Robust Endogenous Growth

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Abstract

This paper shows that Schumpeterian models with endogenous market structure *do not* require linear production structures to generate endogenous growth, defined as steady-state, constant, exponential growth of income per capita. This version of modern growth theory, therefore, is robust: its key result obtains for a thick set of parameter values instead of, as often claimed, for a set of measure zero. The paper, moreover, pays close attention to transitional dynamics, showing not only the existence but also the global stability of the endogenous-growth steady state.

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1 Introduction

Setting its birth at the publication of Romer (1986), modern endogenous growth economics is now in its 30s and has thus reached full maturity. By all measures of scholarly accomplishment it is a success. The field is vibrant and expanding, empirical and policy relevant. Yet, it has lived all its life under the shadow of the so called "linearity" critique.

The critique consists of two claims. (1) To produce endogenous growth — defined as steady-state, constant, exponential growth of income per capita — the early models posit a production structure that is linear in the factor that the economy accumulates. (2) The theory, therefore, is fragile because if production is less than linear growth dies out and income per capita converges to a constant value, while if production is more than linear growth accelerates and income per capita explodes to infinity in finite time. The seeming requirement of *exact* linearity is why the critique is also known as the knife-edge or razor-edge critique.

In this paper I show that Schumpeterian models with endogenous market structure *do not* require such linearity of the production structure to generate endogenous growth. This version of modern growth theory, therefore, is robust in the sense that its key result obtains for a thick set of parameter values instead of, as often claimed, for a set of measure zero.[1] Consequently, researchers can use these models with the confidence that the empirical predictions and policy prescriptions that they produce are not fragile because at their core such models have a robust mechanism that captures critical features of the real-world economy.

To develop the argument, I extend the tractable version of the theory developed in Peretto (2015), which yields a closed form solution for the model's dynamics and thus allows one to see transparently the forces at play. I set up the general framework and notation in Section 3. I then discuss the main result in Section 4. I allow the overall production structure to be more than linear in the growth-driving factor, thus violating the knife-edge assumption that is allegedly necessary to the theory. I nevertheless obtain constant exponential growth.

Because this point is essential to appreciating the paper's contribution, and because the linearity critique is seemingly based on common-sense reasoning, I devote Section 5 to assessing the model's mechanism in the context provided by Jones (2005), which to my knowledge is the clearest articulation of the critique.[2] The punchline of the exercise is that the critique, as articulated there, is inapplicable to this paper's model. Specifically, the critique asserts that "... all growth models that exhibit steady-state growth ultimately rest on an assumption that some differential equation

[1] To my knowledge, these models were originally developed simultaneously and independently by Peretto (1994) and Smulders (1994) in their PhD dissertations. Follow-up publications that develop the approach are Peretto (1996, 1998a, 1998b, 1999), Smulders and van de Klundert (1995, 1996) and Peretto and Smulders (2002). Also relevant to the initial development of this literature are the contributions of Dinopoulos and Thompson (1998) and Howitt (1999). This class of models has received substantial empirical support, especially as an explanation of long-term historical data. For examples, see Zachariadis (2003), Laincz and Peretto (2006), Ha and Howitt (2007), Ulku (2007), Madsen (2008, 2010), Madsen and Ang (2011, 2013), Madsen and Timol (2011), Madsen, Ang and Banerjee (2011), Madsen, Saxena and Ang (2011), Greasley, Madsen and Wohar (2013).

[2] The material in this section evolved in the course of a long email discussion with Chad Jones.

takes the form $\dot{X} = \_X$. Growth models differ primarily according to the way in which they label the X variable and the way in which they fill in the blank in this differential equation" (Jones 2005, p. 1103). The unstated clause in that assertion is that the blank, standing for some model-specific object, is independent of X and of calendar time. I show that this paper's model fills in the blank with an equilibrium object that is a function of, *among other things*, X so that the differential equation is not linear in X. Such equilibrium object, moreover, in steady state becomes constant (i.e., time-invariant and independent of X) so that the differential equation delivers the mathematical description of a constant exponential growth process. But such property is not imposed on any of the model's primitives — in fact, it is not *assumed* anywhere in the paper. It *emerges* as the system converges to the steady state. Let me stress this point. The linearity — in the sense of Jones (2005) — of the core accumulation equation is a steady-state property of the system representing economic decisions. It is not a property of the model's primitives. What makes its emergence possible is that the model interprets X as product quality and fills in the blank with an equilibrium object, i.e., a firm-level investment rate, that depends on X and a very specific other thing that in steady state offsets the growth of X. This very specific other thing is the mass of firms/products, which the model interprets as endogenous market structure.

Of course this is not the first paper that tries to clarify core issues raised in the debate on the robustness of endogenous growth theory. Previous notable contributions are Eicher and Turnovsky (1999) and Daalgard and Kreiner (2003). What distinguishes my attempt is the focus on Schumpeterian theory and decentralized market equilibria.

Before jumping into the technical part of the exposition, it is useful to preview the intuition underlying the paper's central result. The next section reduces the paper's argument to its essentials and isolates the key mechanism driving these models. The exercise should help one appreciate the general idea underlying the approach and see why it delivers the results it does.

2 The insight

The model is of the lab-equipment, or one-sector, class where the final good Y can be consumed, used to produce intermediate goods, invested in the improvement of the quality of existing intermediate goods, or invested in the creation of new intermediate goods. This is a desirable feature that allows one to discuss the model's production structure in terms of only one technology, namely, that used to produce the final good. In symmetric equilibrium, that technology has the following reduced-form representation

$$Y = (\text{constant}) \cdot \ N^{\sigma} Z^{\kappa} \ \cdot \ L,$$

where Y is output, L is labor, and N and Z are, respectively, the variety and the average quality of intermediate goods.

The theory assigns different roles to product variety and product quality. The former provides the proliferation mechanism that sterilizes the market size effect, and thereby the (strong) scale effect, the latter is the economy's engine of growth. The linearity critique says that one must impose $\kappa = 1$ to ensure exact linearity of Y in Z.[3] I show that, instead, this economy converges to a steady state that exhibits constant, exponential growth of income per capita for

$$\kappa \in [1, \kappa^{\max}), \quad \text{with } \kappa^{\max} > 1.$$

The paper's main result therefore is that constant endogenous growth does not require exact linearity of the production structure in the growth-driving factor.

To see why, it is useful to first note that the elasticity of output with respect to labor, 1, is the lower bound below which social returns to quality κ cannot fall without making steady-state constant endogenous growth impossible. Given that one must rule out values $\kappa < 1$, then, the literature's practice has been to impose $\kappa = 1$. It turns out, however, that $\kappa > 1$ is possible because increasing returns to quality in excess of the lower bound 1 are absorbed by the endogenous mass of products.

The second step in the argument is thus to understand the role of product variety. The model's key feature is that there are two activities pushing the technology frontier: (1) creation of new products and the firms that produce and sell such products (horizontal innovation); (2) quality improvement through in-house R&D carried out by firms once in existence (vertical innovation). Accordingly, N is the mass of products/firms competing in the marketplace, a broader concept than just product variety. The model's conditions characterizing agents' behavior say that these returns are functions of the quality-adjusted gross cash flow of the firm:

$$x_i \equiv \frac{X_i (P_i - 1)}{Z_i} = \frac{\text{gross cash flow}}{\text{quality}},$$

where P_i is the price the firm charges, X_i is the quantity it sells, 1 is the marginal cost of production in units of final output, and Z_i is the quality level achieved by the firm. In symmetric equilibrium, this expression becomes

$$x = (\text{constant}) \cdot \underbrace{\frac{Y}{Z}}_{\text{quality-adjusted market size}} \cdot \underbrace{\frac{1}{N}}_{\text{market share}} = (\text{constant}) \cdot \frac{Z^{\kappa-1} L}{N^{1-\sigma}}.$$

If we think of the model's steady state as a situation with unchanging incentives, in this setup this means looking for solutions with constant rate of return to innovation and, therefore, for solutions with constant quality-adjusted gross cash flow.

The first equality in the expression above says that quality-adjusted gross cash flow, x, is constant if the mass of firms, N, grows at the same rate as the quality-adjusted size of the market, Y/Z. That is, *if entry sterilizes the quality-adjusted market size effect.* This *market share effect* is the

[3] One way to see this, popular in the literature (see e.g., Jones 2005), is to postulate that average quality grows according to $\dot{Z} = I$ and to imagine that aggregate investment in quality growth, NI, is a constant fraction s_I of output, so that investment per firm is a $I = s_I Y/N$. The resulting relation $\dot{Z} = s_I Y/N$ is then used to argue that one must assume $\kappa = 1$ to obtain non-explosive growth. It should be clear that such an argument relies crucially on holding N constant.

core of the endogenous market structure mechanism. It is worth emphasizing that it holds independently of the remainder of the model's general equilibrium structure because it is an industry-level mechanism.

The second equality shows how the mechanism plays out in general equilibrium, that is, once we consider the aggregate determinants of quality-adjusted market size. In this setup, these determinants are the supply of labor, L, and the aggregation process that yields the social returns to variety, N, and average quality, Z, which differ from the private returns. The restriction $\sigma < 1$ ensures that the market share effect in the intermediate goods market — N at the denominator of the ratio Y/NZ — dominates over the love-of-variety effect in final production, and thereby that the mass of firms/products ends up at the denominator of the expression for the the quality-adjusted gross cash flow of the firm. This expression then captures the model's main property: firm-level decisions depend on the quality-adjusted gross cash flow, which is proportional to the firm's sales and thus increasing in anything that shifts the firm's demand curve to the right. The offsetting force to such rightward shifts is entry, which, via the dominant market share effect, shifts the firm's demand curve to the left.

