A Appendix

To facilitate the reader, all the equations from the text needed for the proofs are replicated in this document with self-contained numbering.

A.1 Derivation of the return to quality

The usual method of obtaining first-order conditions is to write the Hamiltonian for the optimal control problem of the firm. This derivation highlights the intuition. The firm undertakes R&D up to the point where the shadow value of the innovation, $q_i$, is equal to its cost,

$$1 = q_i \iff I_i > 0.$$  \hfill (A.1)

Since the innovation is implemented in-house, its benefits are determined by the marginal profit it generates. Thus, the return to the innovation must satisfy the arbitrage condition

$$r = \frac{\partial \Pi_i}{\partial Z_i} \frac{1}{q_i} + \frac{\dot{q}_i}{q_i}.$$  \hfill (A.2)

To calculate the marginal profit, observe that the firm’s problem is separable in the price and investment decisions. Facing the isoelastic demand

$$X_i = \left( \frac{\theta}{P_i} \right)^{\frac{1}{\alpha - \theta}} (Z_i^\alpha Z_i^{1-\alpha})^\kappa \frac{L}{N^{1-\sigma}},$$  \hfill (A.3)

and a marginal cost of production equal to one, the firm sets $P_i = 1/\theta$. Substituting this result into the firm’s cash flow,

$$\Pi_i = \left( \frac{1}{\theta} - 1 \right) \frac{1}{\theta^{\frac{1}{\alpha - \theta}}} (Z_i^\alpha Z_i^{1-\alpha})^\kappa \frac{L}{N^{1-\sigma}} - \phi Z_i,$$  \hfill (A.4)
differentiating with respect to $Z_i$, substituting into (A.2) and imposing symmetry yields

$$r = \frac{\alpha \kappa}{Z_i} \cdot \left(\frac{1}{\theta} - 1\right) \theta^{\frac{2}{\phi\zeta}} \left(\frac{Z_i^{\alpha} Z^{1-\alpha}}{N^{1-\sigma}}\right)^{\kappa} \frac{L}{(F-1)X_i} - \phi. \quad (A.5)$$

### A.2 Proof of Proposition 3

The proof is in several steps.

#### A.2.1 Step 1: consumption/saving decision

Recall that consumption, production of intermediates, quality and variety innovation, and resource extraction are all in units of the final good so that the resource allocation problem of this economy is the allocation across its alternative uses of the quantity $Y$ produced according to the technology

$$Y = \theta^{\frac{2}{1-\sigma}} \cdot N^\sigma Z^\kappa \cdot L. \quad (A.6)$$

The consumption flow that results from such allocation is:

$$\frac{C}{Y} \equiv c = \begin{cases} 
(1-\theta) \left[ \theta \left( 1 - \frac{\phi + z}{x} \right) + 1 \right] & n = 0 \quad z \geq 0 \\
(1-\theta) \left[ \theta (\rho - \lambda) \pi + 1 \right] & n > 0 \quad z \geq 0
\end{cases} \quad (A.7)$$

This equation is obtained as follows.

When $n > 0$ assets market equilibrium requires

$$A = NV = \beta Y, \quad (A.8)$$

which says that the wealth ratio $A/Y$ is constant. This result and the saving schedule

$$r = \rho - \lambda + \frac{\dot{C}}{C} \quad (A.9)$$

allow me to rewrite the household budget

$$\dot{A} = rA + wL - C \quad (A.10)$$

as the following unstable differential equation in $c \equiv C/Y$:

$$0 = \rho - \lambda + \frac{\dot{c}}{c} + \frac{1 - \theta - c}{\beta}. \quad (A.11)$$

Accordingly, to satisfy the transversality condition $c$ jumps to the constant value $(\rho - \lambda) \beta + 1 - \theta$.

Using the definition of $\pi$ yields the bottom line of (A.7).

