1 State Space Form and The Kalman Filter

- Let the following system (x_t are the states and z_t observables)

  – Transition equation

$$x_{t+1} = Fx_t + G\omega_{t+1},\ \omega_{t+1} \sim N(0, Q)$$

  – Measurement equation

$$z_t = H_t' x_t + \upsilon_t,\ \upsilon_t \sim N(0, R)$$

- Assume we want to write the likelihood function of $\{z_t\}_{t=1}^{T}$.

2 Why Normal Distribution?

- Assume

$$Z|w = [X'|w \; Y'|w]' \sim N\left(\left[\begin{array}{c} \overline{x} \\ \overline{y} \end{array}\right], \left[\begin{array}{cc} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{array}\right]\right)$$

- then

$$X|y,w \sim N\left(\overline{x} + \Sigma_{xy}\Sigma_{yy}^{-1}\left(y - \overline{y}\right), \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}\right)$$

3 Some Definitions

- $x_{t|t-1}$ is the random variable x_t (state) conditional on $z^{t-1} \equiv \{z_{t-1}, ..., z_{-1}\}$ the history of observables. Define:
  - $\overline{x}_{t|t-1} \equiv E\left(x_t|z^{t-1}\right)$
  - $\Sigma_{t|t-1} \equiv E\left(\left(x_t - \overline{x}_{t|t-1}\right)\left(x_t - \overline{x}_{t|t-1}\right)'|z^{t-1}\right)$

4 Some Derivations

- If $z_{t|t-1}$ is the random variable z_t (observable) conditional on z^{t-1}, then:

  – $\overline{z}_{t|t-1} \equiv E\left(z_t | z^{t-1}\right) =$

  $$E\left(H_t' x_t + \upsilon_t | z^{t-1}\right) = H_t' \overline{x}_{t|t-1}$$

  – $\Omega_{t|t-1} \equiv E\left(\left(z_t - \overline{z}_{t|t-1}\right)\left(z_t - \overline{z}_{t|t-1}\right)' | z^{t-1}\right)$

  $$= E\left(H_t'\left(x_t - \overline{x}_{t|t-1}\right)\left(x_t - \overline{x}_{t|t-1}\right)' H_t + \upsilon_t \upsilon_t' | z^{t-1}\right)$$

  $$= H_t' \Sigma_{t|t-1} H_t + R$$

- Finally:

  – $E\left(\left(z_t - \overline{z}_{t|t-1}\right)\left(x_t - \overline{x}_{t|t-1}\right)' | z^{t-1}\right)$

  $$= E\left(H_t'\left(x_t - \overline{x}_{t|t-1}\right)\left(x_t - \overline{x}_{t|t-1}\right)' + \upsilon_t\left(x_t - \overline{x}_{t|t-1}\right)' | z^{t-1}\right)$$

  $$= H_t' \Sigma_{t|t-1}$$

5 The Kalman filter first iteration

- Given $\overline{x}_{0|-1}$ and $\Sigma_{0|-1}$, assume

$$[x'_{0|-1}\ z'_{0|-1}]' \sim N\left(\begin{bmatrix} \overline{x}_{0|-1} \\ H'_t\overline{x}_{0|-1} \end{bmatrix}, \begin{bmatrix} \Sigma_{0|-1} & \Sigma_{0|-1}H_t \\ H'_t\Sigma_{0|-1} & H'_t\Sigma_{0|-1}H_t + R \end{bmatrix}\right)$$

- Then $x_{0|0} \sim N\left(\overline{x}_{0|0}, \Sigma_{0|0}\right)$ where
  - $\overline{x}_{0|0} = \overline{x}_{0|-1} + \Sigma_{0|-1}H\left(H'\Sigma_{0|-1}H + R\right)^{-1}\left(z_0 - H'\overline{x}_{0|-1}\right)$
  - $\Sigma_{0|0} = \Sigma_{0|-1} - \Sigma_{0|-1}H\left(H'\Sigma_{0|-1}H + R\right)^{-1}H'\Sigma_{0|-1}$
- Therefore, since $x_{1|0} = Fx_{0|0} + G\omega_{1|0}$ and $z_{1|0} = H'_1x_{1|0} + \upsilon_{1|0}$
  - $\overline{x}_{1|0} = F\overline{x}_{0|0}$
  - $\Sigma_{1|0} = F\Sigma_{0|0}F' + GQG'$
  - $\overline{z}_{1|0} = H'_1\overline{x}_{1|0}$
  - $\Omega_{1|0} = H'_1\Sigma_{1|0}H_1 + R$

