \frac{\phi}{x} \right) \right] \cdot Y} = \frac{c}{1 - \theta^2 \left( 1 + \frac{\phi}{x} \right)}$$

and, for given $c$, it falls along a transition with rising firm size $x$.

To summarize, this economy has a vertical production structure that results in the following expression for GDP per worker (equivalently, adult):

$$\frac{G}{M} = \theta^{2\sigma} \left[ 1 - \theta^2 \left( 1 + \frac{\phi}{x} \right) \right] \cdot N^{\sigma} \cdot \left( \frac{R}{M} \right)^{1-\gamma}.$$  

(16)

This expression captures standard ingredients: output per worker rises with efficiency (firms’ average scale), technology (product variety and average quality) and with resource abundance per worker. What is different from the typical construct of growth economics is that the flow of the resource $R(t)$ obeys the Hotelling extraction path characterized by (12)-(13).

### 3.4 Key components of the equilibrium dynamical system

The following results state properties that are useful in the characterization of the economy’s equilibrium dynamics.

**Lemma 1** Denote $n \equiv \dot{N}/N$, $z \equiv \dot{Z}/Z$ and $g \equiv \dot{G}/G - m$. Let also

$$\xi(x) \equiv \frac{\theta^2 \phi/x}{1 - \theta^2 \left( 1 + \frac{\phi}{x} \right)}$$

be the elasticity of GDP with respect to firm size. At any point in time, the interest rate and the growth rate of GDP per worker are, respectively:

$$r = \sigma n + z + \gamma (m + \dot{c}/c + \rho);$$  

$$g = \left( \frac{\sigma n + z + \xi(x) \cdot (\dot{x}/x)}{\text{TFP growth}} \right) - \left( 1 - \gamma \right) \left( m + \dot{c}/c + \rho \right).$$  

(17)  

(18)

**Proof.** See the Appendix. ■

In words, growth of GDP per worker is the growth rate of TFP minus the *growth drag* due to the presence of the natural resource. The drag is equal to the share of the natural resource, $1 - \gamma$,?
times the sum of adult population growth rate \( m \) (i.e., the growth rate of the workforce) and the rate of exhaustion of the (flow) supply of the resource \( \dot{c}/c + \rho \).

It is worth highlighting the difference between variables expressed as per worker versus per capita. Recall that \( b = B/M \) denotes the crude fertility rate, i.e., births per adult. The fertility rate defined as births per capita is \( B/(B + M) \). Similarly, GDP per capita is \( G/(B + M) \). It follows that the growth rate of GDP per capita is

\[
g - \frac{b}{b + 1} \cdot \dot{b}.
\]

There is thus an additional drag at play: when births per adult grow, GDP per capita growth falls below GDP per worker growth. Noting that \( b/(b + 1) = B/(B + M) \) provides the interpretation: this term is the dependency ratio and is itself rising as long as \( b \) rises. It follows that a path with rising births per adult exhibits a widening gap between growth of GDP per worker and growth of GDP per capita because the fraction of the population that does not work is rising.

In the following analysis it is convenient to characterize the evolution of the structure of the intermediate goods sector as follows.

**Lemma 2** Let \( \pi = P - 1 \) be the per-unit gross profit margin. Using the definition of \( x \) in (15), the returns to innovation in (6) and (7) become:

\[
r = \alpha (\pi x - \phi); \tag{19}
\]

\[
r = \frac{1}{\beta} \left( \pi - \frac{\phi + z}{x} \right) + \frac{\dot{x}}{x} + z. \tag{20}
\]

Firm size obeys the differential equation

\[
\frac{\dot{x}}{x} = \gamma m - (1 - \gamma) \left( \frac{\dot{c}}{c} + \rho \right) - \frac{(1 - \sigma) n}{\text{market growth}} - \frac{(1 - \sigma) n}{\text{market fragmentation}}. \tag{21}
\]

**Proof.** See the Appendix. \( \blacksquare \)

These expressions capture the model’s main property: decisions to invest in quality and variety innovation depend on quality-adjusted firm size. The evolution of quality-adjusted firm size, in turn, is driven by the difference between the term \( \gamma m - (1 - \gamma) \left( \frac{\dot{c}}{c} + \rho \right) \), that captures how adult population growth net of resource exhaustion drives the growth of the market for intermediate goods, and the term \( (1 - \sigma) n \), that captures how product proliferation net of the contribution of product variety to TFP growth fragments the overall market in smaller submarkets and thus reduces the profitability of the individual firm.

According to Lemma 1, whether the firms’ investment decisions support positive growth of output per worker depends on whether the resulting rate of growth of TFP is larger than the growth drag; this is the classic condition for sustainability derived by Stiglitz (1974; see also Brock and Taylor 2005), with the difference that in this model TFP growth is endogenous and not necessarily
positive. The reason is that from the perspective of the firm, innovation entails a sunk cost that is economically justified only when the anticipated revenue flow is sufficiently large.

Specifically, the non-negativity constraint on variety growth, \( n \equiv \dot{N}/N \geq 0 \), reveals that there is a threshold of firm size below which entry is zero because the return is too low. The value of the threshold depends on whether entrants anticipate that in the post-entry equilibrium \( z > 0 \) or \( z = 0 \) since it affects the net cash flow that they expect to earn. Similarly, the non-negativity constraint on quality growth, \( z \equiv \dot{Z}/Z \geq 0 \), reveals that there is a threshold of firm size below which incumbents do not do R&D because the return is too low. The value of the threshold depends on whether \( n > 0 \) or \( n = 0 \) since it affects the reservation interest rate of the household. For simplicity, the paper focuses on the case where the threshold for variety innovation, denoted \( x_N \), is smaller than the threshold for quality innovation, denoted \( x_Z \).