When $n = 0$ assets market equilibrium still requires $A = NV$ but it is no longer true that $V = \beta Y/N$ since by definition the free-entry condition does not hold. This means that the wealth ratio $A/Y$ is not constant. However, the relation

$$r = \frac{\Pi_i - I_i}{V_i} + \frac{\dot{V}_i}{V_i} \quad (A.11)$$
holds, since it is the arbitrage condition on equity holding that characterizes the value of an existing firm regardless of how it came into existence in the first place. Imposing symmetry and inserting (A.4), (A.11), and (A.8) into the household budget (A.10) yields

\[ 0 = N [(1/\theta - 1) X - \phi Z - I] + (1 - \theta) Y - C. \]

The definition of \( x \), the R&D technology

\[ \dot{Z}_i = I_i, \quad (A.12) \]

and the fact that \( NX = \theta^2 Y \), allow me to rewrite this expression as the top line of (A.7).

With this result in hand, I now construct the functions \( z(x) \) and \( n(x) \).

### A.2.2 Step 2: innovation rates as functions of the state variable

I begin with the case \( x_N < x_Z \).

**Lemma 1** In the case \( x_N < x_Z \) the rates of variety and quality innovation are:

\[
\begin{align*}
  z(x) &= \begin{cases} 
    0 & \phi \leq x \leq x_N \\
    0 & x_N < x \leq x_Z \\
    \frac{\alpha k x - \phi - \sigma x \phi}{\kappa - \frac{\pi x}{\kappa}} & x < x < \infty
  \end{cases} \\
  n(x) &= \begin{cases} 
    0 & \phi \leq x \leq x_N \\
    \frac{x - \phi}{\pi x} - \rho + \lambda & x_N < x \leq x_Z \\
    \frac{(1 - \alpha) k x - (\kappa x - \phi) (\rho - \sigma)}{\kappa - \frac{\pi x}{\kappa}} - \rho + \lambda & x < x < \infty
  \end{cases}
\end{align*}
\]

where:

\[
\begin{align*}
  x_N &\equiv \frac{\phi}{1 - \pi (\rho - \lambda)}; \\
  x_Z &\equiv \text{arg solve} \left\{ \alpha k x - \phi - \frac{x \phi}{\pi x} = \rho - \sigma (\rho - \lambda) \right\}.
\end{align*}
\]

**Proof.** The ratio \( c \) is constant when there is entry, i.e., when \( n > 0 \), and in such case the return to saving (A.9) becomes \( r = \rho - \lambda + \dot{Y}/Y \). Therefore, we can use the expression for the return to entry,

\[
\begin{align*}
  r &= \frac{\Pi - I}{\beta Y/N} + \frac{\dot{Y}}{Y} - \frac{\dot{N}}{N},
  \quad (A.13)
\end{align*}
\]

(A.4), (A.12) and the definition

\[
\begin{align*}
  x &\equiv (1 - \theta) \frac{\theta Y}{NZ} = (1 - \theta) \theta^{1+\theta} \frac{Z^{k-1} L}{N^{1-\sigma}}, \quad (A.14)
\end{align*}
\]

to obtain

\[
\begin{align*}
  n &= \frac{x - \phi - z}{\pi x} - \rho + \lambda, \quad z \geq 0, \quad (A.15)
\end{align*}
\]
which holds for positive values of the right-hand side. The Euler equation (A.9) and the reduced-form production function (A.6) yield:

\[
r = \rho - \lambda + \frac{\dot{Y}}{Y} = \rho + \kappa z + \sigma n.
\]

Combining this expression with the return to quality

\[
r = \alpha \kappa x - \phi
\]

yields

\[
\alpha \kappa x - \phi = \rho + \kappa z + \sigma n.
\]

Combining this expression with the rate of entry in (A.15) and solving for \(z\) yields

\[
z = \frac{\alpha \kappa x - \phi - \sigma \frac{x - \phi}{\pi x} - [\rho - \sigma (\rho - \lambda)]}{\kappa - \frac{\sigma}{\pi x}}.
\]

Substituting this result back into (A.15) yields

\[
n = \frac{x - \phi - z}{\pi x} - \rho + \lambda = \frac{x - \phi - \frac{z}{\pi x} - \rho + \lambda}{\kappa \pi x - \sigma} = \frac{(1 - \alpha) \kappa x - (\kappa - 1) \phi + [\rho - \sigma (\rho - \lambda)]}{\kappa \pi x - \sigma} - \rho + \lambda.
\]

