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The Manhattan Metaphor

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Abstract Fixed operating costs draw a sharp distinction between endogenous growth based on horizontal and vertical innovation: a larger number of product lines puts pressure on an economy's resources; greater productivity of existing product lines does not. Consequently, the only plausible engine of endogenous growth is vertical innovation whereby progress along the quality or cost ladder does not require the replication of fixed costs. Is, then, product variety expansion irrelevant? No. The two dimensions of technology are complementary in that using one *and* the other produces a more comprehensive theory of economic growth. The vertical dimension allows endogenous growth unconstrained by endowments, the horizontal provides the mechanism that translates changes in aggregate variables into changes in product-level variables, which ultimately drive incentives to push the technological frontier in the vertical dimension. We show that the potential for exponential growth due to an externality that makes entry costs fall linearly with the number of products, combined with the limited carrying capacity of the system due to fixed operating costs, yields *logistic dynamics* for the number of products. This desirable property allows us to provide a closed-form solution for the model's transition path and thereby derive analytically the welfare effects of changes in parameters and policy variables. Our Manhattan Metaphor illustrates conceptually why we obtain this mathematical representation when we simply add fixed operating costs to the standard modeling of variety expansion.

Keywords Endogenous growth · Fixed costs · Technological change · Innovation · Population growth

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1 Introduction

The modern theory of innovation-driven economic growth was born with the development of two classes of models focused on different dimensions of technological advance. In models of horizontal innovation, agents invest resources in R&D to expand the variety of existing goods; in models of vertical innovation agents invest resources to increase the quality and/or reduce the production cost of existing goods. These two approaches are often considered simple transpositions of each other. Grossman and Helpman (1991) and Barro and Sala-i-Martin (2004), for example, argue that they yield *equivalent* aggregate stories and thus that one can use one *or* the other to study innovation-driven growth. In this paper we argue that they are in fact fundamentally different, for a reason that has been in plain sight all along.

In developing his seminal model of horizontal innovation, Romer (1987, 1990) acknowledged that he was borrowing the idea that a larger variety of products leads to higher specialization, and thus to productivity gains, from the static theory of product differentiation. What he did not make explicit—and the literature has failed to appreciate—is that to turn that well-known structure into a model of endogenous growth, he not only introduced fixed, sunk costs of development of new products, he also *eliminated the fixed operating costs* that in static models are responsible for finite product variety in equilibrium.

Far from being a simplifying assumption, the elimination of these costs is *necessary* to obtain endogenous growth through variety expansion. Specifically, steady-state growth driven by product proliferation cannot occur if production of each good entails a fixed cost *even if* the production function for horizontal innovations is linear in the stock of horizontal innovations. (In other words, the linearity of that technology is *not* sufficient for endogenous growth.) Rather than being isomorphic representations of technological advance, vertical and horizontal innovation are fundamentally different.

One way to visualize this difference is what we call the Manhattan Metaphor. A building requires a fixed plot of land. Consequently, the island of Manhattan can accommodate only a finite number of buildings since land is in finite supply. Once the island is saturated with buildings, the only way to progress (i.e., house more people and businesses) is to build taller buildings. This can entail either adding floors to existing buildings or replacing existing building with taller ones (skyscrapers). If we think of the height of buildings as quality/productivity and of the number of building as the number of products, the Manhattan Metaphor illustrates the role of fixed operating costs. Because of the fixed requirement of land per building, at any point in time there is a natural limit to the number of buildings that can exist on the island. Yet, there is no corresponding natural limit to the height of the buildings that take up individual lots. In the vertical dimension, progress is only limited by the state of technology.

In a generic model of variety expansion, the equivalent of the fixed land requirement per building of our Manhattan Metaphor is an operating cost *per product line* that is fixed, in the sense that it does not depend on the volume of production, and is *recurrent*, in the sense that it is borne every period. The key property of these fixed operating costs is that increasing the number of product lines requires their replication. Increasing output of an existing product line, in contrast, does not. Our Manhattan Metaphor highlights how proliferation of buildings puts pressure on the island's finite supply of land, while increasing buildings' height does not. Similarly, in the context of endogenous growth models, a larger number of product lines puts pressure on an economy's resources; greater productivity of existing product lines does not.

The Manhattan Metaphor thus provides us with a sharp and intuitive distinction between horizontal and vertical innovation. In short, if fixed operating costs are not zero, variety expansion (product proliferation) is not a plausible engine of endogenous growth. Consequently, the only dimension in which steady-state endogenous growth can occur is the vertical one wherein progress along the quality or cost ladder does not require the replication of fixed costs.

Is, then, product variety expansion irrelevant? No. The literature has successfully integrated the two dimensions of technological progress in order to eliminate the scale effect. Building on this insight, we argue that the two types of technological change are strongly complementary in that using one *and* the other produces a more comprehensive theory of economic growth. The vertical dimension provides the opportunity of growth unconstrained by endowments, while the horizontal dimension provides the mechanism that translates changes in aggregate variables into changes in product-level variables, which ultimately drive incentives to push the technological frontier in the vertical dimension.

We show that this approach produces models so tractable that we are able to provide a closed-form solution for the entire transition path. This is a very desirable property in that it allows us to derive analytically the welfare effects of changes in parameters and policy variables. Specifically, the potential for exponential growth due the property that entry costs fall linearly with the number of products, combined with the linear crowding process due to the fixed operating costs, yields *logistic dynamics* for the number of products. Our Manhattan Metaphor illustrates conceptually why we obtain this mathematical representation when we simply add fixed operating costs to ingredients that are standard in variety-expansion models. It tells us that fixed operating costs imply saturation and thereby place the emphasis on the limited carrying capacity of the system with respect to the variable—the number of product lines—whose growth requires replication of fixed operating costs.

Before moving on to the main body of the paper, it is worth differentiating our present work from Peretto (1998). That paper developed a model built on several of the ingredients that we use here except that, in line with the standard practice of the time, it set fixed operating costs at zero. The research question was how to sterilize the scale effect through product proliferation, a property that allows one to incorporate population growth in models of endogenous growth without getting counterfactual, explosive behavior. Although the models that we discuss here exhibit this property, in this paper we ask a rather different question. Our primary concern is whether vertical and horizontal innovation are conceptually the same or whether there is some fundamental reason to keep them distinct. A related issue is whether models of endogenous growth driven by product variety expansion are robust. Developing the Manhattan Metaphor as a conceptual apparatus to think about these questions convinced us that they are not, because they cannot survive the (re)introduction of fixed operating costs. The final question arose serendipitously as we worked out the dynamics of the models. We noticed that under relatively straightforward assumptions concerning entry costs we tended to recover reduced-form representations of the equilibrium that had the familiar logistic form. This was good news, as we were able to solve in closed-form the model's transition path, and thereby obtain analytical answers to questions concerning welfare, but we had to wonder whether this simplicity and elegance was just due to luck or something more fundamental. The Manhattan Metaphor again proves useful in understanding that logistic dynamics is inherent to the very structure of these models once we introduce fixed operating costs.

In summary, this paper contributes to the literature on two levels. First, it provides a simple conceptual framework that sheds light on the structure of models of endogenous growth based on the interaction between the vertical and horizontal dimensions of technological advance. In particular, it tells us why the two dimensions are fundamentally different and thus why growth unconstrained by endowments is plausible in one and not in the other. The second contribution is more technical. The introduction of fixed operating costs produces models with logistic dynamics that we can solve in closed form. As a consequence, we can investigate the welfare effects of changes in fundamentals in an extremely tractable way.[1]

The paper is organized as follows. In Sect. 2 we set up the basic model, discuss its preliminary properties and characterize the economy's equilibrium dynamics and steady state. In Sect. 3 we consider different plausible specifications of the entry costs for new firms. We show that for all these extensions the model's dynamics belongs to the logistic class and have a closed-form solution. In Sect. 4 we show how this desirable property allows us to obtain simple analytical answers to the question of what are the welfare effects of specific parameters or policy variables. We conclude in Sect. 5.