6 The Kalman Filter algorithm

- Given $\overline{x}_{t|t-1}$, $\Sigma_{t|t-1}$ and observation z_t
  - $\overline{x}_{t|t} = \overline{x}_{t|t-1} + \Sigma_{t|t-1} H \left(H' \Sigma_{t|t-1} H + R \right)^{-1} \left(z_t - H' \overline{x}_{t|t-1} \right)$
  - $\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1} H \left(H' \Sigma_{t|t-1} H + R \right)^{-1} H' \Sigma_{t|t-1}$
  - $\overline{x}_{t+1|t} = F \overline{x}_{t|t-1}$
  - $\Sigma_{t+1|t} = F \Sigma_{t|t} F' + G Q G'$
  - $\overline{z}_{t+1|t} = H'_{t+1} \overline{x}_{t+1|t}$
  - $\Omega_{t+1|t} = H'_{t+1} \Sigma_{t+1|t} H_{t+1} + R$

7 Writing the likelihood function

We want to write the likelihood function of z^T:

$$\log \ell \left(z^T | F, G, H^T, Q, R \right) = \sum_{t=1}^{T} \log \ell \left(z_t | z^{t-1} F, G, H^T, Q, R \right) =$$

$$-\sum_{t=1}^{T} \left[\frac{N}{2} \log 2\pi + \frac{1}{2} \log \left| \Omega_{t|t-1} \right| + \frac{1}{2} \sum_{t=1}^{T} v_t' \Omega_{t|t-1}^{-1} v_t \right]$$

- $v_t = z_t - \overline{z}_{t|t-1} = z_t - H_t' \overline{x}_{t|t-1}$
- $\Omega_{t|t-1} = H_t' \Sigma_{t|t-1} H_t + R$

8 Initial conditions for the Kalman Filter

1. F has all the eigenvalues inside the unit circle

2. GQG' and R are p.s.d. symmetric

3. $H = \lim_{t\to\infty} H_t$

4.
$$vec\left(\Sigma_{1|0}\right) = (I - F \otimes F)^{-1} vec\,(GQG')$$

$$\Sigma_{t+1|t} \longrightarrow \Sigma$$

$$\Sigma = F\left(\Sigma - \Sigma H\left(H'\Sigma H + R\right)^{-1} H'\Sigma\right) F' + GQG'$$

9 Initial conditions for the Kalman Filter II

1. F has all the eigenvalues inside the unit circle

2. GQG' and R are p.s.d. symmetric, s.p.d.

3. $H = \lim_{t\to\infty} H_t$

4. $\Sigma_{1|0}$ p.s.d. symmetric

$$\Sigma_{t+1|t} \longrightarrow \Sigma$$

Imagine we want to use the following model to explain the joint behavior of $(w_t - p_t, r_t, \triangle p_t, y_t)'$ real wage, nominal interest rate, inflation and output.

Preferences

- Utility function of Household $j \in [0, 1]$:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{G_t \left(C_t^j \right)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \frac{\left(N_t^j \right)^{1+\gamma}}{1+\gamma} + \frac{\eta}{1-\xi} \left(\frac{M_t^j}{P_t} \right)^{1-\xi} \right]$$

  $0 < \beta < 1$ is the discount factor, $\sigma > 0$ the elasticity of intertemporal substitution, $\xi > 1$ the elasticity of money holdings, and $\gamma > 0$ the inverse of the elasticity of labor supply with respect to real wages.

Technology

- Intermediate Goods producer $i \in [0,1]$:

$$Y_t^i = A_t \left(N_t^i\right)^{1-\delta}, \text{where } N_t^i = \left[\int_0^1 \left(N_t^{i,j}\right)^{\frac{\phi-1}{\phi}} dj\right]^{\frac{\phi}{\phi-1}}$$

$\phi > 1$ the elasticity of substitution between different types of labor.

- Final good:

$$Y_t = \left[\int_0^1 \left(Y_t^i\right)^{\frac{\varepsilon_t-1}{\varepsilon_t}} di\right]^{\frac{\varepsilon_t}{\varepsilon_t-1}}$$

$\varepsilon_t > 1$ the elasticity of substitution between intermediate goods, $\Lambda_t = \varepsilon_t / \left(\varepsilon_t - 1\right)$ price markup.

Final Goods Price Setting

- Final good producers are competitive and maximize profits.