The expression for the quality-adjusted gross cash flow, then, shows that a steady state with constant x is feasible for $\kappa > 1$ because the market fragmentation mechanism "tames" potentially explosive growth due to increasing returns that look excessive in terms of the traditional theory. To date, such increasing returns have been deemed impossible by a priori reasoning that extrapolates the properties of one-dimensional models.

To see this, consider the traditional view of the world that ignores endogenous market structure. That is, assume N exogenous and constant, e.g., set $N = 1$. Then, to rule out explosive behavior Y must be exactly linear in Z so to make x independent of Z, i.e., one needs to impose the knife-edge condition $\kappa = 1$. This is not enough, however. As is well known, to rule out explosive behavior one also needs to assume L constant because of the strong scale effect.

Such reasoning does not apply to models with two dimensions of technology that play interdependent, *but distinct*, roles: the vertical dimension provides the engine of growth, the horizontal dimension provides the endogenous market structure that sterilizes the market size effect and thereby its two key manifestations: (1) the strong scale effect, with the associated need to impose constant endowments to rule out explosive behavior due to external forcings (like, e.g., population growth), and (2) the need to impose a knife-edge condition that rules out explosive behavior due to internal "excessive" increasing returns.

3 The model: general setup

The economy is closed. To keep things simple, there is no physical capital. All variables are functions of (continuous) time but to simplify the notation I omit the time argument unless necessary to avoid confusion.

3.1 Households

A representative household consisting of a continuum of identical members supplies labor and trades assets in competitive markets. The mass of household members, i.e., population size, is $L(t) = L_0 e^{\lambda t}$, $L_0 \equiv 1$, where $\lambda \gtreqless 0$ so that I allow for zero or even negative population growth. Each household member is endowed with one unit of time and I abstract from labor-leisure choice. Consequently, $L(t)$ is the household's total endowment of labor, which equals its supply of labor. The household has preferences

$$U(0) = \int_0^\infty e^{-(\rho-\lambda)t} \log\left(\frac{C(t)}{L(t)}\right) dt, \quad \rho > \max\{0, \lambda\} \tag{1}$$

where 0 is the point in time when the household makes decisions, ρ is the individual discount rate and $C(t)$ is aggregate consumption.[4] The household's flow budget constraint is

$$\dot{A} = rA + wL - C, \tag{2}$$

where A is assets holding and r is the rate of return on assets. The intertemporal consumption plan that maximizes (1) subject to (2) then consists of the Euler equation

$$r = \rho - \lambda + \dot{C}/C, \tag{3}$$

the budget constraint (2) and the usual boundary conditions.

3.2 Final producers

A competitive representative firm produces a final good Y that can be consumed, used to produce intermediate goods, invested in the improvement of the quality of existing intermediate goods, or invested in the creation of new intermediate goods. The final good is the numeraire so its price is $P_Y \equiv 1$. The production technology is

$$Y = \int_0^N X_i^\theta \left[\left(Z_i^\alpha Z^{1-\alpha}\right)^\kappa \frac{L}{N^{1-\sigma}}\right]^{1-\theta} di, \quad 0 < \theta, \alpha < 1 \tag{4}$$

where N is the mass of non-durable intermediate goods, X_i is the quantity of good i and L is labor. Given the inelastic labor supply of the household and the one-sector structure of the model, labor market clearing yields that employment in the final sector is equal to population size. Quality is the good's ability to raise the productivity of labor: the contribution of good i depends on its own quality, Z_i, and on average quality $Z = \int_0^N (Z_j/N)\, dj$. I show below that the parameters σ and κ measure social returns to variety and quality. I impose no restrictions on σ and κ for now, stressing that the model requires no a priori linearity assumption with respect to N or Z. The first-order

[4] I work out the paper's argument using log-utility because it is much simpler. See the Appendix for a discussion of the specification $\left[(C/L)^{1-\eta} - 1\right]/(1-\eta)$.

conditions for the profit maximization problem of the final producer yield that each intermediate producer faces the demand curve

$$X_i = \left(\frac{\theta}{P_i}\right)^{\frac{1}{1-\theta}} \left(Z_i^{\alpha} Z^{1-\alpha}\right)^{\kappa} \frac{L}{N^{1-\sigma}}, \tag{5}$$

where P_i is the price of good i. Let w denote the wage. The first-order conditions then yield that the final producer pays total compensation

$$\int_0^N P_i X_i di = \theta Y \quad \text{and} \quad wL = (1-\theta) Y \tag{6}$$

to intermediate goods and labor suppliers, respectively.

3.3 Intermediate producers

The typical intermediate firm operates a technology that requires one unit of final output per unit of intermediate good and a fixed operating cost ϕZ_i, also in units of final output.[5] Using the demand schedule (5) we write the firm's gross profit (i.e., profit before R&D) as

$$\Pi_i = (P_i - 1) \left(\frac{\theta}{P_i}\right)^{\frac{1}{1-\theta}} \left(Z_i^{\alpha} Z^{1-\alpha}\right)^{\kappa} \frac{L}{N^{1-\sigma}} - \phi Z_i. \tag{7}$$

This expression suggests the first restriction on κ that we must impose to have a well-defined maximization problem, namely, that the firm's gross profit be concave in Z_i. A necessary and sufficient condition for this to be the case is $\alpha\kappa < 1$. The firm can increase quality according to the technology

$$\dot{Z}_i = I_i, \tag{8}$$

where I_i is firm-level quality-improving R&D investment in units of final output. The firm chooses the time path of its product's price, $P_i(t)$, and its R&D, $I_i(t)$, to maximize

$$V_i(0) = \int_0^\infty e^{-\int_0^t r(s) ds} \left[\Pi_i(t) - I_i(t)\right] dt \tag{9}$$

subject to (8) and (7), where r is the interest rate and 0 is the arbitrary point in time when the firm makes decisions. The firm takes average quality, Z, in (7) as given. The characterization of the firm's decision yields a symmetric equilibrium where

$$r = \alpha\kappa \frac{(P-1) X}{Z} - \phi \tag{10}$$

is the return to quality innovation (see the Appendix for the derivation).

[5] Modeling the fixed cost as $\phi Z_i^{\varphi} Z^{1-\varphi}$, $0 \leq \varphi \leq 1$, complicates the analysis without changing the results.

3.4 Entry

To start up activity an entrant must sink $\beta Y/N$ units of final output.[6] It is useful to write this assumption in terms of a production function for new products/firms, i.e.,

$$\dot{N} = \left(\frac{N}{\beta Y}\right) \cdot E,$$

where E is aggregate variety-expanding R&D investment in units of the final good. Alternatively, one can refer to E as aggregate investment in entrepreneurship. Because of the sunk setup cost, the new firm cannot supply an existing good in Bertrand competition with the incumbent monopolist but introduces a new good that expands product variety. New firms finance entry by issuing equity and enter at the average quality level (this simplifying assumption preserves symmetry of equilibrium at all times). Entry is positive if the value of the firm is equal to its setup cost, i.e., if the free-entry condition $V_i = \beta Y/N$ holds. Taking logs and time derivatives of the free-entry condition and of the valuation equation (9), imposing symmetry, and using the expression for the gross profit (7) yields the return to variety innovation

$$r = \frac{1}{\beta}\left[\frac{(P-1)\,X}{Z} - \phi - \frac{I}{Z}\right]\frac{NZ}{Y} + \frac{\dot{Y}}{Y} - \frac{\dot{N}}{N}. \tag{11}$$

4 Robust endogenous growth

This section discusses the paper's main result. It studies the economy's allocation of final output Y to consumption, production of intermediates, and, when profitable, vertical and horizontal innovation. It then derives the reduced-form representation of the resulting equilibrium dynamics. Finally, it shows that for all values of κ in the admissible range the process converges to a steady state that exhibits constant exponential growth. The admissible range is $\kappa \in [1, 1/\alpha)$ and thus includes a continuum of values of κ, i.e., $\kappa \in (1, 1/\alpha)$, that because of the linearity critique to date have been deemed inadmissible.

4.1 Structure of the equilibrium

Intermediate producers set $P = 1/\theta$ and in symmetric equilibrium receive $N \cdot PX = \theta Y$ from the final producer. Consequently, $NX = \theta^2 Y$. Imposing symmetry in the production function (4) and using this result to eliminate X yields:

$$Y = \theta^{\frac{2\theta}{1-\theta}} \cdot N^{\sigma} Z^{\kappa} \cdot L. \tag{12}$$

This reduced-form production function is not linear in N, not linear in Z and not linear in the combination of the two, $N^{\sigma}Z^{\kappa}$, in the precise sense that $\sigma + \kappa$ is not restricted to be 1.

[6] I borrow this assumption from Barro and Sala-i-Martin (2005), but see Peretto and Connolly (2007) and Peretto (2015) for more general discussions concluding that the same qualitative results obtain assuming βX, βZ, or similar variations on the basic theme that the entry cost is proportional to a firm-level proxy of scale of operation.