2 The basic model

In this section, we present our basic model and show that fixed operating costs have dramatic consequences for endogenous growth based on variety expansion. For simplicity we limit our formal discussion to models where labor is the only factor of production. The argument is easily generalized to more sophisticated environments where there are more factors of production, some reproducible and some not. In particular, the argument applies also to specifications where fixed costs are in units of a reproducible factor; see, e.g., Peretto (2007a,b). In the next section we consider extensions that allow us to generalize our point and illustrate the great tractability of this class of models.

2.1 Consumption and saving

Consider a closed economy populated by a representative household with identical individual members who supply labor services and consumption loans in competitive markets. We normalize initial population so that at time t population size is $\Lambda e^{\lambda t}$, where λ is the rate of population growth and Λ is a scale parameter. Each individual is endowed with one unit of time, which he supplies inelastically. The household maximizes lifetime utility

$$U = \int_0^\infty e^{-(\rho-\lambda)t} \log C(t)\, dt, \quad \rho > \lambda > 0 \tag{1}$$

where ρ is the individual discount rate. The consumption index C is symmetric over a continuum of differentiated goods,

$$C = \left[\int_0^N C_i^{\frac{\epsilon-1}{\epsilon}} di\right]^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1 \tag{2}$$

where ϵ is the elasticity of product substitution, C_i is the purchase of each differentiated good, and N is the mass of goods. (To simplify the notation, we suppress time from endogenous variables whenever confusion does not arise.)

Individuals face the flow budget constraint

$$\dot{A} = (r - \lambda) A + W - E. \tag{3}$$

All variables are in per capita terms. A is assets holding, r is the rate of return on assets, W is the wage rate, and $E = \int_0^N P_i C_i di$ is consumption expenditure. The wage rate is the numeraire, $W \equiv 1$.

[1] For examples of how easy it is to work out the welfare the implications of policy variables in models that exhibit this property, see, Peretto (2007a,b,c).

The optimal plan for this setup is well known. Individuals save and allocate instantaneous expenditure according to:

$$\frac{\dot{E}}{E} = r - \rho; \tag{4}$$

$$C_i = E \frac{P_i^{-\epsilon}}{\int_0^N P_j^{1-\epsilon} dj}. \tag{5}$$

2.2 Production and innovation

Each consumption good is supplied by one firm. Thus, N also denotes the mass of firms. Each firm produces with the technology

$$X_i = Z_i^{\theta} \left(L_{X_i} - \phi \right), \quad 0 < \theta < 1, \ \phi > 0 \tag{6}$$

where X_i is output, L_{X_i} is labor employment and ϕ is a fixed operating cost. Labor productivity is a function of the firm's accumulated stock of innovations Z_i, with elasticity θ.[2] The firm faces the demand curve $X_i = \Lambda e^{\lambda t} C_i$, where C_i is given in (5) above. With a continuum of goods, one can assume that firms are atomistic and take as given the denominator of (5). Hence, monopolistic competition prevails and firms face isoelastic demand curves.

We assume that innovations are developed by independent inventors that then sell them to firms. Accordingly, the firm maximizes the present discounted value of profit,

$$V_i = \int_0^{\infty} e^{-\int_0^t r(s)ds} \left[P_i X_i - L_{X_i} - P_{Z_i} \dot{Z}_i \right] dt,$$

where $\dot{Z}_i$ is the mass of innovations (patents) purchased by the firm and P_{Z_i} is the price of a patent.

The firm's demand for patents is fully summarized by the asset-pricing equation

$$r = \frac{\partial \Pi_i}{\partial Z_i} \frac{1}{P_{Z_i}} + \frac{\dot{P}_{Z_i}}{P_{Z_i}},$$

where we define $\Pi_i \equiv P_i X_i - L_{X_i} - P_{Z_i} \dot{Z}_i$. The associated pricing strategy is the mark-up rule

$$P_i = \frac{\epsilon}{\epsilon - 1} Z_i^{-\theta}. \tag{7}$$

The firm's instantaneous profit can be written

$$\Pi_i = \frac{\Lambda e^{\lambda t} E}{\epsilon} \frac{Z_i^{\theta(\epsilon-1)}}{\int_0^N Z_j^{\theta(\epsilon-1)} dj} - \phi - P_{Z_i} \dot{Z}_i.$$

Differentiating under the assumption that the firm takes the denominator as given, substituting the resulting expression into the asset-pricing equation derived above, and rearranging terms yields

[2] Cost reduction and quality improvements are isomorphic in this environment where customers care about the services they derive from the products that they purchase. Hence, the vertical dimension of innovation can be either quality (product innovation) or productivity (process innovation) without changing the paper's insight.

$$r = \frac{\Lambda e^{\lambda t} E}{\epsilon} \theta (\epsilon - 1) \frac{Z_i^{\theta(\epsilon-1)-1}}{\int_0^N Z_j^{\theta(\epsilon-1)} dj} \frac{1}{P_{Z_i}} + \frac{\dot{P}_{Z_i}}{P_{Z_i}}.$$

This expression characterizes the demand for vertical (incremental) innovations in sector i.

Innovation projects targeted at sector i produce an improvement of the production process that the inventor can patent and then sell to firm i. The cost of the R&D project is determined by the technology

$$\dot{Z}_i = \alpha Z_i L_{Z_i}, \quad \alpha > 0 \tag{8}$$

where $\dot{Z}_i$ is the flow of innovations generated by employing L_{Z_i} units of labor for an interval of time dt and αZ_i is the productivity of labor in R&D as determined by the exogenous parameter α and by the stock of sector-specific public knowledge, Z_i.[3] Since innovations are carried out by independent inventors, a standard free-entry condition characterizes research targeted at sector i so that $P_{Z_i} = 1/\alpha Z_i$.

Independent inventors can also develop products (and the associated production processes) that are entirely new. When an inventor develops a new good, he can patent the blueprint and sell it to a production firm. Without loss of generality let this firm be firm i and denote the price of this "founding" patent P_{N_i}. Once set up, the new firm operates according to the structure set up above. In particular, the value of the new firm is V_i and follows the standard asset-pricing equation

$$r = \frac{\Pi_i}{V_i} + \frac{\dot{V}_i}{V_i}.$$

Since the inventor can extract the full value of commercializing the new good under monopoly conditions, we have $P_{N_i} = V_i$. The asset-pricing equation then becomes

$$r = \left[\frac{\Lambda e^{\lambda t} E}{\epsilon} \frac{Z_i^{\theta(\epsilon-1)}}{\int_0^N Z_j^{\theta(\epsilon-1)} dj} - \phi - P_{Z_i} \dot{Z}_i \right] \frac{1}{P_{N_i}} + \frac{\dot{P}_{N_i}}{P_{N_i}}.$$

This expression characterizes the demand for horizontal innovations.