Before characterizing the thresholds \( x_N \) and \( x_Z \), it is useful to emphasize that when entry is positive the ratio of consumption to final output, \( c \), and births per adult, \( b \), are constant throughout the transition as well as in steady state. When entry is zero, instead, incumbents earn rents that are increasing in firm size, \( x \), and, since they are distributed to the household as dividends, fuel rising consumption and fertility. The following Lemma states this property formally.

**Lemma 3** There are two regimes, one with entry and one with no entry. The expenditure behavior of the household in the two regimes is

\[
c = \begin{cases} 
\theta^2 \left( \frac{\phi}{\pi} - \frac{\rho}{\pi} \right) + 1 - \theta & \phi/\pi < x \leq x_N \\
\rho \beta^2 + 1 - \theta & x > x_N
\end{cases}
\]

(22)

In the regime with entry, moreover: (i) the ratio of consumption to final output, \( c \), and the crude fertility rate, \( b \), jump to their respective steady-state values \( c^*, b^* \) and remain constant throughout the transition driven by the evolution of firm size \( x \); (ii) the resource input \( R \) follows an exponential process with constant rate of exhaustion \( \rho \), i.e., \( R(t) = S_0 e^{-\rho t} \).

**Proof.** See the Appendix. □

This result says that the ratio of consumption to final output, \( C/Y \), is an increasing function of firm size \( x \) up to the threshold that triggers entry, where it becomes constant. Moreover, the regime with entry exhibits constant — but endogenous — ratio of consumption to final output, birth rate (births per adult and births per capita are proportional to each other) and extraction rate. The questions then are whether the economy converges to such a regime and whether such a regime constitutes a sustainable growth path. For the second question, the key issue is whether endogenous innovation can overcome the fact that the resource stock vanishes at a constant exponential rate. The following Lemma characterizes agents’ innovation behavior.

**Lemma 4** Let \( x_N \) denote the threshold of firm size that triggers variety innovation and \( x_Z \) the threshold of firm size that triggers quality innovation. Assume

\[
\frac{\phi \alpha}{\pi - \rho \beta} < \gamma (m^* + \rho),
\]
where \( m^* \) is the constant, endogenous, growth rate of population in the regime with entry. Then,

\[
x_N = \frac{\phi}{\pi - \rho \beta}
\]

and

\[
x_Z = \arg \max \left\{ (\pi x - \phi) \left( \alpha - \frac{\sigma}{\beta x} \right) < \gamma (m^* + \rho) - \sigma \rho \right\},
\]

with \( x_N < x_Z \). Assume also \( \beta x > \sigma \forall x > \phi \), i.e., \( \beta \phi > \sigma \). Then, for \( x > x_N \) the equilibrium rates of variety and quality innovation are:

\[
n = \begin{cases} 
\frac{1}{\beta} \left( \frac{\pi - \phi}{\sigma} \right) - \rho & x_N < x \leq x_Z \\
(1-\alpha)(\pi x - \phi) - \beta x + \gamma (m^* + \rho) & x > x_Z 
\end{cases}; \tag{23}
\]

\[
z = \begin{cases} 
0 & x_N < x \leq x_Z \\
(\pi x - \phi) \left( \alpha - \frac{\sigma}{\beta x} \right) - \gamma (\rho + m^*) + \sigma \rho & x > x_Z 
\end{cases}. \tag{24}
\]

**Proof.** See the Appendix. ■

This result provides the key building block for the analysis of the transition. It characterizes the innovation rates as functions of firm size \( x \) only.

### 3.5 The equilibrium dynamical system

Lemma 3-4 reduce the system characterizing the economy’s dynamics to the fertility and expenditure rules (10) and (22), the differential equations

\[
\frac{\dot{b}}{b} = \left[ \gamma \left(1 - \theta \right) \right] \frac{b}{(1 - \eta) c(x) - 1} \\
\frac{\dot{x}}{x} = \gamma (b - \delta) - (1 - \gamma) (\dot{c}/c + \rho) - (1 - \sigma) n,
\]

and the initial condition \( x_0 \) plus the transversality condition. There are two cases.

- **Equilibrium with no entry \( (n = 0) \).** A little algebra yields:

\[
\frac{\dot{b}}{b} = \left[ \gamma \left(1 - \theta \right) \right] \frac{b}{(1 - \eta) c(x) - 1} \\
\frac{\dot{x}}{x} = \gamma (b - \delta) - (1 - \gamma) \rho.
\]

- **Equilibrium with entry \( (n > 0) \).** By Lemma 3, after some algebra:

\[
\frac{\dot{b}}{b} = \left[ \gamma \left(1 - \theta \right) \right] \frac{b}{(1 - \eta) (\rho \beta^2 + 1 - \theta)} \\
\frac{\dot{x}}{x} = \gamma m^* - (1 - \gamma) \rho - (1 - \sigma) n(x).
\]
The difference between the two cases is that in the second the unstable differential equations for \( c \) and \( b \) do not depend on \( x \) and, consequently, \( c \) and \( b \) jump to their steady state values \( c^* \) and \( b^* \) and determine \( m^* = b^* - \delta \) at all times. In particular, in the phase diagram in \((x, b)\) space, the \( \dot{b} = 0 \) locus is the saddle path that gives the unique equilibrium trajectory.

4 The transition

This section discusses three scenarios produced by the dynamical system developed thus far: the success story, where the economy makes the full transition to sustainable growth; failure to launch, where the economy remains trapped in a downward spiral of no innovation, resource exhaustion and falling population; premature market saturation, where the economy turns on the engine of innovation only partially and converges to a steady state in which income per capita growth requires population growth.