The definition of gross profit (7) and equations (10)-(11) show that the returns to vertical and horizontal innovation depend on the gross cash flow of the firm $X(P-1)$ — i.e., revenues minus variable production costs — since this is the appropriate measure of profitability for firms that spread fixed costs, including the cost of developing innovations, over their volume of sales. On closer inspection, moreover, one can see that both returns are functions of the quality-adjusted gross cash flow of the firm, $X(P-1)/Z$. It is thus useful to define

$$x \equiv \frac{X(P-1)}{Z} = \frac{\text{gross cash flow}}{\text{quality}}$$

and use it as the model's stationary state variable.

Recall now that $P = 1/\theta$ and $NX = \theta^2 Y$. Equation (12) then yields

$$x = (1-\theta)\frac{\theta Y}{NZ} = (1-\theta)\,\theta^{\frac{1+\theta}{1-\theta}}\frac{Z^{\kappa-1}L}{N^{1-\sigma}}. \tag{13}$$

I assume $\sigma < 1$ to ensure that the *market share effect* in the intermediate goods market — N at the denominator of the ratio $\theta Y/NZ$ — dominates over the love-of-variety effect in final production so that the mass of firms ends up at the denominator of the expression for the the quality-adjusted gross cash flow of the firm.[7] Substituting the expression in (10) and (11) the returns to innovation become:

$$r = \alpha\kappa x - \phi; \tag{14}$$

$$r = \frac{1}{\pi}\left(1 - \frac{\phi + z}{x}\right) + \frac{\dot{x}}{x} + z, \tag{15}$$

where to simplify notation I define

$$\pi \equiv \frac{\beta Y/N}{X(P-1)} = \frac{\beta}{\theta(1-\theta)} = \frac{\text{entry cost}}{\text{gross cash flow}}.$$

Expressions (14)-(15) capture the model's main property: firm-level decisions depend on the quality-adjusted gross cash flow, which is proportional to the firm's sales and thus increasing in labor use in the downstream final sector since production of final goods drives the demand for intermediate goods. It should be clear, thus, that from the viewpoint of the managers of incumbent firms and

[7] For $1 \leq \sigma$ entry is self-sustaining and the model degenerates to Romer-style growth. How to interpret this outcome? The general model with both vertical and horizontal innovation nests variety-driven endogenous growth as a special case. The key is that for $1 < \sigma$ variety is so productive that each new good expands the market by more than it is needed to support itself profitably and thereby fuels further entry. This property is captured by the fact that the mass of firms/products, N, ends up at the numerator of the firm's profitability, x, signaling that each firm/product is actually making all the others *more* profitable. In this case, the model reverts back to the standard two-sector interpretation of the theory that assigns no special role to endogenous market structure. In fact, under $1 \leq \sigma$ the incentives to entry become so strong that vertical innovation does not take place — this is the sense in which the model degenerates to the standard variety-expansion story. A crucial aspect of such degeneration is that, despite the fact that the model so built would not exhibit the scale effect because the entry cost is scaled up by Y (see Barro and Sala-i-Martin 2004, pp. 300-302, on this point), it would nevertheless require the knife-edge condition $1 = \sigma$ in order to prevent the explosive behavior due to the fact that the current mass of firms/products, N, creates the profit opportunity that attracts further entry.

of the entrepreneurs that set up new firms the critical market size variable is quality-adjusted total expenditure on intermediates, $\theta Y/Z$. This structure yields a transparent characterization of how the state of the economy drives incentives for quality and variety innovation and thus it allows me to study the conditions under which a steady state with positive, non-explosive growth exists and whether such steady state is stable. Before proceeding further, however, it is useful to state more precisely the paper's research question.

4.2 The problem: growth on a knife edge or on a wide highway?

If we think of the model's steady state as a situation with unchanging incentives, in this setup this means looking for solutions with constant rate of return to innovation and, therefore, for solutions with constant quality-adjusted gross cash flow, i.e., constant x; see equation (14). The economics of the model then says that the quality-adjusted gross cash flow is constant if the mass of firms, N, grows at the same rate as the quality-adjusted size of the market for intermediate goods, $\theta Y/Z$. That is, *if entry sterilizes the quality-adjusted market size effect*. It follows that the mass of firms absorbs any upward pressure on market size that makes it grow "in excess" of average quality, Z.

It is useful to recall, before studying the mechanism in detail, that thus far the analysis has required only two restrictions:

- $\kappa < 1/\alpha$ to ensure that the firm's profit flow is concave in individual firm quality, Z_i, and therefore the symmetric industry equilibrium well defined;
- $\sigma < 1$ to ensure that the market share effect dominates over the love-of-variety effect so that in equilibrium the firm's the quality-adjusted gross cash flow x_i is decreasing in the mass of firms/products, N.

The second restriction combined with the existence of fixed operating costs implies that *by construction* product variety is not the engine of growth of this economy. The discussion, therefore, concerns the social returns to quality. With this framework in mind, I ask the following two questions about the degree of social returns to quality, κ.

- What is the lower bound on κ that ensures non-decreasing growth?
- What is the upper bound on κ that ensures non-explosive growth?

The first issue is well understood; the second is at the heart of the linearity critique.

The lower bound on κ follows from the straightforward observation that *the return to the accumulation of the growth-driving factor must be non-decreasing in the factor*. In the context of this paper this means that $r = \alpha\kappa x - \phi$ must be non-decreasing in Z, that is x must be non-decreasing in Z. The upper bound on κ follows from similar reasoning: *the return to the accumulation of the growth-driving factor must be non-increasing in the factor*. This means that $r = \alpha\kappa x - \phi$ must be non-increasing in Z, that is, x must be non-increasing in Z.

The key to the paper's argument, therefore, is equation (13), which describes how the quality-adjusted gross cash flow responds to variety, average quality and the other determinants of market size, namely, employment, L. Looking at the equation, two things stand out. First, x is non-decreasing in Z for $\kappa \geq 1$. Hence, $\kappa = 1$ is the lower bound on κ that gives the non-decreasing property because it makes x *independent* of Z. Imposing $\kappa = 1$ to ensure that constant growth is feasible in steady state, several contributions show that in general equilibrium the endogeneity of the mass of intermediate goods sterilizes the effect of employment, so that x is constant in steady state.[8] However, second, $\kappa = 1$ is *not necessary* since the same market fragmentation mechanism that stabilizes x under $\kappa = 1$ can stabilize it under $\kappa > 1$. If the mass of firms can absorb changes in employment, L, then it can also absorb changes in $Z^{\kappa-1}$.

To see this more precisely, let

$$z \equiv \frac{\dot{Z}}{Z} \quad \text{and} \quad n \equiv \frac{\dot{N}}{N}$$

be, respectively, the rates of vertical and horizontal innovation. In equilibrium, these two rates can be written as two functions, $z(x)$ and $n(x)$, that have properties, that I characterize in detail below, dictated by the no-arbitrage condition that variety and quality innovation deliver the same rate of return. The quality-adjusted gross cash flow, x, then obeys the differential equation

$$\frac{\dot{x}}{x} = \Psi(x) \equiv \underbrace{\lambda(\kappa - 1) z(x)}_{\text{market growth}} - \underbrace{(1-\sigma) n(x)}_{\text{market fragmentation}}, \tag{16}$$

A steady state with x constant exists if the equation

$$\Psi(x) = 0 \Rightarrow \lambda + (\kappa - 1) z(x) - (1-\sigma) n(x) = 0$$

has solutions, a property that *does not require* $\kappa = 1$.

To appreciate the mechanism at work here, it is useful to stress that thus far we have been dealing with a variable per firm, not per capita. The difference in denominators is crucial. At any point in time, the growth rate of output per capita is:

$$\frac{\dot{Y}}{Y} - \lambda = \sigma n(x) + \kappa z(x). \tag{17}$$

Comparing (16) to (17) one sees that while the quality-adjusted gross cash flow growth falls with the rate of entry, n, output per capita growth rises with the rate of entry. This difference is due to the $\sigma < 1$ restriction that makes the market share effect dominate the love-of-variety effect in the determination of the firm's quality-adjusted gross cash flow. In other words, product variety expansion is good for output growth, and thus for output per capita growth, whereas, on net, it is bad for the growth of the firm's quality-adjusted gross cash flow. Quality growth, in contrast, is good for both output per capita growth and for per firm profit growth.

[8] The list of such contributions is by now rather long and would include too many self-citations to list here. Let me instead direct the interested reader to my web page: http://public.econ.duke.edu/~peretto/

4.3 The main result

A key step in the characterization of the model's equilibrium is the construction of the two functions mentioned above, $z(x)$ and $n(x)$, that describe the rates of vertical and horizontal innovation. The properties of these functions are dictated by the no-arbitrage condition that variety and quality innovation deliver the same rate of return. Specifically, the functions are built by comparing the rates of return to vertical and horizontal innovation, and checking whether they can be equal or one of the two activities is return-dominated and shuts down. Moreover, the two rates of return must be compared to the reservation rate of return demanded by households. The resulting no-arbitrage conditions yield thresholds of the quality-adjusted gross cash flow, x_N and x_Z, that, respectively, identify regions of the state space $x \in (\phi, \infty)$ where investment in variety innovation and/or investment in quality innovation are zero. Depending on the ordering of the thresholds, there exist two possible transition paths: one where the economy activates variety innovation first and one where it activates quality innovation first. Either way, the economy — under conditions that I shall discuss in a moment — converges to the right-most region of state space, $x > \max\{x_N, x_Z\}$, where both variety and quality innovation take place. In that region it is possible to have the steady state with constant, exponential growth.