The technology for creating new goods is

$$\dot{N} = \beta L_N N, \quad \beta > 0 \tag{9}$$

where $\dot{N}$ is the flow of new products generated by employing L_N units of labor for an interval of time dt and βN is the productivity of labor in R&D as determined by the parameter β and by the stock of public knowledge N. This innovation technology is *qualitatively identical*

[3] If knowledge is sector- or product-specific, there is an inherent advantage to bringing R&D operations in house since it internalizes the intertemporal spillover. For the sake of simplicity we abstract from this consideration in this paper, and eliminate *by construction* the possibility that firms internalize increasing returns to accumulation of sector-specific knowledge by assuming that innovations are undertaken by independent inventors. Since knowledge is sector-specific, however, we retain the property that resources are spread more thinly and knowledge is more specialized in an economy that produces a larger variety of goods so that the scale effect does not arise. This observation might lead one to wonder about *inter-sectoral* spillovers. We could accommodate these without changing the qualitative results of the paper if we posited that R&D costs in each sector depend on the knowledge aggregator

$$K = \int_0^N \frac{1}{N} Z_i di.$$

As discussed in detail in Aghion and Howitt (1998), Peretto (1998) and Peretto and Smulders (2002), as long the knowledge aggregator does not rise with the number of sectors/goods, there is no scale effect.

to that posited for sector-specific incremental innovation. In fact, one can interpret creation of new products as the $(N + 1)$ th R&D sector in this economy. The standard free-entry condition for this research sector yields $P_{N_i} = 1/\beta N$.[4]

Under the assumption that setting up a new firm requires no expenditure other than purchasing the patent for the new good, the free-entry condition above defines the entry cost faced by entrepreneurs in this economy. Introducing additional entry costs would add some realism but it would make the two innovation technologies qualitative different. This feature would mask the differences between the two margins of technological advance, vertical and horizontal, that we wish to emphasize in this paper. We discuss this aspect of the analysis in more detail in Sect. 3.

2.3 The returns to R&D

We focus on symmetric equilibria to keep the analysis as simple as possible.[5] The rates of return to innovation are:

$$r_Z = \left[\frac{\Lambda e^{\lambda t} E}{\epsilon N} \frac{\theta (\epsilon - 1)}{Z} \right] \frac{1}{P_Z} + \frac{\dot{P}_Z}{P_Z};$$

$$r_N = \left[\frac{\Lambda e^{\lambda t} E}{\epsilon N} - \phi - P_Z \dot{Z} \right] \frac{1}{P_N} + \frac{\dot{P}_N}{P_N}.$$

The similarity of the asset-pricing equations highlights the fact that the only difference between the returns to vertical and horizontal R&D arises from the dividend part of the dividend–price ratio. We can make the following observations in this regard.

First, the $N + 1$ innovation technologies operated in this economy are qualitatively identical. Hence, the features of the equilibrium that we describe below do not depend on *technological* differences across R&D activities. Allowing for such differences might add some realism to our description of innovation activities, but it would mask the source of the qualitative differences that we wish to emphasize in this paper.

Second, in the vertical dimension of innovation we posit incremental improvements of products/processes that are brought to market by existing local monopolists. Hence, the Arrow replacement effect is internalized and the return to cost-reducing R&D within each sector depends on *marginal* profit. In contrast, innovation in the horizontal dimension entails the creation of a new firm/product and there is no Arrow replacement effect because there is no existing incumbent to replace. Accordingly, the return to variety-expanding R&D depends on profit. Our assumption that all innovations are developed by independent inventors and then sold to production firms is unrealistic but has the advantage of making clear that an important difference between vertical and horizontal innovation in our scheme stems from the demand side, that is, from the different *users* of the innovations: in the vertical dimension the user is an existing firm, in the horizontal dimension it is a new firm.[6]

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4 Although we do not present the details here, the arguments below go through if horizontal innovation is instead defined as $\dot{N} = \beta L_N f(N)$, where $f(N)$ is a non-linear function. Similarly, we can allow for average vertical knowledge, Z, to play an explicit role in the function $f(\cdot)$, like in Peretto and Smulders (2002), without changing the basic insight of the model.

5 See, e.g., Peretto (1998) for a discussion of the conditions under which models of this class support symmetric equilibria.

6 Introducing creative-destruction along the vertical dimension, whereby all innovations are brought to market by new firms that replace the existing incumbents, would make the analysis much more complicated, but would not change the basic insight of our model. The reason is that the new firm that replaces the old monopolist

Third, the entry cost $1/\beta N$ bounds the change in the number of products per unit of time, $\dot{N}$, while the fixed operating cost ϕ bounds the number of firms that are active at any point in time, N. The difference is crucial. Specifically, the cost of variety-expanding innovation falls linearly with the number of goods so that, from the technological viewpoint, self-sustaining, perpetual variety expansion is possible. The reason why this does not happen in equilibrium is that with fixed costs of production in units of labor, an infinite variety of goods cannot be produced if the labor endowment is finite. Hence, the most important way in which vertical and horizontal innovations differ is that vertical innovations do not replicate fixed production costs while horizontal innovations do. This property has nothing to do with the innovation process but rather with the fact that commercialization of the new product entails setting up production facilities. In this sense the emphasis is on N as the number of *firms/plants*, not as the number of goods/blueprints. Once one takes this into account, it is clear that the rate of return to horizontal innovation falls with the number of firms/plants even if the innovation cost falls linearly with the accumulated stock of horizontal (product) innovations.[7]

More formally, consider the free-entry conditions characterizing patent prices and write

$$r_Z = \alpha \left[\frac{\Lambda e^{\lambda t} E\theta \left(\epsilon - 1\right)}{\epsilon N} - L_Z \right]; \tag{10}$$

$$r_N = \left[\frac{\Lambda e^{\lambda t} E}{\epsilon N} - \phi - L_Z \right] \beta N - \frac{\dot{N}}{N}. \tag{11}$$

The fact that labor productivity in horizontal R&D is linear in N offsets the negative linear dependence of cash flow per firm on N (a market share effect), but cannot offset the fact that a larger number of firms yields a larger demand for labor through the fixed labor requirement so that rising real wages depress profits and thus the return to setting up new firms. To see this point most starkly, imagine an equilibrium with zero vertical R&D so that $L_Z = 0$ and observe that *only* the case $\phi = 0$ yields a constant rate of return to entry with a constant rate of entry; this is shown formally below.

2.4 Aggregate dynamics

General equilibrium is defined by the Euler equation (4), the return to innovation (10), the return to entry (11) and the labor market clearing condition

$$L = N \left[\frac{\Lambda e^{\lambda t} E \left(\epsilon - 1\right)}{\epsilon N} + \phi + L_Z \right] + \frac{\dot{N}}{\beta N}.$$

Footnote 6 continued
enters an already existing product line and does not create a new one. After completion of this paper, we came across Cozzi and Spinesi (2002), which works out a quality-ladder model built on Howitt (1999) with fixed operating costs. Our intuitive argument above is supported by their analytical work. Their paper differs from ours in that we provide a more general interpretation of the role of fixed operating costs in growth theory, we work out in detail the transitional dynamics under different hypotheses concerning horizontal innovation costs, and our model is more tractable. Finally, their research question differs from ours since it focuses on scale effects, which are peripheral to our argument.

[7] One way to check the generality of these statements is to extend the theory to the case of multiproduct firms with plants that can produce more than one good. We are not aware of work in this direction with the exception of a section of Smulders and van de Klundert (1995) and, more recently, Minniti (2006). Both examples support the conclusion that allowing for multiproduct firms does not change the key properties of these models.

On the left hand side there is labor supply, which is given by

$$L = \Lambda e^{\lambda t}.$$

For the purposes of the analysis below, it is useful to set $\lambda = 0$. We consider the role of population growth in the next section.

No-arbitrage between returns to vertical and horizontal R&D, or more generally equilibrium of the assets market, requires $r_Z = r_N = r$. We refer to r as the rate of return to investment. Solving the resource constraint for L_Z and substituting the result into the rate of return to horizontal innovation we obtain

$$r = (E - 1)\,\Lambda\beta.$$

Notice that the rate of return to investment is independent of the number of products. This result stems from the assumption that the cost of horizontal innovation is inversely related to N. In fact, this is precisely what delivers endogenous growth in the first-generation models of variety expansion (e.g., Romer 1990; Grossman and Helpman 1991, Ch. 3). Since the rate of return to investment, in this case horizontal innovation, must equal the rate of return to saving, we have

$$(E - 1)\,\Lambda\beta = \frac{\dot{E}}{E} + \rho.$$

This unstable differential equation implies that expenditure per capita jumps to the steady-state value

$$E^* = 1 + \frac{\rho}{\Lambda\beta}, \tag{12}$$

while the interest rate is at all times $r^* = \rho$.