4.1 The success story

The paper’s main result is the characterization of a path consisting of the three phases discussed in the Introduction:

1. an initial phase where agents build up the economy by exploiting exhaustible natural resources to support population growth;

2. an intermediate phase where agents turn on the Schumpeterian engine of endogenous innovation in response to population-led market expansion;

3. a terminal phase where economic growth becomes fully driven by knowledge accumulation and no longer requires growth of physical inputs.

As discussed earlier, the last phase is crucial because it says that not only economic growth no longer requires growth of physical inputs, but also that technological change compensates for the exhaustion of the natural resource stock. The following proposition states the result formally.

Proposition 5 (Success Story) Assume:

\[
\frac{\rho (1 - \mu)}{1 - \eta} \geq \delta + \frac{1 - \gamma}{\gamma} \rho; \quad (C1)
\]
\[
\frac{(\pi \bar{x}^* - \phi) \left( \alpha - \frac{\sigma}{\beta} \bar{x}^* \right) - \gamma (\rho + m^*) + \sigma \rho}{1 - \frac{\sigma}{\beta} \bar{x}^*} > 0; \quad (C2)
\]
\[
\frac{(1 - \sigma) (1 - \alpha)}{\gamma (m^* + \rho) - \sigma \rho} > \frac{\beta}{1 - \frac{\beta}{\phi}}; \quad (C3)
\]
\[
\alpha \frac{\phi \beta - \pi}{(1 - \sigma) (1 - \alpha) \frac{\gamma (m^* + \rho) - \sigma \rho}{\gamma (m^* + \rho) - \sigma \rho} \pi - \beta} > m^* + \rho. \quad (C4)
\]
Then, there is a unique equilibrium path: the economy chooses the pair \((x_0, b_0)\), where

\[
x_0 = \theta \frac{M_0^2}{N_0^{1-\sigma}} \left( \int_0^{\infty} e^{-x(t)^\rho} dt \right) \frac{1}{S_0} \frac{1}{N_0^{1-\sigma}} < x_N,
\]

and rides the saddle path that converges to \((x^*, b^*)\), where:

\[
x^* = \frac{(1-\sigma)(1-\alpha)}{\gamma(m^*+\alpha)\phi - 1} \frac{x}{\gamma \frac{\pi - \frac{\phi}{x}}{\theta}} > x_Z;
\]

\[
b^* = \frac{\rho(1-\mu)}{\gamma(1-\theta) - \beta} - 1.
\]

**Proof.** See the Appendix. 

---

Figure 1 illustrates the dynamics: the hollow circle denotes the initial choice of consumption, fertility and extraction; the star denotes the sustainable steady state. It is worth stressing again that the initial choice \(x_0\) is not determined solely by the initial stocks but depends on the associated path of consumption.

Condition C1 is the restriction on the parameters that ensures that the economy crosses the threshold \(x_N\) and activates Schumpeterian innovation. Using the rates of innovation given by equations (23)-(24) in Lemma 4, the law of motion of firm size becomes

\[
\frac{\dot{x}}{x} = \begin{cases} 
\frac{\gamma (m^* + \rho) - \sigma \rho - (1 - \sigma) \frac{1}{\beta} \left( \pi - \frac{\phi}{x} \right)}{\gamma (m^* + \rho) - \sigma \rho - (1 - \sigma) \frac{(1-\alpha)(\pi x - \phi - \rho x^* + \gamma (m^* + \rho))}{\beta x - \sigma}} & x_N < x \leq x_Z \\
\gamma (m^* + \rho) - \sigma \rho - (1 - \sigma) \frac{(1-\alpha)(\pi x - \phi - \rho x^* + \gamma (m^* + \rho))}{\beta x - \sigma} & x > x_Z
\end{cases}
\]
Let:
\[ \bar{\nu} \equiv (1 - \sigma) \frac{\pi}{\beta} - \gamma (m^* + \rho) + \sigma \rho; \quad \bar{x}^* \equiv \frac{\phi}{\pi - \frac{\gamma (m^*+\rho)}{1-\sigma} \beta}. \]

The first line of (27) can be written \( \dot{x} = \bar{\nu} \cdot (\bar{x}^* - x) \). Let \( T_N \) be the date when \( x = x_N \). Solving this linear differential equation and then integrating between time \( T_N \) and time \( t \) yields
\[ x(t) = x_N e^{-\bar{\nu}(T_N-t)} + \bar{x}^* \left( 1 - e^{-\bar{\nu}(T_N-t)} \right). \]

There thus exists a value \( T_Z \) such that
\[ x(T_Z) = x_N e^{-\bar{\nu}(T_N-T_Z)} + \bar{x}^* \left( 1 - e^{-\bar{\nu}(T_N-T_Z)} \right) = x_Z, \]
which yields the date when the economy turns on quality growth:
\[ T_Z = T_N + \frac{1}{\bar{\nu}} \log \left( \frac{\bar{x}^* - x_N}{\bar{x}^* - x_Z} \right). \quad (28) \]

This date is finite if and only if \( \bar{x}^* > x_Z \). Using the definitions of \( \bar{x}^* \) and \( x_Z \) yields condition C2, which, intuitively, says that the parameters are such that \( z(\bar{x}^*) > 0 \). An equivalent interpretation of the condition is that the parameters are such that \( \dot{x}(x_Z) > 0 \), that is, that firm size is strictly increasing in the whole range \([x_N, x_Z]\). Thereafter the economy follows the nonlinear differential equation in the second line of (27) and converges to \( x^* \). Condition C3 ensures that this value exists because both the numerator and the denominator of (25) are positive.