Consider now the thresholds. Suppose \(x_N < x_Z\). Then \(n(x) > 0\) for

\[
\frac{x - \phi}{\pi x} - \rho + \lambda > 0,
\]

since \(z = 0\), which yields

\[
x > x_N \equiv \frac{\phi}{1 - \pi (\rho - \lambda)}.
\]

On the other hand, \(z(x) > 0\) for

\[
\alpha \kappa x - \phi - \sigma \frac{x - \phi}{\pi x} - [\rho - \sigma (\rho - \lambda)] > 0,
\]

because entry is already active, which yields

\[
x > x_Z \equiv \arg \text{solve } \left\{ \alpha \kappa x - \phi - \sigma \frac{x - \phi}{\pi x} = \rho - \sigma (\rho - \lambda) \right\}.
\]

The inequality

\[
z(x_N) = \frac{\alpha \kappa x_N - \phi - \sigma \frac{x_N - \phi}{\pi x_N} - [\rho - \sigma (\rho - \lambda)]}{\kappa - \frac{\sigma}{\pi x_N}} < 0
\]

identifies the region of parameter space such that \(x_N < x_Z\).  

Next, I study the case \(x_Z < x_N\).
Lemma 2 In the case \( x_Z < x_N \) the rates of variety and quality innovation are:

\[
z(x) = \begin{cases} 
0 & \phi \leq x \leq x_Z \\
\frac{\kappa \sigma \phi - \sigma \phi}{\kappa - \phi} \frac{\phi - \rho}{\kappa - \sigma} & x_Z < x \leq x_N \\
\frac{\kappa \phi - (\kappa - 1) \phi + \rho}{\kappa \phi} & x_N < x < \infty
\end{cases}
\]

\[
n(x) = \begin{cases} 
0 & \phi \leq x \leq x_Z \\
0 & x_Z < x \leq x_N \\
\frac{(1-\alpha) \kappa x - (\kappa - 1) \phi + [\rho - \sigma (\rho - \lambda)]}{\kappa \phi} - \rho + \lambda & x_N < x < \infty
\end{cases}
\]

where:

\[
\tilde{z}(x) = x \left[ 1 - \frac{1}{\theta} \left( \frac{\hat{c}(x)}{1 - \theta} - 1 \right) \right] - \phi
\]

and \( \hat{c}(x) \) is the solution of the partial differential equation

\[
\frac{dc}{dx} = \frac{\kappa c}{x} \left( \frac{c}{1 - \theta} - 1 \right) - \frac{\rho}{\kappa} c - \frac{(1 - \alpha) x}{\kappa \pi}.
\]

The thresholds are:

\[
x_N \equiv \text{arg solve} \left\{ \frac{(1-\alpha) \kappa x - (\kappa - 1) \phi + [\rho - \sigma (\rho - \lambda)]}{\kappa \pi x - \sigma} = \rho - \lambda \right\};
\]

\[
x_Z \equiv \text{arg solve} \left\{ x \left[ 1 - \frac{1}{\theta} \left( \frac{\hat{c}(x)}{1 - \theta} - 1 \right) \right] = \phi \right\}.
\]

The function \( z(x) \) has zero derivative at \( x = x_Z \), is increasing and has positive derivative at \( x_N \).

**Proof.** As before, over the range \( \phi \leq x \leq x_Z \) the function \( c(x) \) is given by (A.7) evaluated at \( z = 0 \). To characterize it over the range \( x_Z < x \leq x_N \), set the rate of return to vertical innovation equal to the reservation rate of return of savers to obtain:

\[
\rho - \lambda + \frac{\hat{c}}{c} + \kappa z + \lambda = \alpha \kappa x - \phi.
\]

Solving the household budget constraint for \( z \), yields

\[
z = x - \phi - \frac{x}{\theta} \left( \frac{c}{1 - \theta} - 1 \right).
\]

Combining these two expressions yields

\[
\frac{\hat{c}}{c} = (1-\alpha) \kappa x - (\kappa - 1) \phi + \kappa \frac{x}{\theta} \left[ \frac{c}{1 - \theta} - 1 \right] - (\rho - \lambda) - \lambda.
\]