These results allow us to solve (10) for

$$z \equiv \frac{\dot{Z}}{Z} = \alpha L_Z = \frac{\Lambda E^* \alpha\theta\,(\epsilon - 1)}{\epsilon N} - \rho. \tag{13}$$

An important feature of this equation is that $z = 0$ for

$$N \geq \bar{N} \equiv \frac{\Lambda E^* \alpha\theta\,(\epsilon - 1)}{\rho\epsilon}.$$

This threshold is due to the fact that when there are too many products, demand for each product is too small and firms would earn a return on the patents that they purchase that is below the interest rate. Consequently, demand for vertical innovations falls to zero.

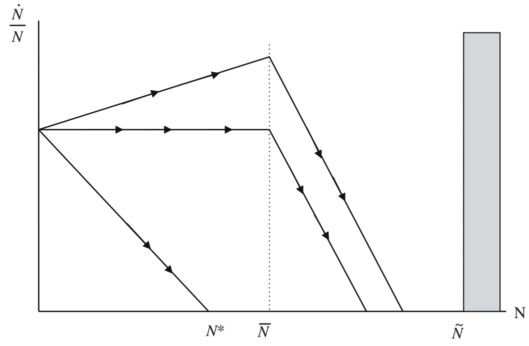
Substitution of these results into the resources constraint yields

$$\frac{\dot{N}}{N} = \begin{cases} \beta\left[\Lambda E^* \frac{1-\theta(\epsilon-1)}{\epsilon} - \left(\phi - \frac{\rho}{\alpha}\right) N\right] - \rho & N < \bar{N} \\ \beta\left[\Lambda E^* \frac{1}{\epsilon} - \phi N\right] - \rho & N \geq \bar{N} \end{cases}.$$

The tractability of this model is evident in the fact that its general equilibrium reduces to a single differential equation in the number of firms. Figure 1 illustrates dynamics.[8] If $\phi > \frac{\rho}{\alpha}$,

[8] For simplicity we ignore the hysteresis due to the non-negativity constraint on $\dot{N}$. In Fig. 1, this is captured by the continuum of steady states between N^* and $\tilde{N}$. The value $\tilde{N}$ denotes the maximum number of firms that the market can accommodate with non-negative profits. Since firms have the option of shutting down vertical R&D operations, the non-negativity constraint on flow profits, which reflects the exit option, yields $\tilde{N} = \Lambda E/\epsilon\phi$.

Fig. 1 General Equilibrium Dynamics



the dynamics always feature a falling entry rate. In contrast, if $\phi < \frac{\rho}{\alpha}$ the entry rate is initially rising, until the economy crosses the threshold $\bar{N}$ whereafter the entry rate is negative. If $\phi = \frac{\rho}{\alpha}$, the entry rate is initially constant and turns negative when vertical R&D shuts down. In all cases, there exists a steady state value

$$N^* = \begin{cases} \frac{\Lambda E^* \frac{1-\theta(\epsilon-1)}{\epsilon} - \frac{\rho}{\beta}}{\phi - \frac{\rho}{\alpha}} & \frac{\Lambda E^*}{\epsilon}\left[1 - \frac{\phi\alpha\theta(\epsilon-1)}{\rho}\right] < \frac{\rho}{\beta} \\ \frac{\Lambda E^* \frac{1}{\epsilon} - \frac{\rho}{\beta}}{\phi} & \frac{\Lambda E^*}{\epsilon}\left[1 - \frac{\phi\alpha\theta(\epsilon-1)}{\rho}\right] \geq \frac{\rho}{\beta} \end{cases}. \tag{14}$$

These dynamics make clear that $\phi > 0$ kills the possibility of endogenous growth through product proliferation regardless of any other feature of the model. Upon reflection, this is obvious: The presence of the term ϕN on the right hand side of the resources constraint implies that the equation cannot hold for a fixed labor endowment if N grows too large. One should note that passed the threshold $\bar{N}$, where vertical innovation shuts down, the model is in all respects identical to that of Grossman and Helpman (1991,Ch. 3) except for the fixed cost ϕ. So, in this exercise the introduction of fixed operating costs is the *only* radical change that we are making with respect to the standard setup.

An interesting feature of this equilibrium is that there is a negative scale effect with the extreme implication that a large population yields zero growth since it implies that $N^* > \bar{N}$. To see this, substitute N^* into (13) to obtain:

$$z^* = (\alpha\phi - \rho)\frac{\alpha\theta(\epsilon - 1)}{1 - \theta(\epsilon - 1) - \frac{\epsilon\rho}{\beta\Lambda E^*}} - \rho,$$

where

$$\beta\Lambda E^* = \Lambda\beta + \rho.$$

According to this expression, steady-state growth depends negatively on the scale parameter Λ. Why? Because the returns to innovation increase with the scale of operations of the firm. The size of the firm is ultimately determined by the share of aggregate demand that the firm captures. Aggregate demand is given by population size times expenditure per capita. Population size reduces these returns, since the number of firms rises more than one-for-one with aggregate demand. This is because the effective cost of entry is $1/\beta\Lambda E^*$, which decreases with Λ. We thus have a crowding-in effect that results in a negative scale effect.

3 The extended model

In this section we discuss the role of different assumptions concerning entry. We first show that these assumptions determine whether the model can handle population growth in a tractable manner. This is important, because a model that requires constant population is overly restrictive theoretically and fails empirically. We then show that for a broad class of specifications, the model's dynamics belong to the logistic class that has a closed-form solution.

We begin with a discussion of why we set $\lambda = 0$ in the analysis of the basic model of the previous section. Let $\lambda > 0$. The differential equation for expenditure is no longer independent of time:

$$(E(t) - 1)\,\Lambda e^{\lambda t}\beta = \frac{\dot{E}(t)}{E(t)} + \rho.$$

This pins down a path $E(t)$ that is not constant. The differential equation for N becomes

$$\frac{\dot{N}(t)}{N(t)} = \begin{cases} \beta\left[\Lambda e^{\lambda t} E(t)\,\frac{1-\theta(\epsilon-1)}{\epsilon} - \left(\phi - \frac{\rho}{\alpha}\right)N(t)\right] & N(t) < N_0(t) \\ \beta\left[\frac{\Lambda}{\epsilon} e^{\lambda t} E(t) - \phi N(t)\right] & N(t) \geq N_0(t) \end{cases},$$

with

$$N_0(t) = \frac{\alpha\Lambda e^{\lambda t}\theta(\epsilon-1)}{\rho\epsilon} E(t).$$

In this setup, demand grows at rate λ while the entry cost does not keep up. Hence, the entry cost *relative* to market size shrinks to zero. This implies that eventually the economy converges to the solution that one would obtain in a model with *no* entry costs and product variety pinned down by fixed operating costs (see, e.g., Smulders and van de Klundert 1995; Peretto 1996).[9] However, the analysis of these dynamics is overly cumbersome, making the model of limited use for applied questions, especially if they concern welfare.

We obtain tractable dynamics if we suitably scale *up* horizontal innovation costs. We discuss two examples that provide insight on the nature of scale effects in models of this class.[10] In particular, we focus on how entry costs regulate how population size and growth affect vertical innovation. In the next section, we focus on how the interaction of the two dimensions of technology drives the overall growth rate of consumption.