Under the mild approximation discussed in Peretto (2015), one can characterize the dynamics of the last phase as a linear process. Let:
\[ \nu \equiv (1 - \sigma) (1 - \alpha) \frac{\pi - \rho \beta}{\beta} - \gamma (m^* + \rho) + \sigma \rho; \quad x^* \equiv \frac{(1 - \alpha) \phi - \gamma (m^* + \rho)}{(1 - \alpha) \pi - \rho \beta - \frac{\gamma (m^*+\rho)}{1-\sigma} \beta}. \]

The second line of (27) can be written \( \dot{x} = \nu \cdot (x^* - x) \). Solving this linear differential equation and then integrating between time \( T_Z \) and time \( t \) yields
\[ x(t) = x_Z e^{-\nu(T_Z-t)} + x^* \left( 1 - e^{-\nu(T_Z-t)} \right). \]

The main advantage of this simplification is that the in the second and third phases the dynamics of firm size follow a piece-wise linear differential equation. Accordingly, the model yields a closed-form solution for all endogenous variables as functions of time, \( t \). This is not essential to the point made in the paper but it is a nice property to have in mind for future applications of the approach.

I have tried, but so far failed, to develop a similar analytical characterization of the dynamics in the first phase. While the saddle path identifies a well-defined function \( b(x) \), all attempts to obtain its analytical expression have failed. Nevertheless, the qualitative analysis provides a clear intuition about the mechanism at play. The path has the property \( \dot{x}/x = \bar{Y}/Y > 0 \). Specifically, under condition C1 agents choose consumption, fertility and extraction paths that ensure positive
growth of market size and thus of firm size. Observing that the elasticity $\xi (x)$ defined in Lemma 1 has the property $d\xi (x)/dx < 0$ we have:

$$\frac{db (x)}{dx} > 0;$$

$$\frac{d (\dot{x}/x)}{dx} = \frac{d}{dx} \left( \frac{\gamma (b(x) - \delta) - (1 - \gamma) \rho}{1 + (1 - \gamma) \xi (x)} \right) > 0.$$ 

In words, the equilibrium path exhibits rising fertility and accelerating firm size growth. Using the extraction path (12), we obtain

$$g = \xi (x) (\dot{x}/x) - (1 - \gamma) (m + \dot{c}/c + \rho) = [1 + \xi (x)] (\dot{x}/x) - b (x) + \delta.$$ 

The rising growth of firm size, $\dot{x}/x$, is critical for the sign of the growth rate of output per worker. First, $g$ is positive if an only if economies of scale are sufficiently strong, i.e., if and only if $[1 + \xi (x)] (\dot{x}/x) > b (x) - \delta$. Second, the elasticity $\xi (x)$ is decreasing in $x$ because static economies of scale are bounded above. Therefore, the growth rate of output per worker can be positive throughout the first phase if and only if the net effect of economies of scale exhaustion and rising firm size growth dominates the rising crude birth rate. This calculation complements the phase diagram’s visual message and says that the initial phase does not necessarily exhibit falling output per worker but that the relentless downward pressure due to growth drag eventually must result in falling output per worker if the economy takes too long to activate innovation. A similar calculation makes another point not apparent from the phase diagram: the rate of exhaustion is falling over time as the rate of growth of the ratio $c$ falls toward zero. The downward pressure from exhaustion is nevertheless relentless because the exhaustion rate has a strictly positive floor given by the discount rate $\rho$.

Now refer back to condition C1, which ensures that $\dot{Y}/Y > 0$ and thus that the economy crosses the threshold $x_N$ at time $T_N$. The condition actually says that the economy follows a version of the Hartwick rule (Hartwick 1977; see also Solow 1974): agents transform natural resources into productive adults and the net effect is aggregate economic growth. Although the Hartwick rule has been derived in models of physical capital accumulation, the mechanism at its heart operates in this model. Stripping away the normative interpretation of the rule, since we are characterizing a market equilibrium, what we have here is that (i) households invest the revenues from extraction of the exhaustible resource in the accumulation of a productive assets and (ii) the net effect of such extraction-reinvestment process is overall growth of output. Although for simplicity the model abstracts from education, it treats the reproduction decision as a costly investment in future wage earners (adult humans) and thus it would be perfectly appropriate to say that the key component of the first phase is the transformation of natural capital into human capital.

To complete the characterization of the main features of this scenario, note that in the second and third phases the growth rate is given by equation (18) in Lemma 1 while the rates of innovation are given by equations (23)-(24) in Lemma 4. Since the transition features rising firm size, $x$, it features a rising rate of variety innovation (entry), $n (x)$. Under conditions C2-C3, the economy
crosses the threshold $x_Z$ at time $T_Z$, displays rising rates of variety innovation, $n(x)$, and quality innovation, $z(x)$, and converges from below to the (fully) endogenous growth rate

$$g^* = \alpha (\pi x^* - \phi) - m^* - \rho.$$ 

Note that because crude fertility, $b$, is constant, this is the rate of growth of both output per worker and output per capita. Similarly, because both the ratios of consumption to final output, $c$, and of final output to GDP, $Y/G$, are constant, this is the growth rate of consumption per capita. The associated sustainability condition is condition C4, which says that

$$g^* > 0 \iff \alpha (\pi x^* - \phi) = \frac{\phi \beta - \pi}{\gamma (m^* + \rho) - \sigma \rho \pi - \beta} > m^* + \rho.$$ 

This inequality holds for small values of $m^* + \rho$, that is, given $\rho$ it holds for sufficiently slow population growth. In fact, this growth rate is compatible with zero, or even negative, population growth. Formally, it holds for $m^* \in (m^*_{\min}, m^*_{\max})$ with $m^*_{\min} < 0$ and $m^*_{\max} > 0$. This interval includes 0 and allows for negative population growth.

### 4.2 Failure to launch?

There are two potential pitfalls on the path of this economy. The first is that failure to launch is possible. When condition C1 in Proposition 5 fails, either the $\dot{x} = 0$ locus intersects the $\dot{b} = 0$ locus from above for some value $\hat{x} \in [\phi/\pi, x_N]$, or it is above the $\dot{b} = 0$ locus for all $x \in [\phi/\pi, x_N]$. The latter is just a special case of the former and thus the following discussion focuses only on the case where the intersection $\hat{x}$ exists.