The \( \hat{c} \geq 0 \) locus is thus

\[
c \geq (1 - \theta) \left[ 1 + \theta \rho - (1 - \alpha) \kappa x + (\kappa - 1) \phi \right].
\]
In this region, the law of motion of $x$ is
\[
\frac{\dot{x}}{x} = \lambda + (\kappa - 1) z
\]
\[
= \lambda + (\kappa - 1) \left[ x - \phi - \frac{x}{\theta} \left( \frac{c}{1 - \theta} - 1 \right) \right].
\]
Recall, however, that $z \geq 0$ so that $\dot{x}/x$ is strictly positive. There is then a unique equilibrium trajectory: the economy jumps on the saddle path in $(x,c)$ space that converges to $(x^*,c^*)$ with smooth pasting. Writing
\[
\frac{\dot{c}}{x} = \frac{dc}{dx} = \frac{c (1 - \alpha) \kappa x - (\kappa - 1) \phi - \rho + \kappa \frac{c}{1 - \theta} \left( \frac{c}{1 - \theta} - 1 \right)}{\lambda + (\kappa - 1) \left[ x - \phi - \frac{x}{\theta} \left( \frac{c}{1 - \theta} - 1 \right) \right]}
\]
yields a partial differential equation that doesn’t have a closed-form solution. However, we can show that the function $\tilde{c}(x)$ that solves it has the same derivative from the left and the right at $x = x_Z$ and approaches the value $c^*$ with zero derivative at $x = x_N$:
\[
\frac{dc(x_Z^-)}{dx} = \frac{dc(x_Z^+)}{dx},
\]
\[
\frac{dc(x_N)}{dx} = 0.
\]
In other words, it is increasing, concave and has no kinks. The associated expression for $z$ is
\[
\tilde{z}(x) = x \left[ 1 - \frac{1}{\theta} \left( \frac{\tilde{c}(x)}{1 - \theta} - 1 \right) \right] - \phi.
\]
Once again, we can show that $\tilde{z}(x)$ starts out at $x = x_Z$ with zero derivative and approaches the line that holds for $x > x_N$ with positive derivative:
\[
\frac{dz(x_Z)}{dx} = 1 - \frac{1}{\theta} \left( \frac{c(x_Z)}{1 - \theta} - 1 \right) - \frac{x_Z}{\theta} \frac{dc(x_Z)}{dx} \frac{dx}{1 - \theta} = 0;
\]
\[
\frac{dz(x_N)}{dx} = 1 - \frac{1}{\theta} \left( \frac{c(x_N)}{1 - \theta} - 1 \right) - \frac{x_N}{\theta} \frac{dc(x_N)}{dx} \frac{dx}{1 - \theta} > 0.
\]
The function $\tilde{z}(x)$ exhibits a kink at $x = x_N$ because when entry begins quality innovation attracts only a fraction of the economy’s saving flow, which is now a constant fraction of final output.

According to these results, the only difference between the two cases is the middle region. With the functions $z(x)$ and $n(x)$ in hand, I can now prove the main result.

### A.2.3 Step 4: Existence

After some algebra, the equation $\Psi(x) = 0$ yields
\[
a_1 x^2 + a_2 x - a_3 = 0,
\]
I am thus looking for values of steady state Having established that the condition in the text of the proposition is sufficient for existence of the values \(x\) that equation has two solutions in the region \(x > \max\{x_N, x_Z\}\). I proceed in two steps. First I show that \(\Delta' (\kappa) < 0\) for all \(\kappa\). Then I study the components of the function \(\Delta (\kappa)\) at the boundary values 1 and \(1/\alpha\).

- To show that \(\Delta (\kappa)\) is decreasing, note that: \(a_1' (\kappa) > 0; a_2' (\kappa) < 0; a_3' (\kappa) < 0\). Consequently, \[\Delta' (\kappa) = 2a_2 (\kappa) a_2' (\kappa) + 4a_1' (\kappa) a_3 (\kappa) + 4a_1 (\kappa) a_3' (\kappa) < 0.\]

- Next, the boundary values. First:

\[
\lim_{\kappa \to 1} a_1 (\kappa) = \lim_{\kappa \to 1} (\kappa - 1) \kappa \alpha \pi = 0;
\]

\[
\lim_{\kappa \to 1} (a_2 (\kappa))^2 = \{[\rho - \sigma (\rho - \lambda)] \pi + (1 - \sigma) (1 - \alpha \kappa)\}^2 > 0;
\]

\[
\lim_{\kappa \to 1} a_3 (\kappa) = [\rho - \sigma (\rho - \lambda)] > 0.
\]