3.1 Example I: entry cost increasing in population

Suppose horizontal innovation costs increase linearly with population size, perhaps because inventing a new good is harder the more people to whom it must appeal. With an entry cost of

$$\frac{\Lambda e^{\lambda t}}{\beta N},$$

[9] Models thus built have the same qualitative steady-state properties as those that we consider here. They just compress the transitional dynamics to a jump to the new steady state by positing that the number of products/firms can change discretely.

[10] The case $\dot{N} = \beta L_N$, where entry costs do not depend on accumulated experience in horizontal product development, that the literature has already analyzed (see, e.g., Peretto 1998), does not support potential exponential growth of the number of firms and thus does not yield logistic dynamics in the presence of fixed operating costs.

the rate of return to horizontal innovation becomes

$$r=\left[\frac{\Lambda e^{\lambda t}E}{\epsilon N}-\phi-L_Z\right]\frac{\beta N}{\Lambda e^{\lambda t}}-\frac{\dot{N}}{N}+\lambda.$$

Proceeding as before, we get

$$r=(E-1)\,\beta+\lambda.$$

Asset market clearing then yields

$$(E-1)\,\beta+\lambda=\frac{\dot{E}}{E}+\rho$$

and again we have that E jumps to a steady-state level

$$E^*=1+\frac{\rho-\lambda}{\beta}, \tag{15}$$

with $r^*=\rho$ at all times. Expenditure no longer depends on the population *level*, but instead depends negatively on population *growth*. There are two reasons for this. First, we have eliminated the positive dependence of the rate of return to assets on population size. Second, we now have constant steady-state entry, which is an investment that must be financed through saving. Specifically, $\frac{\lambda}{\beta}$ is the rate of investment in entry in steady state.

To see this more clearly, let $n\equiv N/\Lambda e^{\lambda t}$ denote the number of firms per capita. Vertical growth is

$$z=\frac{E^*\alpha\theta\,(\epsilon-1)}{\epsilon n}-\rho. \tag{16}$$

Again we have that $z=0$ for

$$n\geq\bar{n}\equiv\frac{E^*\alpha\theta\,(\epsilon-1)}{\rho\epsilon}.$$

The entry condition reduces to

$$\frac{\dot{n}}{n}=\begin{cases}\beta\left[E^*\frac{1-\theta(\epsilon-1)}{\epsilon}-\left(\phi-\frac{\rho}{\alpha}\right)n\right]-\rho & n<\bar{n}\\ \beta\left[E^*\frac{1}{\epsilon}-\phi n\right]-\rho & n\geq\bar{n}\end{cases}.$$

Dynamics are qualitatively similar to those discussed in the previous section with the difference that in a neighborhood of the steady state there is no hysteresis since population growth allows $\dot{n}<0$. The steady state value of n is

$$n^*=\begin{cases}\frac{1}{\phi-\frac{\rho}{\alpha}}\left[E^*\frac{1-\theta(\epsilon-1)}{\epsilon}-\frac{\rho}{\beta}\right] & \frac{E^*}{\epsilon}\left[1-\frac{\phi\alpha\theta(\epsilon-1)}{\rho}\right]<\frac{\rho}{\beta}\\ \frac{1}{\phi}\left[\frac{E^*}{\epsilon}-\frac{\rho}{\beta}\right] & \frac{E^*}{\epsilon}\left[1-\frac{\phi\alpha\theta(\epsilon-1)}{\rho}\right]\geq\frac{\rho}{\beta}\end{cases}. \tag{17}$$

Steady-state vertical growth spending is

$$z^*=(\alpha\phi-\rho)\,\frac{\theta\,(\epsilon-1)}{1-\theta\,(\epsilon-1)-\frac{\epsilon\rho}{\beta E^*}}-\rho, \tag{18}$$

where

$$\beta E^*=\beta+\rho-\lambda.$$

The scale parameter Λ now drops out, implying no scale effect. Vertical growth, however, now depends on population growth, λ, through expenditure per capita because financing constant product creation requires a higher saving rate.

The dynamical system is in fact simpler than the graphical analysis of Fig. 1 suggests. To explore this property, we focus on the region of parameter space that yields a steady state with positive vertical R&D and work with

$$\frac{\dot{n}}{n} = \eta - \beta\left(\phi - \frac{\rho}{\alpha}\right)n, \quad \eta \equiv \beta E^* \frac{1-\theta(\epsilon-1)}{\epsilon} - \rho.$$

This is a logistic equation with growth coefficient η and crowding coefficient $\beta\left(\phi - \frac{\rho}{\alpha}\right)$.[11] Using the value n^* in (17), also called the system's carrying capacity, we can rewrite it as

$$\frac{\dot{n}}{n} = \eta\left(1 - \frac{n}{n^*}\right),$$

which has solution

$$n(t) = \frac{n^*}{1 + e^{-\eta t}\left(\frac{n^*}{n_0} - 1\right)}, \tag{19}$$

where n_0 is the initial condition.

3.2 Example II: entry cost proportional to initial variable cost of production

Another way to scale up the entry cost is to assume that it is proportional to the initial variable cost of production. The idea is that a firm setting up operations incurs the cost of building prototypes of the new products.[12] This assumption also captures the notion that the setup costs increase with the anticipated volume of output. With an entry cost of

$$\frac{\Lambda e^{\lambda t} E(\epsilon - 1)}{\beta \epsilon N},$$

the rate of return to horizontal innovation becomes

$$r = \left[\frac{\Lambda e^{\lambda t} E}{\epsilon N} - \phi - L_Z\right]\frac{\beta \epsilon N}{\Lambda e^{\lambda t} E(\epsilon - 1)} - \frac{\dot{N}}{N} + \lambda + \frac{\dot{E}}{E}.$$

Proceeding as before, we get

$$r = \left(1 - \frac{1}{E}\right)\frac{\beta\epsilon}{\epsilon - 1} + \lambda + \frac{\dot{E}}{E}.$$

Asset market clearing yields

$$\left(1 - \frac{1}{E}\right)\frac{\beta\epsilon}{\epsilon - 1} + \lambda = \rho$$

and again we have that E jumps to a steady-state level

$$E^* = \left[1 - \frac{(\rho - \lambda)(\epsilon - 1)}{\beta\epsilon}\right]^{-1}, \tag{20}$$

[11] For an exhaustive discussion of the logistic equation, its properties, solution and applications, see, e.g., Banks (1994).

[12] We borrow this argument from Etro (2004,p. 299). See also Barro and Sala-i-Martin (2004,Ch. 6).

with $r^* = \rho$ at all times. Again, expenditure no longer depends on the population *level* but depends negatively on population *growth.* Vertical growth is

$$z = \frac{E^* \alpha \theta (\epsilon - 1)}{\epsilon n} - \rho. \tag{21}$$

Again we have that $z = 0$ for

$$n \geq \bar{n} \equiv \frac{E^* \alpha \theta (\epsilon - 1)}{\rho \epsilon}.$$

The entry condition reduces to

$$\frac{\dot{n}}{n} = \begin{cases} \frac{1}{E^*} \frac{\beta \epsilon}{\epsilon - 1} \left[E^* \frac{1 - \theta(\epsilon - 1)}{\epsilon} - \left(\phi - \frac{\rho}{\alpha} \right) n \right] - \rho & n < \bar{n} \\ \frac{1}{E^*} \frac{\beta \epsilon}{\epsilon - 1} \left[E^* \frac{1}{\epsilon} - \phi n \right] - \rho & n \geq \bar{n} \end{cases}.$$