**Proposition 6 (Failure to Launch)** Assume

$$\frac{\rho (1 - \mu)}{1 - \eta} - 1 < \delta + \frac{1 - \gamma}{\gamma} \rho.$$ 

and consider the case where the $\dot{x} = 0$ locus intersects the $\dot{b} = 0$ locus from above at the value $\hat{x} \in [\phi/\pi, x_N]$. Then, two outcomes are possible. If the economy has a sufficiently large endowment $S_0$, it chooses a pair $(x_0, b_0)$ with $x_0 \in (\hat{x}, x_N)$ and places itself on the saddle path that converges to $x^*$. If, instead, the economy has an insufficient endowment $S_0$, it must choose a pair $(x_0, b_0)$ with $x_0 \in (\phi/\pi, \hat{x})$ and is thus doomed to collapse.

**Proof.** See the Appendix. ■

Figure 2 illustrates the case in which society fails to build up the economy. The square denotes the unstable steady state in the no innovation region. The cross denotes the economic collapse point where firms become non-viable. The hollow circle denotes the initial choice of fertility, consumption and extraction when the path that leads to the sustainable steady state is not accessible.
Many factors enter the conditions for this scenario to occur. Most prominent is the size of the initial endowment. When $S_0$ is too small, holding constant all the other determinants of fertility and consumption behavior, the economy might be constrained to an initial choice of $R_0$ resulting in $x_0 < \tilde{x}$. A second prominent factor not immediately apparent from the phase diagram is the regime of property rights over the natural resource. Recall that the model posits a continuum of mass one of household each one with endowment $S_0$. What this means is that the model posits decentralized resource management by atomistic agents with full property rights. It follows that the initial value $R_0$ does not allow for (i) coordination among agents and (ii) over-exploitation in the sense of the Tragedy of the Commons (Hardin 1968).

Coordination is potentially crucial because the scenario discussed here hinges on a clear externality: in their extraction decisions agents do not account for the dynamics of aggregate market size and thus extract less than what would allow the economy to cross the threshold $\tilde{x}$. A potentially paradoxical implication is that weaker property rights that result in some form of the tragedy of the commons — in the sense of more aggressive extraction motivated by the expectation that failure to extract today leaves nothing to extract tomorrow — might allow the economy to cross the threshold $\tilde{x}$ for unchanged parameters. In this light, the model poses interesting questions and sheds a different light on issues that traditionally have had straightforward interpretations.

One way to think about these dynamics is that the existence of the threshold $\tilde{x}$ opens the door to temporary changes in extraction behavior that have permanent effects on the growth path of the economy. A simple example could be a temporary suppression of property rights. Obviously, it cannot be desirable to engineer a full blown tragedy of the commons whereby $R_0 = S_0$. So, a temporary intervention has to achieve higher extraction but not complete exhaustion. Thinking about schemes that might accomplish it, two come to mind. The first is temporary nationalization of the resource.
The second is temporary subsidies. Both schemes achieve coordination on a more aggressive extraction path but they have different features that yield different potential costs. Nationalization, interpreted as total suppression of property rights, might turn out to be irreversible and might result in less efficient resource management, both for political-economy reasons. Subsidization also can turn out to be irreversible and produce inefficiencies of its own for political-economy reasons. A third scheme with similar trade-offs is regulation, e.g., extraction mandates. The debate on such issues is very old and very lively. However, it has mostly taken place in a context where the market failure is typically taken to be over-exploitation. The scenario discussed here, in contrast, is one of under-exploitation with potentially fatal long-term consequences.

Another surprising implication of the dynamics driving this scenario is the following. Consider an economy that at time zero can select \( x_0 > \tilde{x} \) and start on the path that leads to success. Now imagine that such economy at some future date is hit by a shock, say an epidemic, that kills a large fraction of the population. Because of its past extraction, the economy at the time of the shock has a smaller endowment and therefore is vulnerable in the following sense. The fall in the size of the workforce resets the state variable \( x \) at a smaller value. Say that such value is below \( \tilde{x} \). The economy now needs to make a new set of initial decisions but, because the endowment is smaller, might well be unable to set the new initial \( R \) at a value that yields \( x > \tilde{x} \) and therefore be doomed to collapse. One could think of this scenario as far-fetched. In fact, it is consistent with the most recent re-interpretation of the history of Easter Island proposed in archeology (Hunt and Lipo 2011). As is well known, Easter Island is typically proposed as the archetypical example of a society that collapsed due to over-exploitation of its resource base (a colorful expression often used is eco-suicide). The recent evidence suggests instead that it collapsed because (i) upon first contact with Europeans the native population crashed from diseases against which it had no defense and (ii) the local environment had suffered greatly from the spread of the rats that came with the Europeans.

The reflections above should not be pushed too far, since at this stage the analysis is mostly qualitative and much more work is called for to fully flesh out the implications and the empirical validity of the mechanism at the heart of this model. They clearly suggest, however, that the model does offer a new perspective on important issues.