I begin with the case in which the economy activates variety innovation first. The expressions for the two functions are as follows.

Proposition 1 *In the case $x_N < x_Z$ the rates of variety and quality innovation are:*

$$z(x) = \begin{cases} 0 & \phi \leq x \leq x_N \\ 0 & x_N < x \leq x_Z \\ \frac{\alpha\kappa x - \phi - \sigma\frac{x-\phi}{\pi x} - [\rho - \sigma(\rho-\lambda)]}{\kappa - \frac{\sigma}{\pi x}} & x_z < x < \infty \end{cases};$$

$$n(x) = \begin{cases} 0 & \phi \leq x \leq x_N \\ \frac{x-\phi}{\pi x} - \rho + \lambda & x_N < x \leq x_Z \\ \frac{(1-\alpha)\kappa x - (\kappa-1)\phi + [\rho - \sigma(\rho-\lambda)]}{\kappa\pi x - \sigma} - \rho + \lambda & x_z < x < \infty \end{cases},$$

where:

$$x_N \equiv \frac{\phi}{1 - \pi(\rho - \lambda)};$$

$$x_Z \equiv \arg\text{solve}\left\{\alpha\kappa x - \phi - \sigma\frac{x-\phi}{\pi x} = \rho - \sigma(\rho - \lambda)\right\}.$$

Proof. See the Appendix. ∎

Figure 1 illustrates the properties of the two functions. $z(x)$ is initially zero, becomes positive at x_Z, leaving that point with positive derivative, can be either concave or convex, and becomes asymptotically linear. $n(x)$ is initially zero, becomes positive at x_N, leaving that point with positive derivative, is always concave and converges from below to a finite upper bound.

When the economy activates quality innovation first things are a bit different. The expressions for the two functions are as follows.

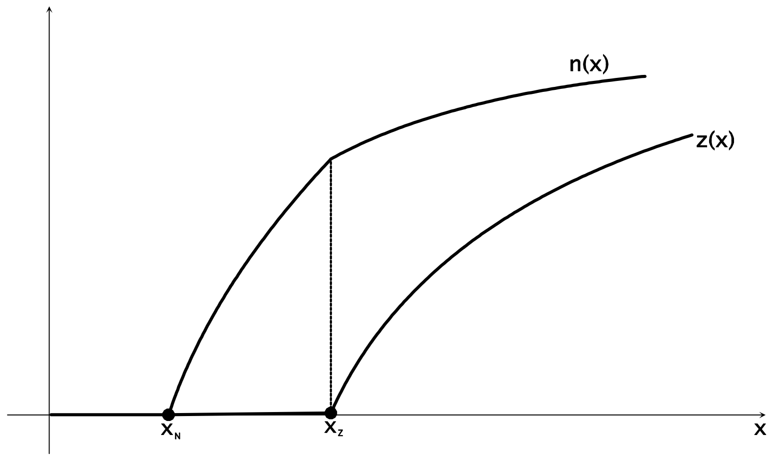

Figure 1: Innovation rates as functions of the quality-adjusted gross cash flow in the case $x_N < x_Z$.

Proposition 2 *In the case $x_Z < x_N$ the rates of variety and quality innovation are:*

$$z(x) = \begin{cases} 0 & \phi \leq x \leq x_Z \\ \tilde{z}(x) & x_Z < x \leq x_N \\ \frac{\alpha\kappa x - \phi - \sigma\frac{x-\phi}{\pi x} - [\rho - \sigma(\rho-\lambda)]}{\kappa - \frac{\sigma}{\pi x}} & x_z < x < \infty \end{cases};$$

$$n(x) = \begin{cases} 0 & \phi \leq x \leq x_Z \\ 0 & x_Z < x \leq x_N \\ \frac{(1-\alpha)\kappa x - (\kappa-1)\phi + [\rho - \sigma(\rho-\lambda)]}{\kappa\pi x - \sigma} - \rho + \lambda & x_N < x < \infty \end{cases},$$

where:

$$\tilde{z}(x) = x\left[1 - \frac{1}{\theta}\left(\frac{\tilde{c}(x)}{1-\theta} - 1\right)\right] - \phi$$

and $\tilde{c}(x)$ is the solution of the partial differential equation

$$\frac{dc}{dx} = \frac{\kappa c}{x} \frac{\frac{x}{\theta}\left(\frac{c}{1-\theta} - 1\right) - \frac{\rho}{\kappa} - (1-\alpha)x}{\lambda + (\kappa-1)\left[x - \phi - \frac{x}{\theta}\left(\frac{c}{1-\theta} - 1\right)\right]}.$$

The thresholds are:

$$x_N \equiv \arg\text{solve}\left\{\frac{(1-\alpha)\kappa x - (\kappa-1)\phi + [\rho - \sigma(\rho-\lambda)]}{\kappa\pi x - \sigma} = \rho - \lambda\right\};$$

$$x_Z \equiv \arg\text{solve}\left\{x\left[1 - \frac{1}{\theta}\left(\frac{\tilde{c}(x)}{1-\theta} - 1\right)\right] = \phi\right\}.$$

The function $z(x)$ has zero derivative at $x = x_Z$, is increasing and has positive derivative at x_N.

Proof. See the Appendix. ■

Figure 2 illustrates the properties of the two functions. $z(x)$ becomes positive at x_Z leaving that point with zero derivative, is convex up to x_N and then has the same properties as in the

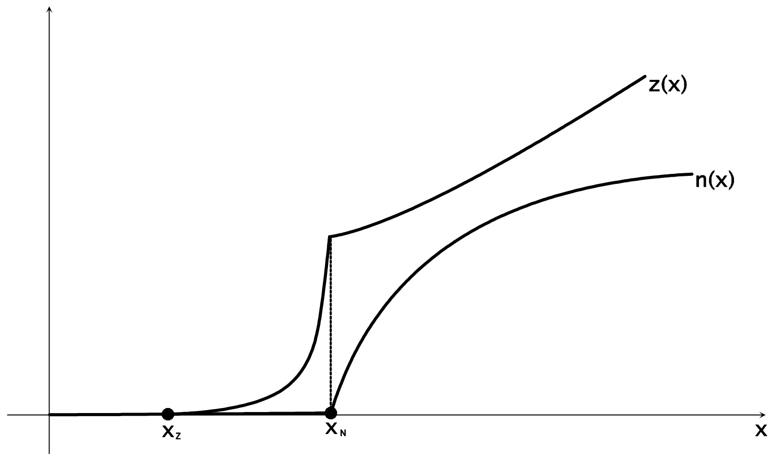

Figure 2: Innovation rates as functions of the quality-adjusted gross cash flow in the case $x_Z < x_N$.

previous case. $n(x)$ becomes positive at x_N and has the same properties as in the previous case, i.e., is increasing, concave and bounded above.

The following proposition states the main result of the paper.

Proposition 3 *The equation $\Psi(x) = 0$ has two real roots x^* and $\bar{x}$, with $x^* < \bar{x}$, for any $\kappa \in [1, 1/\alpha)$ if*

$$\frac{\rho - \sigma(\rho - \lambda)}{\phi} > \left(\frac{1}{\alpha} - 1\right). \tag{18}$$

The roots x^* and $\bar{x}$ identify two steady states with constant, exponential, endogenous growth, one stable, x^*, and the other unstable, $\bar{x}$. For initial condition $x(0) \in (\phi, \bar{x})$, the steady state x^* is the attractor of the economy's dynamics if

$$\frac{1 - \sigma}{\rho - \sigma(\rho - \lambda)} \leq \pi. \tag{19}$$

For initial condition $x(0) > \bar{x}$ the economy's dynamics is explosive.