This system's qualitative behavior is identical to that of the previous example. The steady state value of n is

$$n^* = \begin{cases} \frac{E^*}{\phi - \frac{\rho}{\alpha}} \left[\frac{1 - \theta(\epsilon - 1)}{\epsilon} - \frac{\rho(\epsilon - 1)}{\beta \epsilon} \right] & 1 - \frac{\phi \alpha \theta (\epsilon - 1)}{\rho} < \frac{(\epsilon - 1)\rho}{\beta} \\ \frac{E^*}{\phi} \left[\frac{1}{\epsilon} - \frac{\rho(\epsilon - 1)}{\beta \epsilon} \right] & 1 - \frac{\phi \alpha \theta (\epsilon - 1)}{\rho} \geq \frac{(\epsilon - 1)\rho}{\beta} \end{cases}. \tag{22}$$

The solution $n^* < \bar{n}$ can be written

$$\frac{E^*}{n^*} = \left(\phi - \frac{\rho}{\alpha} \right) \left[\frac{1 - \theta (\epsilon - 1)}{\epsilon} - \frac{\rho (\epsilon - 1)}{\beta \epsilon} \right]^{-1}.$$

This is important because it says that firm size is independent of population size and growth. Steady-state vertical growth then is

$$z^* = \left(\phi - \frac{\rho}{\alpha} \right) \left[1 - \theta (\epsilon - 1) - \frac{\rho (\epsilon - 1)}{\beta} \right]^{-1} \theta (\epsilon - 1) - \frac{\rho}{\alpha}, \tag{23}$$

which is independent of both population size and growth.

Once again, we can write the transition path in a neighborhood of this steady state as governed by

$$\frac{\dot{n}}{n} = \nu - \frac{1}{E^*} \frac{\beta \epsilon}{\epsilon - 1} \left(\phi - \frac{\rho}{\alpha} \right) n \quad \nu \equiv \beta \frac{1 - \theta (\epsilon - 1)}{\epsilon - 1} - \rho.$$

Using the value for n^* in (22) we can rewrite this equation as

$$\frac{\dot{n}}{n} = \nu \left(1 - \frac{n}{n^*} \right),$$

which has solution

$$n(t) = \frac{n^*}{1 + e^{-\nu t} \left(\frac{n^*}{n_0} - 1 \right)},$$

where n_0 is the initial condition.

An interesting feature of the specifications discussed above is that the qualitative differences in how steady-state vertical growth depends on population size and growth are due solely to different assumptions concerning the entry cost, that is, the technology for horizontal innovation which regulates how changes in market size translate into changes in firm size. As we noticed in the introduction, the integration of the two dimensions of technological change—vertical and horizontal—produces a more comprehensive theory of economic growth. The vertical dimension provides the opportunity of growth unconstrained by endowments, the horizontal dimension provides the mechanism that translates changes in aggregate variables into changes in firm-level variables. Our examples show how the latter ultimately drive incentives to push the technological frontier in the vertical dimension.

The question we tackle next is: Why logistic growth in the number of firms? The answer is straightforward and intuitive. The primitives of the two models outlined above (and of the basic model of Sect. 2) have two things in common. First, the technology for product development features a cost of entry that is decreasing linearly in the number of products. As is well known, this feature yields the potential for exponential growth through product proliferation. Second, fixed operating costs imply that the economy has limited carrying capacity with respect to the number of active product lines—or, better, the number of active firms. In particular, the crowding process is linear in the number of products/firms. Now, exponential growth plus linear crowding yield logistic growth (Banks 1994), a process with well-known properties and a closed-form solution for the transition path. What is remarkable is that this logistic process arises naturally in our environment where to a set of standard, well-understood ingredients we only add fixed operating costs. Our Manhattan Metaphor captures conceptually why we obtain this mathematical representation by placing the emphasis on the limited carrying capacity of the system with respect to the variable—the number of products/firms—whose growth requires replication of fixed operating costs.

We close this section by noting that it is possible to extend the analysis of this model to the case of endogenous population growth along the lines of Connolly and Peretto (2003). The resulting model displays constant product proliferation driven by constant *endogenous* population growth, together with constant productivity growth in the vertical dimension.

4 The advantage of tractability

In this section we discuss the main strength of the models proposed above. We focus on the fact that they admit a closed-form solution for the transition path and illustrate how this property allows us to study analytically the welfare effects of changes in fundamentals.

For the purposes of this discussion we disentangle the elasticity of product substitution, ϵ, from social increasing returns to variety in our preferences by rewriting (2) as

$$C = N^{\omega - \frac{\epsilon}{\epsilon-1}} \left[\int_0^N C_i^{\frac{\epsilon-1}{\epsilon}} di \right]^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1, \omega \geq 0, \tag{24}$$

where ω measures social returns to variety. Because this is an external effect, it does not affect the behavior of agents. The demand curve (5), the price strategy (7) and symmetry then yield that at any point in time

$$C^* = \frac{\epsilon - 1}{\epsilon} E^* N^{\omega} Z^{\theta}. \tag{25}$$

One can reinterpret (24) as a production function for a final homogenous good assembled from intermediate goods—and thus think of C^* as output *per capita*—and define aggregate total factor productivity (TFP) as

$$T = N^{\omega} Z^{\theta}. \tag{26}$$

In steady state this gives us

$$\left(\frac{\dot{T}}{T}\right)^{*} = \omega\lambda + \theta z^{*} \equiv g^{*}, \tag{27}$$

which says that TFP growth has two components: one fully endogenous, θz^{*}, and one semi-endogenous, $\omega\lambda$.

We focus on the model developed in Example II since it is the one that, with the modification outlined above, fits best two sets of well-documented facts that are particularly relevant to this class of models.[13] The first concerns cross-country data: (i) the correlation between income per capita growth and population size is zero; (ii) the correlation between income per capita growth and population growth is also zero. The second concerns time-series data: (iii) the aggregate R&D share of labor exhibits no trend; (iv) the ratio of the number of firms to total employment exhibits no trend; and (v) average total employment and average R&D employment per firm exhibit no trend.

To see this, observe that in both models the typical incumbent firm employs in production operations and R&D, respectively,

$$L_X = \frac{E}{n}\frac{\epsilon - 1}{\epsilon} + \phi$$

and

$$L_Z = \frac{E}{n}\frac{\theta(\epsilon - 1)}{\epsilon} - \frac{\rho}{\alpha}$$

workers. Both models predict that in steady state $n \equiv N/L$ is constant, so they both fit fact (iv). Both models predict that in steady state the ratio E/n is constant, so they both fit facts (iii) and (v) since L_X, L_Z and $L_X + L_Z$ are all constant. The model of Example I, however, predicts that in steady state E/n is decreasing in population growth and thus fails to fit fact (ii) since it predicts a negative correlation between λ and z^{*}. The model of Example II, instead, predicts that in steady state E/n is independent of population growth and thus predicts that the correlation between λ and z^{*} is zero, fitting fact (ii) *provided that* ω is sufficiently close to zero. Fact (iii) concerns the aggregate share of R&D, which in the model of Example II is

$$\frac{NL_Z + L_N}{L} = n\left(L_Z + \frac{\lambda}{\beta}L_X\right)$$

and therefore constant in steady state.

Now define

$$\Delta \equiv \frac{n^{*}}{n_0} - 1.$$

This is the percentage change in product variety that the economy experiences along the transition to the steady state n^{*} starting from initial condition n_0. With this definitions in hand, it is then possible to establish the following.