Hence,

\[
\lim_{\kappa \to 1} \Delta (\kappa) = \lim_{\kappa \to 1} (a_2 (\kappa))^2 > 0.
\]

Similarly:

\[
\lim_{\kappa \to (1/\alpha)^-} a_1 (\kappa) = \left(\frac{1}{\alpha} - 1\right) \pi > 0;
\]

\[
\lim_{\kappa \to (1/\alpha)^-} (a_2 (\kappa))^2 = \left\{[\rho - \sigma (\rho - \lambda)] \pi + (1 - \sigma) - \left(\frac{1}{\alpha} - \sigma\right) - \left(\frac{1}{\alpha} - 1\right) \phi \pi\right\}^2 > 0;
\]

\[
\lim_{\kappa \to (1/\alpha)^-} a_3 (\kappa) = \rho - \sigma (\rho - \lambda) - \left(\frac{1}{\alpha} - 1\right) \phi.
\]

- Since \(\Delta (\kappa)\) is monotonically decreasing, it is sufficient to assume \(\lim_{\kappa \to (1/\alpha)^-} \Delta (\kappa) > 0\) to obtain \(\Delta (\kappa) > 0\) for all \(\kappa \in [1, 1/\alpha]\). Inspecting the expression, we see that it is the case if \(\lim_{\kappa \to (1/\alpha)^-} a_1 (\kappa) a_3 (\kappa) > 0\), that is, if \(\lim_{\kappa \to (1/\alpha)^-} a_3 (\kappa) > 0\). We thus obtain the condition:

\[
\rho - \sigma (\rho - \lambda) > \left(\frac{1}{\alpha} - 1\right) \phi
\]

It is worth stressing the point: this condition is sufficient for \(\Delta (\kappa) > 0\) for all \(\kappa \in [1, 1/\alpha]\).

More generally, let

\[\bar{\kappa} = \text{arg solve} \left\{(a_2 (\kappa))^2 + 4a_1 (\kappa) a_3 (\kappa) = 0\right\}\]

so that \(\Delta (\kappa) > 0\) for

\[1 \leq \kappa \leq \min \left\{\bar{\kappa}, \frac{1}{\alpha}\right\}.
\]

Having established that the condition in the text of the proposition is sufficient for existence of the steady state \(x^*\), I now ask whether the economy converges to such steady state.
A.2.4 Step 5: Stability

Figures 3-4 in the text illustrate the dynamics. Consider first the case $x_N < x_Z$, in which the economy activates first variety innovation. For $x \leq x_N < x_Z$ the growth rate of profitability is $\dot{x}/x = \lambda$ and the economy crosses the threshold for entry in finite time. For $x_N < x < x_Z$ the growth rate is

$$\frac{\dot{x}}{x} = \lambda - (1 - \sigma) \left( \frac{x - \phi}{\pi x} - \rho + \lambda \right).$$

This expression identifies a steady-state value

$$x_N^* \equiv \arg \text{solve} \left\{ (1 - \sigma) \frac{x - \phi}{\pi x} = \rho - \sigma (\rho - \lambda) \right\}.$$

Now recall that

$$x_Z \equiv \arg \text{solve} \left\{ \alpha \kappa x - \phi - \sigma \frac{x - \phi}{\pi x} = \rho - \sigma (\rho - \lambda) \right\}$$

and note that a sufficient condition for $x_Z < x_N^*$ is that $x_N^* \to \infty$, that is, that

$$(1 - \sigma) \frac{x - \phi}{\pi x} \leq \rho - \sigma (\rho - \lambda) \quad \forall x \in (\phi, \infty).$$

Since the left-hand side is an increasing, concave, bounded above function, the condition we seek is

$$(1 - \sigma) \frac{1}{\pi} \leq \rho - \sigma (\rho - \lambda) \Rightarrow \frac{1 - \sigma}{\rho - \sigma (\rho - \lambda)} \leq \pi.$$ 

Interestingly, this condition does not depend on $\kappa$, since we are looking for parameter combinations that boost incentives to variety growth when quality growth is still zero. The intuition for this condition is that it prevents premature market saturation.