Proof. See the Appendix. ■

The result is best understood qualitatively; see the proof of the proposition for the algebraic details. First, recall that we are characterizing the dynamical system

$$\frac{\dot{x}}{x} = \Psi(x) = \lambda + (\kappa - 1) z(x) - (1 - \sigma) n(x),$$

where the key ingredients are the two functions $z(x)$ and $n(x)$ discussed above. Figures 3-4 illustrate the dynamics. Consider the case $x_N < x_Z$ in which the economy activates variety innovation first. For $x \leq x_N < x_Z$, we have $z(x) = n(x) = 0$ because both returns are too low. Accordingly, the growth rate of the quality-adjusted gross cash flow is $\dot{x}/x = \lambda > 0$ and the economy crosses

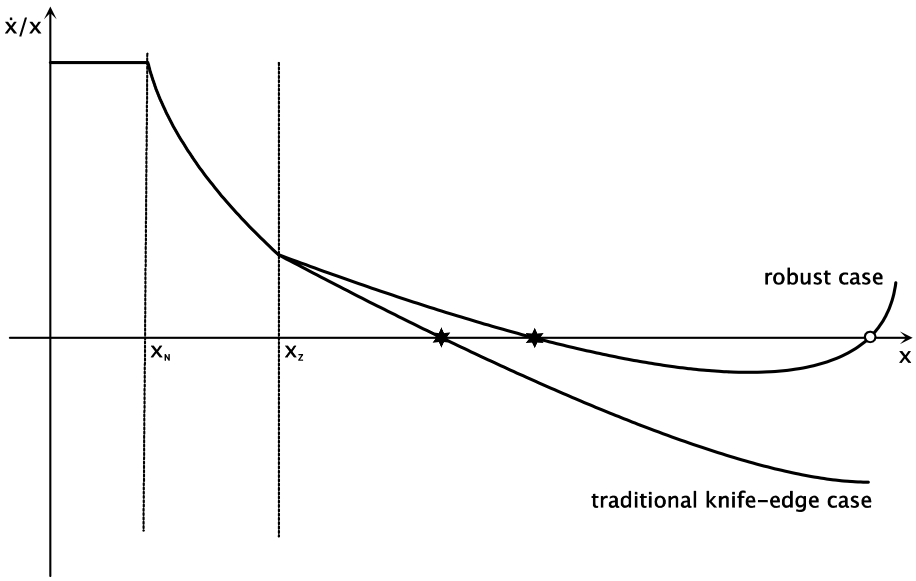

Figure 3: Dynamics when $x_N < x_Z$ in the traditional case $\kappa = 1$ and in the robust case $\kappa > 1$. The stars denote the stable steady states; the hollow circle the unstable one.

the threshold for entry in finite time. For $x_N < x < x_Z$, $z(x) = 0$ and the growth rate of the quality-adjusted gross cash flow is

$$\frac{\dot{x}}{x} = \lambda - (1-\sigma)\, n(x).$$

This is a converging process. The system, therefore, crosses the threshold for quality innovation in finite time only if $\dot{x}/x = \lambda - (1-\sigma)\, n(x_Z) > 0$, that is, if the quality-adjusted gross cash flow is still growing at $x = x_Z$. The proof of the proposition shows that to obtain this outcome it is *sufficient* to assume that condition (19) holds. The intuition is that product proliferation is not so extreme that the quality-adjusted gross cash flow stops growing before the activation of vertical innovation. In other words, the economy avoids premature market saturation. The case $x_N > x_Z$, in which the economy activates quality innovation first, is rather different. Instead of a deceleration of the rate of growth of the quality-adjusted gross cash flow, at $x = x_Z$ we have an acceleration and therefore we don't need any restriction to ensure that the economy crosses the threshold x_N in finite time. Only for $x > x_N$ the quality-adjusted gross cash flow growth starts slowing down and converging to zero.

For comparison with the existing literature, it is instructive to look at

$$\begin{aligned}\lim_{x\to\infty} \Psi(x) &= \lim_{x\to\infty} \left[\lambda + (\kappa-1)\, z(x) - (1-\sigma)\, n(x)\right] \\ &= \lambda + \lim_{x\to\infty} (\kappa-1)\, z(x) - \lim_{x\to\infty} (1-\sigma)\, n(x).\end{aligned}$$

The proof of the proposition shows that as $x \to \infty$, $n(x) \to \frac{1-\alpha-\pi(\rho-\lambda)}{\pi}$ while $z(x) \to \alpha x$. Hence, the restriction for the return to innovation to be non-decreasing in Z, $\kappa \geq 1$, implies that the

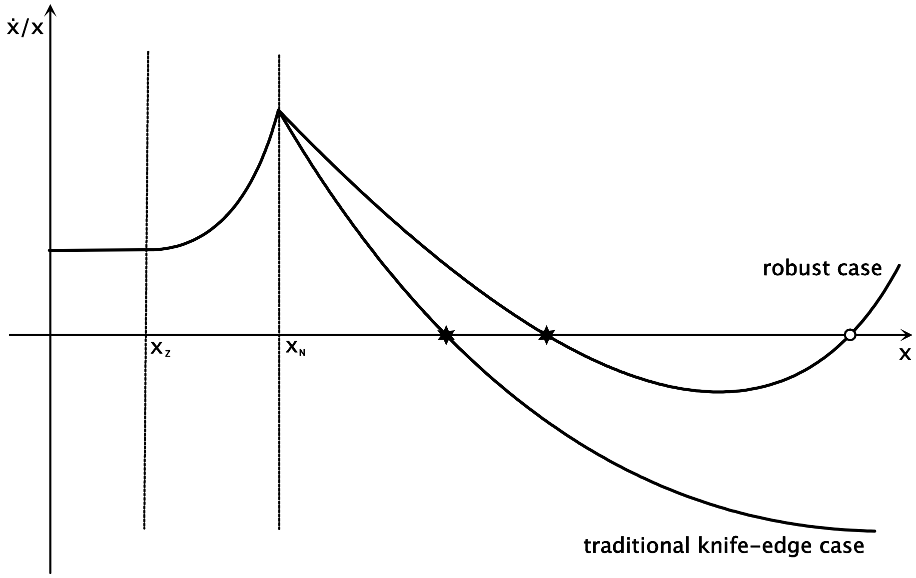

Figure 4: Dynamics when $x_Z < x_N$ in the traditional case $\kappa = 1$ and in the robust case $\kappa > 1$. The stars denote the stable steady states; the hollow circle the unstable one.

quality-adjusted gross cash flow explodes to infinity unless one imposes the knife-edge condition $\kappa = 1$ because doing so kills the term $z(x)$. This then allows one to ensure that a solution $\Psi(x) = 0$ exists by simply assuming

$$\lambda - \lim_{x \to \infty} (1 - \sigma) n(x) = \lambda - \frac{1 - \alpha - \pi(\rho - \lambda)}{\pi} < 0,$$

which yields $\lim_{x \to \infty} \Psi(x) < 0$. The literature has thus far considered only this case: linearity of production in the growth-driving factor, Z in this case, to ensure that constant steady-state growth is both (1) technologically feasible and (2) non-explosive, and restrictions on the other parameters to ensure that, once agents' behavior is taken into account, the steady state with constant growth exists as a market equilibrium and is the attractor of the associated dynamical system. Feasibility in this context means that the economy exhibits non-decreasing returns to Z.

This approach, however, exploits a *sufficient* condition, which, as such, can be relaxed. So, what happens if $\kappa > 1$? Proposition 3 states that there are two sufficient conditions that yield the following outcome:

- the economy converges to the steady state with both quality and variety innovation;
- such steady state exhibits constant, exponential growth of income per capita for any $\kappa \in [1, 1/\alpha)$, that is, for all values of κ in the admissible range.

The proof shows that the equation $\Psi(x) = 0$ yields the quadratic form $a_1 x^2 + a_2 x - a_3 = 0$, with coefficients that are functions of the primitive parameters of the model, including κ. The condition for existence of the stable steady state with constant exponential growth is then that two real roots exist in the interval $x \in (x_Z, +\infty)$. To check when this is the case, the proof looks for values of κ in the admissible range $[1, 1/\alpha)$ such that $\Delta(\kappa) \equiv (a_2(\kappa))^2 + 4a_1(\kappa)\,a_3(\kappa) > 0$. The exercise reveals that condition (18) is sufficient for this to be true for all $\kappa \in [1, 1/\alpha)$.

The intuition behind this result — namely, that non-explosive endogenous growth does not require exact linearity of the production structure — is as follows. The elasticity of output with respect to labor, 1, is the lower bound below which social returns to quality κ cannot fall without making steady-state constant endogenous growth impossible. But why is $\kappa > 1$ possible? The answer is that $\kappa > 1$ yields that quality growth fuels market growth "in excess" of each firm's quality growth but that such "excess" market growth is absorbed by entry. If social returns to quality are larger than what is needed to support constant exponential growth, the "excess" growth in market size is absorbed by entry.

To summarize, Proposition 3 says that $\kappa > 1$ is feasible because there is a thick region of parameters' space where the market fragmentation mechanism "tames" potentially explosive growth due to increasing returns that look excessive in terms of the traditional theory. To date, such increasing returns have been deemed impossible by a priori reasoning that extrapolates the properties of one-dimensional models. Such reasoning does not apply to models with two dimensions of technology that play interdependent, but distinct, roles: the vertical dimension provides the engine of growth, the horizontal dimension provides the endogenous market structure that *sterilizes the market size effect* and thereby its manifestations — the (strong) scale effect and the need to impose a knife-edge condition that rules out explosive behavior under "excessive" increasing returns.

4.4 The insight: further thoughts

The sufficient condition for stability in Proposition 3 says that the economy enters the right-most region of the phase diagram in finite time when $\pi \equiv \frac{\beta}{\theta(1-\theta)} = \frac{\beta Y/N}{(P-1)X}$ is sufficiently high, that is, when entry is sufficiently costly relative to the firm's anticipated gross cash flow. Recall that this condition is needed only to rule out premature market saturation in the variety-first case. Similarly, the sufficient condition for existence says that non-explosive growth obtains when ϕ is not too high. Why? Because large fixed operating costs tend to generate large firms that wish to invest heavily in quality growth. This tends to shift upward the function $\Psi(x)$ and can result in it being everywhere above the horizontal axis. To summarize, the sufficient conditions for models of this class to produce non-explosive endogenous growth are that (1) there is no incentive to saturate the market with excessive product proliferation and (2) there is no incentive for firms to be excessively large and grow too fast through in-house R&D.