Readers familiar with Unified Growth Theory (Galor 2011), for example, might want to note that allowing for a Malthusian feedback such that population size responds to a physical resource constraint would only make things worse. If fertility falls as the natural resource runs out, eventually it must fall below the mortality rate yielding shrinking population. The paper abstracts from these forces to keep things simple but reflecting on them adds perspective to the paper’s main point, namely, that a period of rising population that seemingly exacerbates natural resource scarcity is the key to success. If achieving such rising population requires overcoming Malthusian constraints, success is harder to achieve but not necessarily impossible. The topic deserves careful analysis and
I leave it to future work.\footnote{Some specifics might help readers familiar with UGT. Introducing exhaustible resources in such models yields that initially the rate of technological change not only must be positive to fuel population growth and the associated growth acceleration, but it must be sufficiently strong to more than offset the tendency of the shrinking resource base to drag along the population. In UGT models all dynamic feedbacks are initially positive so that if the rate of technological change is an increasing function of population and population shrinks because of resource exhaustion not compensated by technological change, then the rate of technological change slows down and the whole process results in extinction. This paper’s model does not exhibit this property because the initial phase is not Malthusian in the UGT sense, that is, agents are not at a corner solution where the fertility rate is dictated by the budget constraint but, rather, are obeying the first-order conditions of their forward-looking maximization problem and are thus able build up the economy even in the absence of growth of the effective resource base.}

4.3 Premature market saturation

Figure 3 illustrates the second potential pitfall on the path of this economy: premature market saturation. The dark circle denotes the steady state with no quality innovation. This is when condition C2 does not hold and thus the economy fails to cross the threshold for vertical innovation, $x_Z$, and converges instead to the steady state $(\bar{x}^*, b^*)$. This steady state exhibits the semi-endogenous growth rate

$$\bar{g}^* = \sigma \cdot \frac{\gamma (m^* + \rho) - \rho}{1 - \sigma} - (1 - \gamma) (m^* + \rho).$$

The associated sustainability condition is

$$\bar{g}^* > 0 \iff \sigma > (1 - \gamma) \frac{m^* + \rho}{m^*},$$

Figure 3: premature market saturation
which, intuitively, holds for small values of $m^* + \rho$. Note, however, that the condition is possible in the first place only if $\sigma > 1 - \gamma$. Moreover, since the right-hand side is decreasing in $m^*$, given $\sigma, \gamma, \rho$, the condition holds the faster is population growth, i.e., for

$$m^* > \frac{\rho}{\sigma (1-\gamma)} - 1.$$  

The implication is that, given a non-zero exhaustion rate, sustainable growth requires sufficiently fast population growth.

This is an important point in light of the evidence and arguments discussed in, among others, Strulik et al. (2013). Sustainability predicated on population growth runs counter first-principles and to facts because (i) an infinite population is not possible on a finite planet and (ii) population growth is not only slowing down everywhere, but in many countries it is negative. At most, one should expect it to settle at zero in the long run. The difference between the two scenarios, therefore, is that the semi-endogenous growth outcome ensures sustainability only under implausible conditions. The fully endogenous growth outcome, in contrast, does not need population growth and therefore does not tie sustainability to implausible assumptions about the interdependence of population and resources.

### 5 Conclusion

This paper has proposed a Schumpeterian approach to the study of the interactions among population, technology and exhaustible resources. Relative to the traditional approach of resource economics based on the DHSS (Dasgupta, Heal, Solow, Stiglitz) foundation, the focus on firms’ incentives and the endogeneity of the structure of the market in which they operate provides a novel view of the interplay of population and resources and stresses the key role of market size.

The framework is remarkably tractable and allows one to obtain a transparent characterization of dynamics that are typically very complex. Several points emerge from the analysis of sustainability, defined as the ability of the economy to achieve positive, constant, exponential growth of consumption per capita in the long run.

First, it is not just the rate of technological change that matters for sustainability, but also the type. The concept of de-coupling, often used in the popular debate on sustainability, is broader than simply overcoming resource exhaustion: it refers to a qualitative change in economic activity, from economic growth based on larger use of natural inputs to economic growth divorced as much as the laws of nature allow from such inputs. Moreover, the same first-principles that drive the concerns about increasing scarcity of physical inputs drive the concerns about the planet’s ability to withstand a perpetually growing population — which, after all, is a physical input subject to physical constraints. Therefore, de-coupling requires divorcing economic growth from demographic growth as well. In this perspective, these first-principles reject semi-endogenous growth as a viable solution to the sustainability problem. The paper’s analysis shows that the concept of productivity growth as the amplification of the growth of the number of people operates in the opposite direction.

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of what the notion of scarcity at the heart of the sustainability debate entails. In the language of
the model, quality innovation can deliver sustainable growth while variety innovation cannot. The
reason is that the former stands in for the accumulation of intangibles and the increase in the flow
of services that we obtain from goods for unchanged use of physical resources. The latter, instead,
stands in for innovation whose implementation requires the accumulation of tangible productive
assets (firms, plants), which requires larger use of physical resources.

The adopted definition of sustainability might strike some as too narrow. Similarly, the harsh-
ness of the environment postulated in the paper might strike some as extreme. It is not hard to
extend the framework to: (i) regeneration in the resource dynamics (renewable), which would allow
for a steady state with constant, positive stock of the resource; (ii) a backstop technology triggered
by sufficiently high resource price, which would make the analysis much more difficult and yield
conclusions in line with what we already know, namely, that at some point the economy switches
to the alternative source. The meaningful counter-argument to such observations, however, is not
that such modifications are feasible but that they assume scarcity away with respect to the pa-
er’s baseline case, which instead sets the highest possible bar for technology to clear to deliver
sustainability. This is not a crucial reason not to consider such extensions, but it suggests that the
paper strikes at the heart of the sustainability question precisely because it strips away all forces
that weaken the scarcity problem.

One possible weakness of the paper is that the model is agnostic with respect to the population
level: it simply predicts a constant proportional rate of change. That is, the model lacks an internal
mechanism that forces such rate of change to be zero. It is of course possible to extend the framework
to allow for feedbacks that stabilize population. The cost is that it complicates the analysis while
the paper privileges transparency. Moreover, an important caveat applies: as argued above, the
stabilizing mechanism cannot be Malthusian, in the sense that population becomes proportional
to the resource base, since the latter is always shrinking. In other words, in studying population-
resources interdependence one must be very careful: potential exhaustion changes drastically the
nature of the problem. This is an important topic that I leave to future work.