Proposition 1 *Let* $\log C^{*}(t)$ *and* U^{*} *be, respectively, the instantaneous utility index* (24) *and welfare function* (1) *evaluated at* E^{*}*. Then, a path starting at time* $t = 0$ *with initial*

[13] See, e.g., Barro and Sala-i-Martin (2004), Laincz and Peretto (2006) and Ha and Howitt (2007) for details on these facts and the data sets and techniques used to uncover them.

condition n_0 *and converging to the steady state* n^* *is characterized by:*

$$\log T(t) = \log\left(Z_0^{\theta} n_0^{\frac{1}{\epsilon-1}}\right) + g^* t + \frac{\gamma\Delta}{\nu}\left(1 - e^{-\nu t}\right) + \omega \log \frac{1+\Delta}{1+\Delta e^{-\nu t}}, \tag{28}$$

where

$$\gamma \equiv \theta \frac{\alpha\theta(\epsilon-1)}{\epsilon}\frac{E^*}{n^*} = \theta\left(z^* + \rho\right),$$
$$\nu \equiv \beta \frac{1-\theta(\epsilon-1)}{\epsilon-1} - \rho.$$

Therefore,

$$\log C^*(t) = \log E^* + g^* t + \frac{\gamma\Delta}{\nu}\left(1 - e^{-\nu t}\right) + \omega \log \frac{1+\Delta}{1+\Delta e^{-\nu t}}, \tag{29}$$

which yields

$$U^* = \frac{1}{\rho-\lambda}\left[\log E^* + \frac{g^*}{\rho-\lambda} + \frac{\gamma\Delta}{\rho-\lambda+\nu}\right] + \omega\int_0^{\infty} e^{-(\rho-\lambda)t} \log\frac{1+\Delta}{1+\Delta e^{-\nu t}} dt. \tag{30}$$

Proof See the Appendix. □

We can now investigate in a straightforward manner how specific parameters affect productivity and welfare. An interesting one, that has been the subject of much debate, is population size. A change in population size shows up as a shock to the model's initial condition. Specifically, imagine an economy in steady state n_0 and let Λ increase to Λ'. Such an increase has no effect on E^* and on steady-state firm size E^*/n^*. Therefore, at the end of the transition we have $n^* = n_0$. Recall that by definition $n_0 = N_0/e^{\lambda t}\Lambda$ and $n^* = N^*/e^{\lambda t}\Lambda'$. The transition to the new steady state starts from $n_0' = N_0/e^{\lambda t}\Lambda' = n_0\Lambda/\Lambda'$. We thus have that

$$\Delta = \frac{n^*}{n_0'} - 1 = \frac{\Lambda'}{\Lambda} - 1 > 0$$

is the percentage increase in population. We observe next that the increase in Λ has no effect on γ, ν, g^*, E^*, and differentiate (28) to write:

$$\frac{d\log T}{dt} = \frac{\dot{T}}{T} = g^* + \Delta e^{-\nu t}\left[\gamma + \frac{\omega\nu}{1+\Delta e^{-\nu t}}\right];$$
$$\frac{d^2\log T}{dt^2} = \frac{d}{dt}\left(\frac{\dot{T}}{T}\right) = -\Delta\nu e^{-\nu t}\left[\gamma + \frac{\omega\nu}{\left(1+\Delta e^{-\nu t}\right)^2}\right].$$

We then conclude that a helicopter drop of people raises welfare because it induces a temporary acceleration of productivity growth.

In fact, we can say by how much. The change in welfare given by a generic change in population size is

$$U^* - U_0 = \frac{1}{\rho - \lambda}\left[\log E^* + \frac{g^*}{\rho - \lambda} + \frac{\gamma \Delta}{\rho - \lambda + \nu}\right]$$
$$+\omega \int_0^\infty e^{-(\rho-\lambda)t} \log \frac{1+\Delta}{1+\Delta e^{-\nu t}} dt$$
$$-\frac{1}{\rho - \lambda}\left[\log E_0 + \frac{g_0}{\rho - \lambda} + 0\right]$$
$$-\omega \int_0^\infty e^{-(\rho-\lambda)t} \log \frac{1}{1} dt$$
$$= \frac{1}{\rho - \lambda}\frac{\gamma \Delta}{\rho - \lambda + \nu} + \omega \int_0^\infty e^{-(\rho-\lambda)t} \log \frac{1+\Delta}{1+\Delta e^{-\nu t}} dt,$$

since $E^* = E_0$ and $g^* = g_0$. If, to fit the fact that the correlation between income per capita growth and population growth is zero, we postulate $\omega = 0$, then

$$U^* - U_0 = \frac{1}{\rho - \lambda}\frac{\gamma}{\rho - \lambda + \nu}\Delta$$

and we can refer to

$$\frac{1}{\rho - \lambda}\frac{\gamma}{\rho - \lambda + \nu} = \frac{1}{\rho - \lambda}\frac{\theta\left(z^* + \rho\right)}{\rho - \lambda + \nu}$$

as the welfare multiplier of an increase in population size. This expression tells us that this multiplier is larger in an economy with growth favoring fundamentals because it translates into a larger temporary acceleration of vertical growth that yields a higher steady-state level of productivity than in the baseline case. In other words, the economy makes a transition to a steady-state growth path with the same slope as, but a higher intercept than, the starting one. The welfare multiplier calculated above tells us by how much the latter level effect raises welfare. It also tells us that the welfare increase is smaller, the faster the transition—this is captured by the presence in the denominator of the growth coefficient of the logistic equation, ν. In other words, the temporary acceleration of TFP growth is due to the fact that the number of firms adjusts slowly so that there is a temporary scale effect on incumbent firms. If we worked with a model where entry costs are zero (i.e., $\beta \to \infty$), then we would have $\nu \to \infty$ and the welfare multiplier would be zero because firm size would adjust instantaneously to the larger market.

Another important parameter is population growth. Again, imagine an economy in steady state n_0 and let λ increase to λ'. According to (22) and (20) this change reduces steady-state n so that we now have that at the end of the transition $n^* < n_0$. The transition starts from n_0 since population is pre-determined and its acceleration does not affect its level at time $t = 0$. Recall, moreover, that a change in population growth does not affect firm size E/n. Consequently, we have that

$$\Delta = \frac{n^*}{n_0} - 1 = \frac{E^*}{E_0} - 1 = \frac{\frac{(\lambda-\lambda')(\epsilon-1)}{\beta\epsilon}}{1 - \frac{(\rho-\lambda')(\epsilon-1)}{\beta\epsilon}} < 0$$

is the percentage decrease in expenditure. We observe next that the increase in λ has no effect on γ, ν, and notice that according to the derivatives calculated above $\Delta < 0$ implies that the path of $\log T$ is convex. We then have two cases. If population growth has no long-run effect on TFP growth because $\omega = 0$, then the path entails a temporary slowdown. If, instead, we allow for the semi-endogenous growth component by setting $\omega > 0$, then the path of TFP entails either a temporary slowdown followed by a permanent acceleration or just an acceleration.

Regardless of the specific path of TFP, however, we can be quite precise about the effects of the change in population growth on welfare. Inspection of (30) reveals that λ enters in four places. First, faster population growth reduces the effective discount rate $\rho - \lambda$ and thus raises welfare directly. Second, faster population growth lowers E^*; see (20). Third, the lower E^* yields a lower n^*, which implies $\Delta < 0$. Fourth, faster population growth raises g^* through the semi-endogenous growth component $\omega\lambda$. On the other hand, γ and ν remain the same. We can then organize the welfare effects in two components. The change in welfare given by a generic change in population growth is

$$\begin{aligned}
U^* - U_0 = & \frac{1}{\rho-\lambda'}\left[\log E^* + \frac{g^*}{\rho-\lambda'} + \frac{\gamma\Delta}{\rho-\lambda'+\nu}\right] \\
& +\omega\int_0^\infty e^{-(\rho-\lambda')t}\log\frac{1+\Delta}{1+\Delta e^{-\nu t}}dt \\
& -\frac{1}{\rho-\lambda}\left[\log E_0 + \frac{g_0}{\rho-\lambda} + 0\right] \\
& -\omega\int_0^\infty e^{-(\rho-\lambda)t}\log\frac{1}{1}dt \\
= & \frac{1}{\rho-\lambda'}\left[\log E^* + \frac{g^*}{\rho-\lambda'}\right] - \frac{1}{\rho-\lambda}\left[\log E_0 + \frac{g_0}{\rho-\lambda}\right] \\
& +\frac{1}{\rho-\lambda'}\frac{\gamma\Delta}{\rho-\lambda'+\nu} + \omega\int_0^\infty e^{-(\rho-\lambda')t}\log\frac{1+\Delta}{1+\Delta e^{-\nu t}}dt.
\end{aligned}$$

The first line of the final expression gives the welfare effect of changing the steady state, the second gives the effect due to the transition to the new steady state. Observe that $\Delta < 0$ implies $\frac{1+\Delta}{1+\Delta e^{-\nu t}} < 1$ so that the transition reduces welfare. For population growth to be welfare improving then it *must* be that the steady-state effect is positive and sufficiently strong.