A further implication of this mechanism is that steady-state entry is possible even when population growth is zero or even *negative*, i.e., $\lambda \leq 0$.[9] To see this note that in steady state

$$n(x) = \frac{1}{1-\sigma} \cdot [\lambda + (\kappa - 1)\, z(x)],$$

[9] To avoid possible confusion, let me stress that this observation applies to the region $x > \max\{x_N, x_Z\}$. I have already argued that to get there population growth might be necessary.

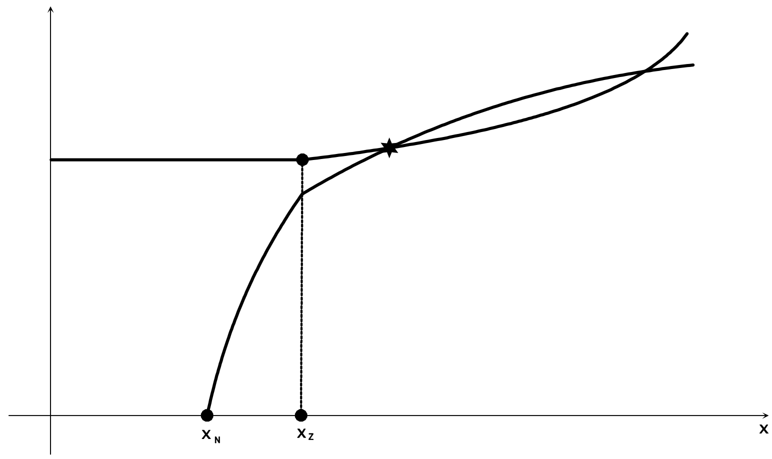

Figure 5: Solution of $\Psi = 0$ in the case $x_N < x_Z$.

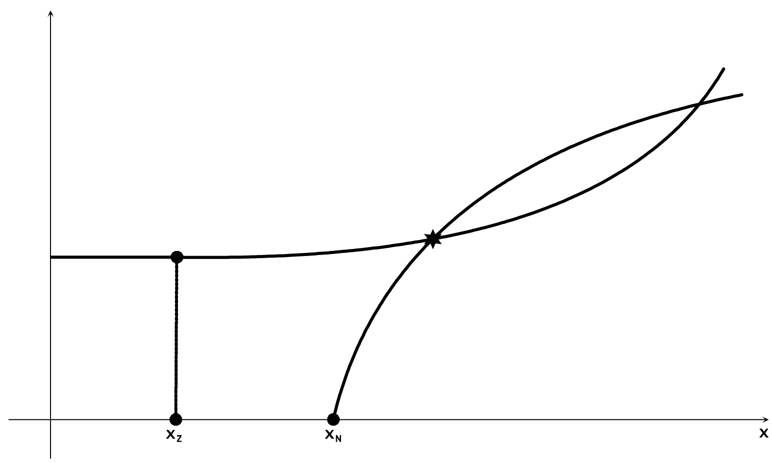

Figure 6: Solution of $\Psi = 0$ in the case $x_Z < x_N$.

which says that entry is positive as long as quality growth compensates for the potentially negative net contribution of physical factors to market growth, the term λ. In fact, this result says that quality growth *causes* variety expansion in that it creates the profit opportunity for entrants. Figures 5-6 show the solution of this equation and complement the dynamic analysis of Figures 3-4. In the case $\lambda = 0$, we have

$$n(x) = \frac{\kappa - 1}{1 - \sigma} \cdot z(x),$$

that is, variety expansion is proportional to quality growth.

It is also useful to stress that in this structure there is no add-up-to-one restriction on the parameters regulating returns to scale, as in standard two-sector models, because this is not a two-sector model. As stated, the key to the insight is that in the determination of the return to innovation quality and variety end up at the opposite ends of the ratio $\theta Y/NZ$ that drives the dynamics of the quality-adjusted gross cash flow, x. Why is this difference so important? Because in a standard two-sector model the quantity of one factor, say human capital, *raises* the marginal product of the other factor, say physical capital. Therefore, in such a view of the world the factors can only reinforce each other in the determination of the returns to investment so that to have non-decreasing, non-explosive growth the overall production structure must have the well-known property that the degrees of social returns in the two sectors add up to one. In the Schumpeterian view of the world, in contrast, the mass of firms/products is not just another factor of production but the measure of endogenous market structure that *lowers* the quality-adjusted gross cash flow of the firm through the dominant market share effect. Consequently, more product variety, while it raises final output, *lowers* the return to vertical (in-house) innovation.

5 Interpretation: Where are the linearities?

It is often claimed that endogenous growth models are linear. To my knowledge, the clearest articulation of this view is Jones (2005), who asserts that

> "... all growth models that exhibit steady-state growth ultimately rest on an assumption that some differential equation takes the form
>
> $$\dot{X} = \_X. \tag{20}$$
>
> Growth models differ primarily according to the way in which they label the X variable and the way in which they fill in the blank in this differential equation" (Jones 2005, p. 1103).

The primitives that I use in deriving the paper's results do not exhibit such property, i.e., they are not linear in this sense. One can thus ask: How is it possible to obtain endogenous growth without *assuming* the necessary functional form?

5.1 Answering the question: first-principles

I italicized the word "assuming" in the question above because it is at the heart of the linearity critique, as per the quote, and its relevance to this paper's contribution. In short, the answer to the question is that the model developed in this paper *does not assume* the required form but it *produces it endogenously.* It pays to be specific here.

One reading of the linearity critique, as worded in the quote above, is that we need not worry about what the blank is and where it comes from, but we can focus exclusively on the fact that, whatever it is, it must be independent of calendar time t and, crucially, independent of X itself otherwise the whole argument unravels immediately because the differential equation would be, in fact, non-linear in X.[10] Henceforth, I shall use the word "constant" to refer to a model-specific

[10] The critique as worded in the quote conflates the two implicit claims on the blank — independent of X and independent of calendar time t — because it is an attempt at summarizing a very special class of models, namely,

filler of the blank with this property, i.e., time-invariant and independent of the accumulated endogenous variable. On this reading, the linearity critique boils down to asserting in fancier words that endogenous growth models exhibit the mathematical property that the variable X grows at a constant exponential rate in steady state. It is not clear what emphasizing this self-evident fact teaches us about the real-world growth process or growth theory.

I thus argue that the critique has meaning only as it relates to the blank and that the discussion must be about, and only about, the mechanism that in the context of a specific model delivers the property that the blank is constant (as defined above). It is only in this very specific sense that one can say that equation (20) represents a core element of the model. If, for example, the blank stands for the economy's intensity of investment in the growth of X, then the critique's point is that such intensity must be independent of X and of calendar time t. To determine whether this is the case requires working out how the economy determines such intensity, i.e., it requires to actually solve the model. If this process of unpacking the blank reveals that the model indeed *assumes* that $\dot{X}$ is proportional to X, with coefficient of proportionality independent of calendar time t, by postulating such property as a primitive, then the critique, as formulated in the quote, has bite. If not, the critique reduces to the tautological claim that models of exponential growth processes work with the mathematical representation of exponential growth processes.

In what follows, I shall therefore interpret the model developed in this paper in terms of the second reading of the linearity critique and focus on the model's mapping from the primitives to the state-space representation of the equilibrium. Also, to dispel any possible misunderstanding, let me state that, *as per the quote*, everything I said in the two paragraphs above refers to the model's steady state.

With these stipulations in place, let us now turn to this paper's model and restrict attention to average product quality, Z, since it is the dimension in which the model exhibits endogenous growth. In equilibrium agents invest according to a function $I(\cdot)$ that *over time* converges to the form

$$I = Z \cdot (\text{constant})$$

so that in steady state, and *only in steady state*,

$$\dot{Z} = I = Z \cdot (\text{constant}) .$$

The goal of this section is to develop this argument in detail. The main step is the characterization of the properties of the function $I(\cdot)$, namely, (1) the identification of what, exactly, are the arguments inside it and (2) the determination of its shape. The crucial claim is going to be that $I(\cdot)$ is, among other things, a function of Z itself so that the model's core equation violates equation (20). The exception is the steady state where the function takes the form mathematically necessary to be describing a constant growth rate.

those whose entire economic structure can be reduced to the accumulation of a single stock. It is self-evident from the equation that to be independent of calendar time t the blank must be independent of the variable X since X is tasked with growing exponentially over time.

Before I develop the argument, it is useful to note that with the focus of the discussion on what properties delivers endogenous growth, one might think that the variety dimension of technology, N, is not subject to the critique discussed above because product variety is not tasked with supporting endogenous growth — in fact, it is tasked with "taming" explosive growth. Nevertheless, the mechanism that yields the properties of the $\dot{Z}$ equation applies to the $\dot{N}$ equation as well, as I show below. The reason is that N exhibits the semi-endogenous growth property and thus *must* have the steady-state mathematical representation $\dot{N} = (\text{constant}) \cdot N$ for the same reason why average quality *must* have the steady-state mathematical representation $\dot{Z} = (\text{constant}) \cdot Z$.