The case where $x_N < x_Z$ features an acceleration of the rate of growth of profitability at $x = x_Z$ so that the economy crosses the threshold $x_N$ in finite time. I conclude, therefore, that the condition stated in the proposition is sufficient for convergence to the steady state $x^*$ for any initial condition $x(0) \in (\phi, \bar{x})$.

For the comparison with the existing literature, it is instructive to look at

$$\lim_{x \to \infty} \Psi(x) = \lim_{x \to \infty} \left[ \lambda + (\kappa - 1) z(x) - (1 - \sigma) n(x) \right],$$

where, using the expressions above:

$$\lim_{x \to \infty} z(x) = \lim_{x \to \infty} \frac{\alpha \kappa x - \phi - \sigma \frac{x - \phi}{\pi x} - [\rho - \sigma (\rho - \lambda)]}{\kappa - \frac{\sigma}{\pi x}} = \lim_{x \to \infty} \alpha x;$$

$$\lim_{x \to \infty} n(x) = \lim_{x \to \infty} \left( 1 - \alpha \right) \kappa x - (\kappa - 1) \phi + [\rho - \sigma (\rho - \lambda)] - \rho + \lambda = \lim_{x \to \infty} \frac{1 - \alpha}{\pi} - \rho + \lambda.$$ 

This concludes the long proof of Proposition 3.
A.3 A CIES economy

I now sketch the results for a generic CIES economy with welfare function

\[ U(0) = \int_{0}^{\infty} e^{-(\rho-\lambda)t} \frac{1}{1-\eta} \left[ \left( \frac{C(t)}{L(t)} \right)^{1-\eta} - 1 \right] dt, \quad \rho > \lambda \geq 0, \quad \eta > 0. \]

The associated Euler equation is

\[ r = \rho + \eta \left( \frac{\dot{C}}{C} - \lambda \right). \]

I shall focus this discussion on the steady-state properties of this generalization for two reasons. First, it is sufficient to make the point that the robust endogenous growth studied in the paper is in fact a general property and that the assumption of log-utility in the main text just a simplification. Second, the analysis of the transitional dynamics in the general case is much more cumbersome because the model loses the nice feature that the ratio \( C/Y \equiv c \) is constant at all times and therefore one needs to study the dynamical system in two dimensions, \( c \) and \( x \). The main difficulty is that the thresholds for activation of vertical innovation become non-linear locus in \((x,c)\) space. The phase diagram is doable but much more cumbersome to present. The key advantage of the log-utility specification, therefore, is that it yields a transparent characterization of the dynamics.

To work out the steady state of the CIES economy, I start by using the reduced-form production function (??) to rewrite the Euler equation as

\[ r = \rho + \eta \left( \frac{\dot{Y}}{Y} - \lambda \right) + \eta \left( \frac{\dot{C}}{C} - \frac{\dot{Y}}{Y} \right) = \rho + \eta (\kappa z + \sigma n) + \eta \frac{\dot{c}}{c}. \] (A.17)

In steady state \( c \) is constant and I can proceed as in the proof of Proposition 3. Combining (A.17) with the return to quality (A.16) yields

\[ \kappa \alpha x - \phi = \rho + \eta (\kappa z + \sigma n). \] (A.18)

The return to entry (A.13), the definition of \( x \) in (A.14) and the Euler equation (A.17) yield

\[ n = \frac{x - \phi - z}{\pi x} - \rho + \lambda + (1 - \eta) (\kappa z + \sigma n), \quad z \geq 0, \]

which holds for positive values of the right-hand side. Solving this expression for \( n \) yields

\[ n = \frac{x - \phi - \rho + \lambda}{1 - (1 - \eta) \sigma} + \frac{\kappa (1 - \eta) - 1/\pi x}{1 - (1 - \eta) \sigma} z. \]