The comparison of the two steady states reveals an interesting trade-off. Faster population growth reduces steady-state consumption per capita, because the economy must support faster entry, but faster entry translates into faster growth of income per capita. The importance of the latter component of this trade-off depends on the strength of social increasing returns to variety. In light of the empirical fact that the correlation between income per capita growth and population growth *is* zero, one could argue that we should write the model with ω approximately zero, so that $g^* = g_0$ and faster population growth potentially reduces welfare because it requires a sacrifice of steady-state consumption not justified by the contribution to productivity of faster product proliferation. The only force left to counteract this negative effect is the fact that faster population growth effectively makes people more patient. This, in turn, suggests that the form of preferences that one posits is crucial. If we posited preferences whereby individuals do not care about their family members, then we would have that faster population growth reduces welfare because it requires a sacrifice of steady state consumption that is not justified by its direct and indirect (through productivity growth) contribution to individual utility.

5 Discussion

What do the simple models outlined in the previous sections teach us about R&D-driven growth? First, product proliferation and productivity improvement are *not* isomorphic under realistic characterizations of production operations. Second, linearity of the accumulation equation for horizontal innovations is *not* sufficient to obtain perpetual product proliferation.

The implicit—and crucial—assumption at the heart of the variety expansion model of endogenous growth is that fixed production costs are zero. This is equivalent to assuming that there are no constraints to the proliferation of production facilities. Our Manhattan Metaphor is designed to illustrate how thinking about such constraints reveals the importance of apparently innocuous assumptions and draws a sharp distinction between models that are thought to be equivalent. More importantly, it tells us that working with a framework that does not confuse innovation and entry produces a more plausible and useful representation of technology. Finally, it suggests that the emphasis on the knife-edge properties of models of endogenous growth that have at their heart some linear accumulation equation is misplaced. In our view, the linearity of that equation is largely a matter of analytical convenience. The important assumption is less about the form of the primitive equation governing the accumulation of some variable but *whether there are natural limits to the expansion of that variable* that should be taken into account.

The Manhattan Metaphor emphasizes the notion of saturation. A natural way to incorporate it in models of endogenous growth that allow for product variety expansion is to introduce a fixed operating cost per product. We find that when we do so a logistic representation of equilibrium dynamics emerges. This is remarkable for two reasons. First, the logistic equation has a well-known closed-form solution, a fact that makes this specification of the theory very powerful in tackling welfare questions since we know the exact form of the model's transition path. Second, and in our view, more important, the logistic representation emerges because of the combination of two ingredients: the technology for product development features a cost of entry per firm that is decreasing linearly in the number of firms; fixed operating costs imply that the economy has limited carrying capacity with respect to the number of firms and that the associated crowding process is linear in the number of firms. The first ingredient is standard in growth models driven by product-variety expansion; the second is standard in static specialization models. The logistic dynamics of the number of firms that we uncover—a simple and elegant representation of the process of product proliferation—is a natural outcome of using *both* ingredients, instead of just one or the other as the literature typically does.

Upon reflection, using both is a natural thing to do as fixed operating costs and fixed entry costs are *not* the same. Given the resources endowment, the former imply a limit on the number of firms that the economy can accommodate, the latter a limit on the speed with which the economy can reach that limit. In plainer language, the former imply a limit on *how far* the economy can travel in the variety dimension of technology, the latter imply a limit on *how fast*. Logistic dynamics are a natural implication of their interaction.

Appendix: Proof of Proposition 1

Taking logs of (26) yields

$$\log T(t) = \theta \log Z_0 + \theta \int_0^t z(s)\, ds + \omega \log N(t).$$

Using the definition $n \equiv Ne^{-\lambda t}$, the expression for g^* in (27) and adding and subtracting z^* from $z(s)$, we obtain

$$\log T(t) = \theta \log Z_0 + g^* t + \theta \int_0^t \left[z(s) - z^*\right] ds + \omega \log n(t).$$

Using (16), (19) and the definition of Δ we rewrite the third term as

$$\begin{aligned}\theta \int_0^t \left(z(s) - z^*\right) ds &= \theta \frac{\alpha\theta(\epsilon - 1)}{\epsilon} \int_0^t \left(\frac{E^*}{n(s)} - \frac{E^*}{n^*}\right) ds \\ &= \gamma \int_0^t \left(\frac{n^*}{n(s)} - 1\right) ds \\ &= \gamma\Delta \int_0^t e^{-\nu s} ds \\ &= \frac{\gamma\Delta}{\nu}\left(1 - e^{-\nu t}\right),\end{aligned}$$

where

$$\gamma \equiv \theta \frac{\alpha\theta(\epsilon - 1)}{\epsilon} \frac{E^*}{n^*} = \theta\left(z^* + \rho\right).$$

Using (19) and the definition of Δ we rewrite the last term as

$$\begin{aligned}\omega \log n(t) &= \omega \log \frac{n^*}{1 + \Delta e^{-\nu t}} \\ &= \omega \log n_0 + \omega \log \frac{\frac{n^*}{n_0}}{1 + \Delta e^{-\nu t}} \\ &= \omega \log n_0 + \omega \log \frac{1 + \Delta}{1 + \Delta e^{-\nu t}}.\end{aligned}$$

These results yield (28).

We now use (28) and the definition of Δ to write (24) as

$$\begin{aligned}\log u^*(t) &= \log \frac{\epsilon - 1}{\epsilon} + \log E^* + \log T(t) \\ &= \log\left(\frac{\epsilon - 1}{\epsilon} Z_0^{\theta} n_0^{\frac{1}{\epsilon - 1}}\right) + \log E^* + g^* t \\ &\quad + \omega \log \frac{1 + \Delta}{1 + \Delta e^{-\nu t}} + \frac{\gamma\Delta}{\nu}\left(1 - e^{-\nu t}\right).\end{aligned}$$

Without loss of generality, we set

$$\frac{\epsilon - 1}{\epsilon} Z_0^{\theta} n_0^{\frac{1}{\epsilon - 1}} = 1$$

and obtain (29). We then substitute this expression into (1) and write

$$\begin{aligned}U^* = &\int_0^\infty e^{-(\rho - \lambda)t}\left[\log E^* + g^* t\right] dt \\ &+ \frac{\gamma\Delta}{\nu} \int_0^\infty e^{-(\rho - \lambda)t}\left(1 - e^{-\nu t}\right) dt \\ &+ \omega \int_0^\infty e^{-(\rho - \lambda)t} \log \frac{1 + \Delta}{1 + \Delta e^{-\nu t}} dt.\end{aligned}$$

The first and second integrals have straightforward closed form solutions. The third is solvable as well, but it entails a complicated expression containing the hypergeometric function that is not worth using since it adds no insight and does not simplify the algebra in the analysis below. Hence, we obtain (30).

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