5.2 Answering the question: details

We start from the model's key primitives written in the following terms:

$$\dot{N} = \frac{N}{\beta Y} \cdot E; \tag{21}$$

$$\dot{Z}_i = I_i. \tag{22}$$

In this notation, E is aggregate investment in entry and I_i is investment in quality by firm i. To ease the exposition, henceforth I focus on the symmetric equilibrium and drop the subscript i in the firm-level equation. Now recall that this is a one-sector model, that is, the terms E and I are interpreted as composite inputs produced with the same technology as that for the final good Y. In other words, the model is understood as taking a shortcut with respect to the much more cumbersome version that one could write, where on the right-hand side of equations (21) and (22) one specifies functions of the quantities of intermediates and labor allocated to, respectively, horizontal and vertical innovation. Such functions would have the same mathematical form as the right-hand side of equation (4). Therefore, the right-hand side of the *primitives* (21) and (22) is non-linear in N and non-linear in Z.

It is now useful to rewrite equations (21) and (22) as:

$$\dot{N} = N \cdot \frac{E}{\beta Y};$$

$$\dot{Z} = Z \cdot \frac{I}{Z}.$$

On the right-hand side we now have two measures of investment intensity, $E/\beta Y$ and I/Z. Recall that in solving the model I defined

$$n \equiv \frac{\dot{N}}{N} = \frac{E}{\beta Y} \quad \text{and} \quad z \equiv \frac{\dot{Z}}{Z} = \frac{I}{Z}$$

and then obtained from the equilibrium conditions expressions that give these two growth rates — or, equivalently, the investment intensity in variety and quality, $E/\beta Y$ and I/Z — as functions of the state variable x; see Propositions 1 and 2. For notational convenience I called such functions $n(x)$ and $z(x)$, where, from its definition and the model's equilibrium conditions,

$$x = \frac{(P-1)X}{Z} = \frac{P-1}{P}\frac{PX}{Z} = \frac{P-1}{P}\frac{\theta Y}{NZ} = (1-\theta)\,\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}, \quad \sigma < 1 \leq \kappa. \tag{23}$$

Recall that x is the quality-adjusted gross cash flow of the firm and that using it as my state variable is suggested by the equations characterizing behavior, since they say that the returns to vertical and horizontal innovation are functions of the quality-adjusted gross profit earned by the typical firm in symmetric equilibrium. For the purposes of this discussion, it is insightful to summarize this step of the argument as saying that we can write x as a function $x(N, Z, L)$, which, clearly, is non-linear in Z, non-linear in N and non-linear in the combination of the two in the precise sense that the exponents $\kappa - 1$ and $1 - \sigma$ do not add up to one.

With this notation in hand, we can write:

$$\dot{N} = N \cdot n(x), \quad x = x(N, Z, L); \tag{24}$$

$$\dot{Z} = Z \cdot z(x), \quad x = x(N, Z, L). \tag{25}$$

At the most basic level, therefore, addressing the "missing linearities question" reduces to pointing out that the terms $n(x)$ and $z(x)$ in these expressions are: (1) the blanks that the linearity critique talks about (see equation (20) above); (2) functions of the state vector (N, Z, L); (3) constant in steady state. It is more interesting, however, to develop the argument in a form that brings to the forefront the role played by agents' behavior and the model's equilibrium.

First, note that comparing expressions (24)-(25) to the primitives (21)-(22) highlights that the model works with:

$$E = \beta Y \cdot n(x) \Leftrightarrow n(x) = \frac{E}{\beta Y};$$

$$I = Z \cdot z(x) \Leftrightarrow z(x) = \frac{I}{Z}.$$

In other words, the equilibrium objects $n(x)$ and $z(x)$ represent the investment decisions expressed in terms familiar to most readers. Next, note that the terms $E/\beta Y$ and I/Z that I am interpreting as measures of investment intensity can be mapped into, respectively, the share of output allocated to entry and the firm-level R&D/sales ratio. Specifically, we can define $s_N \equiv E/Y$ and $s_Z \equiv I/PX$ and write:

$$\dot{N} = N \cdot \frac{1}{\beta} \cdot s_N;$$

$$\dot{Z} = Z \cdot \frac{PX}{Z} \cdot s_Z.$$

We now recall the definition $x \equiv \frac{(P-1)X}{Z}$ and write

$$\dot{N} = N \cdot \frac{1}{\beta} \cdot s_N;$$

$$\dot{Z} = Z \cdot \frac{Px}{P-1} \cdot s_Z, \quad P = \frac{1}{\theta}.$$

The mapping between what I have done in the paper and this representation is the following:

$$s_N(x) \equiv \beta \cdot n(x);$$

$$s_Z(x) \equiv \frac{1-\theta}{x} \cdot z(x).$$

Accordingly, the new terms s_N and s_Z are not arbitrary objects but are specific functions of the state variable x constructed through manipulation of the model's equilibrium conditions.[11] Determining which representation is better is purely a matter of expositional convenience and idiosyncratic taste. What I did here is simply decompose the core objects constructed earlier, the functions $n(x)$ and $z(x)$, in terms that translate into the representation that many readers might be more accustomed to. Neither the economics nor the math have changed.

5.3 Answering the question: the punchline

Now refer back to equations (21)-(22) with their right-hand sides interpreted according to the observations just made, i.e., as measures of investment intensity. Simply looking at the graphical representation of the functions $n(x)$ and $z(x)$ makes the point that we are dealing with non-linear functions. Hence, we only need to highlight that (1) x converges to a constant value x^* and (2) x is a function of the model's state vector (N, Z, L). Since this is the punchline of this section, let me reproduce here the key string of relations that I have used above,

$$x = \frac{(P-1)X}{Z} = \frac{P-1}{P}\frac{PX}{Z} = \frac{P-1}{P}\frac{\theta Y}{NZ} = (1-\theta)\,\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}, \quad \sigma < 1 \leq \kappa.$$

The full system that we are working with is:

$$\dot{N} = N \cdot \frac{1}{\beta} \cdot s_N(x) = N \cdot \frac{1}{\beta} \cdot s_N\left((1-\theta)\,\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}\right);$$

$$\dot{Z} = Z \cdot \frac{x}{1-\theta} \cdot s_Z(x) = Z \cdot \left(\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}\right) \cdot s_Z\left((1-\theta)\,\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}\right);$$

$$\dot{L} = L \cdot \lambda.$$

Section 4 of the paper simply uses the more compact notation:

$$\dot{N} = N \cdot n(x) = N \cdot n\left((1-\theta)\,\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}\right);$$

$$\dot{Z} = Z \cdot z(x) = Z \cdot z\left((1-\theta)\,\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}\right);$$

$$\dot{L} = L \cdot \lambda.$$

The two representations are not just equivalent but two ways to say the same thing.

The structure of the model's equilibrium system is thus as follows. The growth rates of variety, $\dot{N}/N$, and average quality, $\dot{Z}/Z$, are given by two functions $n(x)$ and $z(x)$ that are non-linear in x, which is itself non-linear in Z, non-linear in N and non-linear in the combination of the two in the

[11] The interested reader might want to notice that $s_Z = I/PX = NI/NPX = NI/\theta Y$ so that the argument can be cast in terms of aggregate spending on quality R&D as a fraction of final output.

sense tat $\sigma+\kappa \neq 1$. It follows that the $\dot{N}$ equation above is non-linear in N, non-linear in Z and non-linear in the combination of the two. Similarly, the $\dot{Z}$ equation above is non-linear in N, non-linear in Z and non-linear in the combination of the two. The only linear differential equation is the third. For the purposes of this argument, however, the law of motion of population is an uninteresting object since it is only a forcing process and we are trying to understand the endogenous forces at work in the model. We can thus set $\lambda = 0$ and strip the system down to only the endogenous state variables N and Z. This leaves us with a pair of autonomous non-linear differential equations with a full Jacobian matrix.[12] More precisely, not only the differential equations in the model's equilibrium system system do not have the form $\dot{N} = \_N$ and/or $\dot{Z} = \_Z$ with the blanks independent of N and Z, but we are also dealing with a system that does not allow us to think about one equation in isolation from the other. The model is about the interdependence of Z and N. Another way to say this is that one cannot abstract from the cross-equations restrictions that make the equilibrium system a system. The exception, as already said, is the steady state. I discuss this point further below, when I assess the model in the context of an extension of the linearity critique due to Growiec (2007).

5.4 Answering the question: interpretation

We can now reconsider the question motivating this exercise appropriately refined, namely: How is it possible to obtain endogenous growth without *assuming* the necessary functional form $\dot{Z} = \_Z$ with the blank independent of Z and of calendar time t? The answer is remarkably simple. No assuming is needed. Along the dynamics, as $x \rightarrow x^*$ *both* equations *become* of the class mathematically consistent with constant exponential growth, namely, $\dot{N} = N \cdot (\text{constant})$ and $\dot{Z} = Z \cdot (\text{constant})$. But such property is not imposed on any of the model's primitives — in fact, it is not assumed anywhere in the paper.[13] It *emerges* as the system converges to the steady state. Let me stress this point: the linearity — in the sense of Jones (2005) — of the accumulation equations for N and Z is a steady-state property of the system representing economic decisions and the interaction among the two dimensions of technology. It is not a property of the primitives per se. As such, it cannot be intuited, or guessed at, by simply inspecting one of the primitives in isolation from the others and without reference to the model's mechanism.