Substituting this result in (A.18) and solving for \( z \) yields

\[ z(x) = \frac{(\kappa \alpha x - \phi - \rho) [1 - (1 - \eta) \sigma] - \eta \sigma \left( \frac{x - \phi}{\pi x} - \rho + \lambda \right)}{\eta (\kappa - \sigma / \pi x)}. \]
Substituting this solution back in the expression for \( n \) yields

\[
\begin{align*}
n(x) &= \frac{x - \phi}{\pi x} - \rho + \lambda + \frac{\kappa (1 - \eta) - 1/\pi x}{1 - (1 - \eta) \sigma} z(x) \\
&= \left( \frac{x - \phi}{\pi x} - \rho + \lambda \right) \frac{\kappa}{\kappa - \sigma/\pi x} + \frac{\kappa (1 - \eta) - 1/\pi x}{\eta (\kappa - \sigma/\pi x)} [\kappa \alpha (x - \phi) - \rho].
\end{align*}
\]

The definition of \( x \) in (A.14) and the reduced-form production function (??) yield

\[
\frac{\dot{x}}{x} = \lambda + (\kappa - 1) z(x) - (1 - \sigma) n(x).
\]

The key issue then is whether the equation

\[
0 = \lambda + (\kappa - 1) z(x) - (1 - \sigma) n(x),
\]

which yields again a quadratic form

\[
0 = a_1 x^2 + a_2 x - a_3
\]

with

\[
a_1 \equiv [\kappa \eta - 1 + (1 - \eta) \sigma] \kappa \alpha \pi,
\]

\[
a_2 \equiv [\kappa \eta - 1 + (1 - \eta) \sigma] \kappa \alpha \phi + \rho (1 - \sigma) + \eta \sigma \lambda] \pi + (1 - \sigma) \kappa \alpha - \eta (\kappa - \sigma),
\]

\[
a_3 \equiv \eta \lambda \sigma + (1 - \sigma) \rho + \eta \sigma \phi - [\eta - (1 - \sigma) \alpha] \kappa \phi,
\]

has solutions in the range where it holds. The sufficient condition is now \( a_1 a_3 \geq 0 \). This is the same as throughout the paper since in all the cases studied the condition was that the two coefficients be of the same sign. The difference is that now \( a_1 \) can be negative. Suppose it is positive, i.e., \( \eta > (1 - \sigma)/(\kappa - \sigma) \). Then the upper bound on \( \kappa \) is

\[
\kappa < \frac{\eta \lambda \sigma + (1 - \sigma) \rho + \eta \sigma \phi}{[\eta - (1 - \sigma) \alpha] \phi}.
\]

If \( x_N < x_Z \) the threshold for \( n(x) > 0 \) is the same as in the log-utility case since the function \( n(x) \) reduces to

\[
n(x) = \frac{x - \phi}{\pi x} - \rho + \lambda
\]

and thus yields

\[
x > x_N \equiv \frac{\phi}{1 - (1 - \eta) \sigma}.
\]

Then, \( z(x) > 0 \) for

\[
(\kappa \alpha x - \phi - \rho) [1 - (1 - \eta) \sigma] - \eta \sigma \left( \frac{x - \phi}{\pi x} - \rho + \lambda \right) > 0,
\]

which yields

\[
x > x_Z \equiv \text{arg solve} \left\{ 1 - (1 - \eta) \sigma = \frac{\eta \sigma \left( \frac{x - \phi}{\pi x} - \rho + \lambda \right)}{\kappa \alpha x - \phi - \gamma \rho} \right\}.
\]

The assumption

\[
z(x_N) = \frac{\kappa \alpha (x_N - \phi) - \rho}{} [1 - (1 - \eta) \sigma] - \eta \sigma \left( \frac{x_N - \phi}{\pi x_N} - \rho + \lambda \right) < 0
\]

ensures that \( x_N < x_Z \).
A.4 Derivation of the returns to innovation in the extended model of Section 6

The typical firm’s Hamiltonian is:

$$CVH_i = \left( \frac{1}{\theta} - 1 \right) \theta^{\frac{2}{1-\sigma}} \left( X_i Z_i^{\gamma} \right)^{\kappa} \frac{L}{N^{1-\sigma}} - \phi Z_i D (Z_i; Z, N) - I_i + q_i I_i \frac{1}{D (Z_i; Z, N)}.$$  