A similar possible weakness, especially in empirical perspective, is that the model incorporates
the simplest model of exhaustible resource dynamics. As is well known, because such model equates
the Hotelling rents to the spot market price of the extracted resource, it produces counterfactual
behavior: it says that the price of the resource grows all the time at the rate of interest. It is possible
to extend that component of the model to more sophisticated versions (especially versions that allow
technological change in extraction) and obtain conclusions quite similar to those described above.
The cost is added complexity, the benefit is better fit with the data. Since the added complexity is
notoriously substantial and the goal of this paper is to illuminate mechanisms rather than fit the
data, I leave the elaboration of such extensions to future work.
6 Appendix

6.1 Derivation of the return to quality

The usual method of obtaining first-order conditions is to write the Hamiltonian for the optimal control problem of the firm. This derivation highlights the intuition. The firm undertakes R&D up to the point where the shadow value of the innovation, \( q_i \), is equal to its cost,

\[
1 = q_i \Leftrightarrow I_i > 0. \tag{29}
\]

Since the innovation is implemented in-house, its benefits are determined by the marginal profit it generates. Thus, the return to the innovation must satisfy the arbitrage condition

\[
r = \frac{\partial \Pi_i}{\partial Z_i} \frac{1}{q_i} + \frac{\dot{q}_i}{q_i}. \tag{30}
\]

To calculate the marginal profit, observe that the firm’s problem is separable in the price and investment decisions. Facing the isoelastic demand (2) and a marginal cost of production equal to one, the firm sets \( P_i = 1/\theta \). Substituting this result into (4), differentiating with respect to \( Z_i \), substituting into (30) and imposing symmetry yields (6).

6.2 Proof of Lemma 1

Log-differentiate (16) with respect to time and then use the Euler equation (9) and the extraction path (12).

6.3 Proof of Lemma 2

Use (15) directly and in log-differentiated form to rewrite (6) and (7). Then log-differentiate the definition of \( x \) with respect to time and use the Euler equation (9) to obtain (21).

6.4 Proof of Lemma 3

Write the household budget (8) as

\[
\dot{A} = rA + D_Y + wM + pR - C,
\]

where \( D_Y \) is the flow of dividends paid by the final sector. Under the paper’s assumptions \( D_Y = 0 \) and so omitting it from (8) does not change the analysis. However, for the purposes of this proof it is useful to include it and recognize that it is \( D_Y = Y - wM - pR - N \cdot PX \) (recall that in equilibrium \( L = M \)). Dividing through by \( A \) yields

\[
\frac{\dot{A}}{A} = r + \frac{D_Y + wM + pR - C}{A}.
\]
When $n = 0$ assets market equilibrium requires $A = NV$ but $V < \beta Y/N$ since by definition the free-entry condition does not hold. Differentiating (5) with respect to time and substituting the result in the expression for the budget derived above yields

$$0 = \frac{\Pi - I}{V} + \frac{Y - wM - pR - N \cdot PX + wM + pR - C}{NV},$$

which simplifies to

$$0 = \frac{\Pi - I}{V} + \frac{Y - N \cdot PX - C}{NV}.$$

Using the definition of $\Pi$ and rearranging terms yields

$$C = N \left[ (P - 1) X - \phi Z - I \right] + (1 - \theta) Y.$$  

The definitions of $c$ and $x$, the R&D technology (3), and the fact that $NX = \theta^2 Y$, then yield

$$c = \theta^2 \left( \pi - \frac{\phi + z}{x} \right) + 1 - \theta.$$

Finally, set $z = 0$ in this expression, since it holds for $x \leq x_N < x_Z$, to obtain the top line of (22).

When $n > 0$ assets market equilibrium requires $A = NV = N\beta X = \beta \theta^2 Y$, which says that the wealth ratio $A/Y$ is constant. Using the definition of $c$ and the saving schedule (9), rewrite the budget constraint, after rearranging terms, as

$$c = \left( \rho + \frac{\rho}{c} \right) \beta \theta^2 + 1 - \theta.$$

This unstable differential equation says that $c$ jumps to its steady-state value $c^* = \rho \beta \theta^2 + 1 - \theta$, which is the value in the bottom line of (22).

Next, eliminating $h_A$ and labor income $wM$ reduces the first order condition for population $M$ to

$$\dot{h} = \rho h - \eta - \frac{\gamma (1 - \theta) - c}{(1 - \mu) (1 - \eta) c}.$$  

Finally, using the fertility rule (10) to eliminate $h$ yields the differential equation

$$\frac{\dot{b}}{b} = \left[ \frac{\gamma (1 - \theta)}{(1 - \eta) c(x) - 1} \right] \frac{b}{1 - \mu} - \rho.$$  

where $c(x)$ is given by (22). This equation holds for all $x > \phi$.

In the region $x > x_N$, since $c(x) = c^*$ the fertility rate jumps to its own steady state

$$b^* = \frac{\rho (1 - \mu)}{\gamma (1 - \theta) (1 - \eta) c^*}.$$  

Moreover, integrating the equilibrium extraction path (12) yields

$$R(t) = R_0 e^{-\rho t}.$$
Substituting this result into the constraint

\[ S_0 = \int_0^\infty R(t) \, dt \]

yields \( R_0 = \rho S_0 \). Therefore:

\[ R(t) = \rho S_0 e^{-\rho t}; \quad S(t) = S_0 e^{-\rho t}. \]