To explore further this point, one could say that each equation in the system (21)-(22) consists of two parts. The first is the technology (i.e., the primitive) part, the second is the behavior part. For example, in the case of quality, the technology part is $\dot{Z} = I$ and the behavior part is $I = Z \cdot z(x)$, where $z(x)$ is simply compact notation for a function obtained from the equilibrium conditions. In words, this says that agents invest in quality according to a function of the state

[12] Let me emphasize that the system becomes autonomous only *after* we kill population growth. The argument I am making is general and deals with time dependence as well, as I stressed in my definition of the term "constant" that I use to refer to Jones's blank.

[13] The only linearity assumption that the model makes, based on the traditional replication argument, is constant returns to scale with respect to the rival factors of production X_i and L in the production function for the final good Y.

vector (N, Z, L) that can be decomposed in the product of two terms, Z and $z(x)$, with x given by (23). The decomposition is not arbitrary, but stems from the property that agents care about rates of returns and that rates of return are functions of the quality-adjusted gross cash flow x and of the rates of innovation z and n. Similarly, one can write $\dot{N} = (N/\beta Y) \cdot E$ as the technology (i.e., primitive) part and $(N/\beta Y) \cdot E = N \cdot n(x)$ as the behavior part, where one can think in terms of a function $E(N, Z, L)$ that because of the properties of the model's equilibrium takes the specific form $E = \beta Y \cdot n(x)$ with $Y = \theta^{2\theta} N^{\sigma} Z^{\kappa} L$.

To conclude this discussion, let me turn to the extension of the linearity critique due to Growiec (2007). Consider a vector of variables X whose growth rate, denoted $\hat{X}$, is governed by the equation $\hat{X} = F(X)$. A balanced growth path exists if $d\hat{X}/dt = DF(X) \cdot \dot{X} = 0$, that is, if either $\dot{X} = 0$ or the matrix DF is singular. Since we are interested in $\hat{X} > 0$, the necessary condition for steady-state growth is singularity of DF, which requires either at least one of the differential equations in the system to be linear or a subset of the differential equations to be in linear combination. As this is a theorem, Growiec (2007) argues, such a condition is imposed either explicitly or implicitly in all growth models. Is this the case also for the model developed in this paper? No. The condition is not imposed. What happens is that the theorem's condition is trivially satisfied by the fact that *all* differential equations in the system (21)-(22) become linear in the sense of Jones (2005). Moreover, in steady state the Jacobian matrix of the system is diagonal and thus the theorem (and any such heavy-duty mathematical tool) becomes redundant since it is designed to apply to systems with cross-equations restrictions and in this model — one could argue — the steady-state representation of the dynamics is no longer a system in the sense of interdependent equations.

More importantly, however, the theorem applies to the reduced-form representation of the economy's dynamics and therefore — exactly as the one-dimensional $\dot{X} = \_X$ articulation of the linearity critique — has nothing to say about the model's primitives, which is what this paper is about. The claim of the paper is not that the differential equation for the equilibrium evolution of Z in steady state is not of the form $\dot{Z} = Z \cdot$ (constant) — such a claim would be a mathematical absurdity! The claim of the paper is that the form $\dot{Z} = Z \cdot$ (constant) need not be assumed as a property of the primitive $\dot{Z} = I$ representing how agents (i.e., firms) combine factors of production to obtain an increase in the quality of the product that they sell. Rather, the form $\dot{Z} = Z \cdot$(constant) emerges as agents follow an equilibrium investment rule that takes the form $I = Z \cdot z(x)$, where $z(x)$ is compact notation for a function obtained from the equilibrium conditions and the variable x, which is a summary statistic for how the state vector (N, Z, L) drives the the quality-adjusted gross cash flow of the firm, is constant in steady state.

In this light, replacing $z(x)$ with a blank reduces the critique to the tautological claim that a model of an exponential growth process must be based on the mathematical representation of an exponential growth process. What such a claim adds to our understanding of the model and of its contribution to the field is unclear to me. The economics of the model is about how it obtains such exponential representation, i.e., about what the blank is and where it comes from.

6 An extension: difficulty index

It is often argued that to match the evidence models of endogenous innovation must allow for rising difficulty of innovation. The most recent example is Bloom et al. (2017). To address such claims, I now generalize the model by allowing for a richer cost structure in innovation.

Specifically, recall that I_i is the firm's total expenditure on purchasing the inputs required to support a growth rate (a.k.a. rate of innovation) z_i, while ϕZ_i is the firm's total expenditure on purchasing the inputs required to stay in operation (a.k.a. fixed operating costs or, equivalently, management costs). In both cases, total expenditure is the product of the price/cost per unit of the input times the number of units purchased to carry out the activity. It is then natural to think that the unit cost is the same if we think of R&D and management as activities using the same inputs. As one looks at the expressions, it is also natural to think that there is no compelling reason why the unit cost should exhibit any particular returns to scale. It follows that one can think of a generic function common to the two activities. As I wrote the model I can thus extend it to:

$$\text{unit cost in R\&D} = Z_i \cdot D\left(Z_i; Z, N\right);$$

$$\text{unit cost in management} = Z_i \cdot D\left(Z_i; Z, N\right).$$

These expressions say that the unit cost consists of an internal (firm-specific) component due to Z_i and an external component due to Z and N.

It is useful to be more specific and write:

$$\text{unit cost in R\&D and management} = Z_i \cdot D\left(Z_i; Z, N\right) = Z_i \cdot Z_i^{\delta_1} Z^{\delta_2} N^{\delta_3}.$$

I leave the parameters δ_1, δ_2 and δ_3 unrestricted for now. Proceeding as in the previous analysis, I obtain the following expressions for the rates of return to innovation in the symmetric equilibrium (see the appendix for the derivation):

$$r = \alpha\kappa \frac{(P-1)X}{ZD\left(Z; Z, N\right)} - \delta_1 z - \phi\left(1+\delta_1\right) + \frac{\dot{q}}{q}, \quad q = D\left(Z; Z, N\right);$$

$$r = \left[1 - \frac{\phi + z}{\frac{(P-1)X}{ZD(Z;Z,N)}}\right] \frac{\theta\left(P-1\right)}{\beta P} + \frac{\dot{V}}{V}, \quad V = \frac{\beta Y}{N}.$$

Inspecting these expressions, it is evident that we have the same mechanism as the basic model with the only difference that we now define

$$x \equiv \frac{(P-1)X}{ZD\left(Z; Z, N\right)}.$$

With the functional form posited above, this becomes:

$$x \equiv \frac{(P-1)X}{Z^{1+\delta_1+\delta_2} N^{\delta_3}}.$$

Endogenous growth is now possible for $\kappa \geq 1+\delta_1+\delta_2$, that is, if social increasing returns to quality in final production more than compensate for the rising difficulty of innovation, which occurs for $\delta_1 > 0$ and/or $\delta_2 > 0$. Accordingly, the relevant region of parameter space where steady-state exponential endogenous growth is feasible is $1 + \delta_1 + \delta_2 \leq \kappa \leq 1/\alpha$. Note that for $\delta_3 > 0$, the restriction $\sigma < 1$ needed to ensure the dominant market share effect is relaxed.

An interesting aspect of the structure proposed here is that nothing dictates that both δ_1 and δ_2 be positive so that one is free to believe in the difficulty of innovation rising in the firm's own knowledge Z_i but decreasing in average knowledge Z. Alternatively one can believe that the difficulty of innovation rises in both Z_i and Z. Or one can believe that both Z_i and Z reduce the cost of innovation. The core mechanism is robust to all such alternatives.

The only difference of substance between this extension and the basic model developed in the paper is that the analysis of the dynamics in this case is considerably more algebra-intensive. The reason is that we no longer work with $q = 1$ but with $q = D(Z; Z, N) = Z^{\delta_1+\delta_2} N^{\delta_3}$ since to allow for rising difficulty of innovation we sacrifice the transparency and tractability of the one-sector structure. The mechanism and the core insight concerning the conditions under which endogenous growth is a robust proposition, however, do not change. From the perspective of this paper, therefore, one can reasonably argue that extensions such as this are expensive — in the sense that they cost a ton of extra calculations — while they add next to nothing to the main point of the analysis. Under the one-sector structure of the basic model, in fact, a rising cost of innovation that makes endogenous growth unfeasible as in Bloom et al. (2017) can be captured in a straightforward manner by setting $\kappa < 1$. There is no need to complicate matters by pursuing such a property in the roundabout fashion inherent to the difficulty of innovation index.

7 Conclusion

In this paper I have shown that Schumpeterian models with endogenous market structure *do not* require a linear production structure to generate endogenous growth defined as steady-state, constant, exponential growth of income per capita. This version of modern growth theory, therefore, is robust in the sense that its key result obtains for a thick set of parameter values instead of, as often claimed, for a set of measure zero. The mechanism delivering this property is the market fragmentation process, originally studied by Peretto (1994) and Smulders (1994), that "tames" potentially explosive growth due to increasing returns that look excessive in terms of the traditional theory. To date, such increasing returns have been deemed impossible by a priori reasoning that extrapolates the properties of one-dimensional models. Such reasoning does not apply to models with two dimensions of technology that play interdependent, but distinct, roles: the vertical dimension provides the engine of growth, the horizontal dimension provides the endogenous market structure that sterilizes the market size effect and thereby its two key manifestations: (1) the strong scale effect, with the associated need to impose constant endowments to rule out explosive behavior due to external forcings, and (2) the need to impose a knife-edge condition that rules out explosive behavior due to internal “excessive” increasing returns.

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