This yields:

$$r = \frac{\partial \Pi_i}{\partial Z_i q_i} - \frac{\partial D (Z_i; Z, N)}{\partial D_i} \frac{Z_i}{D (Z_i; Z, N) Z_i D (Z_i; Z, N)} + \frac{\dot{q}_i}{\dot{q}_i}, \quad q_i = D (Z_i; Z, N),$$

where

$$\frac{\partial \Pi_i}{\partial Z_i} = \frac{\partial \left( (X_i Z_i^{\gamma})^{\kappa} \right)}{\partial Z_i} \cdot \left( \frac{1}{\theta} - 1 \right) \theta^{\frac{2}{1-\sigma}} \left( X_i Z_i^{\gamma} \right)^{\kappa} \frac{L}{N^{1-\sigma}} (P_i - 1) X_i$$

$$- \phi \left( D (Z_i; Z, N) + Z_i \frac{\partial D (Z_i; Z, N)}{\partial D_i} \right)$$

$$= \alpha \kappa (P_i - 1) X_i - \phi \left( D (Z_i; Z, N) + Z_i \frac{\partial D (Z_i; Z, N)}{\partial D_i} \right).$$

With the functional form

$$D (Z_i; Z, N) = Z_i^{\delta_1} Z_i^{\delta_2} N^{\delta_3}$$

I have:

$$r = \frac{\alpha \kappa (P_i - 1) X_i}{D (Z_i; Z, N)} - \delta_1 z_i - \phi (1 + \delta_1) + \frac{\dot{q}_i}{\dot{q}_i}, \quad q_i = D (Z_i; Z, N);$$

$$r = \frac{(P_i - 1) X_i - \phi Z_i D (Z_i; Z, N) - I_i}{V_i} + \frac{\dot{V}_i}{V_i}, \quad V_i = \frac{\beta Y}{N}.$$

The price-dividend ratio in the return to entry can be written:

$$\frac{(P_i - 1) X_i - \phi Z_i D (Z_i; Z, N) - I_i}{V_i} = \frac{(P_i - 1) X_i}{Z_i D (Z_i; Z, N) - \frac{F_i + I_i}{\beta Y}} Z_i D (Z_i; Z, N)$$

$$= \frac{(P_i - 1) X_i}{Z_i D (Z_i; Z, N) - \frac{\phi - z_i}{\beta Y}} Z_i D (Z_i; Z, N).$$

Imposing symmetry and recalling that $NPX = \theta Y$ yields:

$$r = \frac{\alpha \kappa (P - 1) X}{Z D (Z; Z, N)} - \delta_1 z - \phi (1 + \delta_1) + \frac{\dot{q}}{q}, \quad q = D (Z; Z, N) = Z^{\delta_1 + \delta_2} N^{\delta_3};$$

$$r = \left[ 1 - \frac{\phi + z}{Z D (Z; Z, N)} \right] \frac{\theta (P - 1) X}{\beta P} + \frac{\dot{V}}{V}, \quad V = \frac{\beta Y}{N}.$$  

These show that we have the same mechanism as the basic model with the only difference that we now define

$$x = \frac{(P - 1) X}{Z D (Z; Z, N)} = \frac{(P - 1) X}{Z^{\delta_1 + \delta_2} N^{\delta_3}}.$$  

11
Concavity of the revenue function holds for

\[
\frac{\partial^2}{\partial Z_i^2} \left[ \left( \frac{1}{\theta} - 1 \right) \theta^{\frac{1}{\theta}} \frac{L}{N^{1-\sigma}} (Z_1^{1-\alpha})^\kappa \cdot Z_i^{\alpha \kappa} - \phi Z_i^{\delta_2} N^{\delta_3} \cdot Z_i^{1+\delta_1} \right] < 0,
\]

that is, for

\[
\left( \frac{1}{\theta} - 1 \right) \theta^{\frac{1}{\theta}} \frac{L}{N^{1-\sigma}} (Z_1^{1-\alpha})^\kappa \cdot \alpha \kappa (\alpha \kappa - 1) Z_i^{\alpha \kappa - 2} - \phi Z_i^{\delta_2} N^{\delta_3} \cdot (1 + \delta_1) \delta_1 Z_i^{\delta_1 - 1} < 0.
\]

We thus get the sufficient condition:

\[
\alpha \kappa \leq 1.
\]

Quasi-convexity of the innovation plus management cost component holds for \( \delta_1 \geq 0 \). Nothing else is needed.