### 6.5 Proof of Lemma 4

Equation (17) and Lemma 1 allow one to rewrite the return to variety innovation in (20) as

\[ n = \begin{cases} \frac{1}{\beta} \left( \pi - \frac{\phi + z}{\phi} \right) - \rho & z > 0 \\ \frac{1}{\beta} \left( \pi - \frac{\phi}{\phi} \right) - \rho & z = 0 \end{cases} \quad (31) \]

and the return to quality innovation in (19) as

\[ z = \begin{cases} \alpha (\pi x - \phi) - \sigma n - \gamma (m^* + \rho) & n > 0 \\ \alpha (\pi x - \phi) - \gamma (m + \hat{c}/c + \rho) & n = 0 \end{cases} \quad (32) \]

The threshold \( x_N \) follows directly from (31), which says that when agents anticipate \( z = 0 \) entry is positive for

\[ x > x_N \equiv \frac{\phi}{\pi - \rho \beta}. \]

Solving (31) and (32) for \( z \) then yields (24), which says that in the region \( x > x_N \) quality innovation is positive for

\[ (\pi x - \phi) \left( \alpha - \frac{\sigma}{\beta x} \right) > \gamma (m^* + \rho) - \sigma \rho. \]

Since the left-hand side is increasing in \( x \), we have a unique value \( x_Z \). Finally, \( x_Z > x_N \) if

\[ (\pi x_N - \phi) \left( \alpha - \frac{\sigma}{\beta x_N} \right) < \gamma (m^* + \rho) - \sigma \rho, \]

which yields

\[ \phi \alpha \frac{\rho \beta}{\pi - \rho \beta} < \gamma (m^* + \rho). \]

### 6.6 Proof of Proposition 5

Refer to Figure 1. In the region \( x \in [\phi/\pi, x_N] \), the \( \dot{b} = 0 \) and \( \dot{x} = 0 \) loci are, respectively:

\[ b = \frac{\rho (1 - \mu)}{\gamma (1 - \phi_0) (1 - \eta) (1 - \epsilon)}; \]

\[ b = \frac{(1 - \gamma) \rho}{\gamma} + \delta. \]
The $\dot{b} = 0$ locus is increasing and concave in $x$. Three cases are possible. (i) The $\dot{x} = 0$ locus is below the $\dot{b} = 0$ locus for all $x \in [\phi/\pi, x_N]$. (ii) The $\dot{x} = 0$ locus intersects the $\dot{b} = 0$ locus for some value $\ddot{x} \in [\phi/\pi, x_N]$. (iii) The $\dot{x} = 0$ locus is above the $\dot{b} = 0$ locus for all $x \in [\phi/\pi, x_N]$. Case (i) requires that at $x = \phi/\pi$ the $\dot{x} = 0$ locus be lower than the $\dot{b} = 0$ locus, that is,

$$\frac{\rho(1 - \mu)}{\gamma - 1} > \frac{(1 - \gamma) \rho}{\gamma} + \delta.$$  

This is condition C1, which ensures that the economy leaves the region $x \in [\phi/\pi, x_N]$ and activates horizontal innovation because the only trajectories originating in the no innovation region that do not violate boundary conditions are those that enter the innovation region.

Note now that, of all the trajectories that enter the innovation region, those that connect with an explosive trajectory cannot be equilibria because they eventually violate boundary conditions. Consequently, there exists only one trajectory leaving the no innovation region that satisfies all boundary conditions: this is the trajectory that connects with smooth pasting with the saddle path that applies for $x > x_N$. To see this, note two things. First, the ratio of the $\dot{b}$ and $\dot{x}$ equations yields

$$\frac{db(x_N)}{dx} = \frac{b}{x_N} [1 + (1 - \gamma) \xi(x)] \frac{\gamma(1 - \theta)}{(1 - \theta) \xi(x)} - 1 \frac{b^*}{1 - \mu} - \rho = 0.$$  

Second, recall that the economy chooses an initial pair $(x_0, b_0)$, so that

$$x_0 = \theta^{1-\sigma} M_0^2 \left( \left[ \int_0^\infty e^{-\varepsilon(t) dt} \right]^{-1} S_0 \right)^{1-\gamma},$$  

where

$$\varepsilon(t) \equiv \frac{1}{t} \int_0^t (\dot{c}(s)/c(s) + \rho) ds.$$  

This choice incorporates the fact that by choosing the initial value of the extraction flow, $R_0$, to satisfy the lifetime constraint

$$S_0 = \int_0^\infty R(t) dt,$$

the economy in fact chooses the initial value of firm size subject to this constraint.

After the economy leaves to innovation region and starts riding the saddle path $b = b^*$, what happens next depends on condition C2. It the condition holds, the economy crosses the threshold $x_Z$ and activates vertical innovation. It then converges to the steady state $x^*$, which exists and is positive if:

$$\frac{(1 - \sigma)(1 - \alpha) \phi}{\gamma (m^* + \rho) - \sigma \rho} - 1 > 0;$$  

$$\frac{(1 - \sigma)(1 - \alpha) \pi}{\gamma (m^* + \rho) - \sigma \rho} - \beta > 0.$$  

Combining these two inequalities and observing that $\frac{\beta}{\pi} \phi > 1$ yields the existence condition C3:

$$\frac{(1 - \sigma)(1 - \alpha)}{\gamma (m^* + \rho) - \sigma \rho} > \frac{\beta}{\pi} > \frac{1}{\phi}.$$  

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Condition C4 ensures that steady-state growth at \( x = x^* \) is positive:

\[
g^* = \alpha (\pi x^* - \phi) - m^* - \rho > 0 \Rightarrow \alpha \frac{\phi \beta - \pi}{(1-\sigma)(1-\alpha)} > m^* + \rho.
\]

Finally, observe that

\[
\frac{d (x'/x)}{dx} > 0
\]

follows from the fact that, from the phase diagram,

\[
\frac{db(x)}{dx} > 0,
\]

that is the saddle path is upward sloping, and \( \xi'(x) < 0 \) since as firm size grows static economies of scale are gradually exhausted.

References