- The input demand functions associated with this problem are

$$Y_t^i = \left[\frac{P_t^i}{P_t}\right]^{-\varepsilon_t} Y_t \qquad \forall i,$$

- The zero profit condition $\Rightarrow$ the price of the final good

$$P_t = \left[\int_0^1 P_t^{i1-\varepsilon_t} di\right]^{\frac{1}{1-\varepsilon_t}}$$

Intermediate Goods Price Setting Process

- Operate in a monopolistic competition environment.
- Reset prices if signal to do so ($1-\theta_p$).
- They choose the price that maximizes:

$$E_t \sum_{\tau=0}^{\infty} \theta_p^{\tau} Q_t^{t+\tau} \left[P_t^i Y_{t+\tau}^i - W_{t+\tau} \left(\frac{Y_{t+\tau}^i}{A_{t+\tau}} \right)^{\frac{1}{1-\delta}} \right]$$

subject to

$$Y_t^i = \left[\frac{P_t^i}{P_t} \right]^{-\varepsilon_t} Y_t \qquad \forall i,$$

- Solution

$$E_t \sum_{\tau=0}^{\infty} \theta_p^{\tau} Q_t^{t+\tau} \left\{ \left[\frac{P_t^{i,*}}{P_{t+\tau}} - \Lambda_t M C_t^i \right] Y_t^i \right\} = 0,$$

- The evolution of the price level is:

$$P_t = \left[\theta_p \left(P_{t-1} \right)^{1-\varepsilon_t} + \left(1-\theta_p \right) \left(P_t^* \right)^{1-\varepsilon_t} \right]^{\frac{1}{1-\varepsilon_t}}$$

Consumers problem

$$G_t {C_t}^{-\frac{1}{\sigma}} = \beta E_t \{ G_{t+1} C_{t+1}^{-\frac{1}{\sigma}} R_t \frac{P_t}{P_{t+1}} \}$$

$$G_t {C_t}^{-\frac{1}{\sigma}} \frac{W_t^j}{P_t} = \vartheta \left(N_t^j \right)^{\gamma}$$

Dynamics

$$a_t + (1-\delta)n_t - y_t = 0$$

$$mc_t - (w_t - p_t) + y_t - n_t = 0$$

$$\frac{1}{\sigma}c_t + \gamma n_t - g_t - mrs_t = 0$$

$$\rho_r r_{t-1} + (1-\rho_r)\left[\gamma_\pi \Delta p_t + \gamma_y y_t\right] + ms_t - r_t = 0$$

$$-y_t + c_t = 0$$

$$mrs_t - (w_t - p_t) = 0$$

$$E_t\left[-\sigma r_t + \sigma \Delta p_{t+1} - \sigma g_{t+1} + \sigma g_t - c_t + c_{t+1}\right] = 0$$

$$E_t\left[\kappa_p mc_t + \kappa_p \mu_t - \Delta p_t + \beta \Delta p_{t+1}\right] = 0$$

where

$$\kappa_p = (1-\delta)(1-\theta_p\beta)(1-\theta_p)/(\theta_p(1+\delta(\bar{\varepsilon}-1)))$$

11 Evaluation the Likelihood function

Step 1 Solve the Model (Uhlig Algorithm)

$$0 = As_t + Bs_{t-1} + Ce_t + Dz_t$$

$$0 = E_t[Fs_{t+1} + Gs_t + Hs_{t-1} + Je_{t+1} + Ke_t + Lz_{t+1} + Mz_t]$$

$$z_{t+1} = Nz_t + \varepsilon_{t+1} \qquad \varepsilon_{t+1} \sim N(0, \Sigma)$$

– $s_t = (w_t - p_t, r_t, \triangle p_t, y_t)'$ is the endogenous state, $e_t = (n_t, mc_t, mrs_t, c_t)'$ are endogenous variables, and $z_t = (a_t, ms_t, \mu_t, g_t)'$ is the exogenous state.

– $N = \begin{bmatrix} \rho_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_g \end{bmatrix}$

– Solution

$$s_t = PPs_{t-1} + QQz_t$$
$$e_t = RRs_{t-1} + SSz_t$$

```
PP =

         0   -1.8579         0         0
         0    0.4591         0         0
         0   -0.3415         0         0
         0   -0.5122         0         0


QQ =

   -0.5527   -2.6391   -0.1233    0.0346
   -0.1657    0.6521    0.0305    0.0407
   -0.3599   -0.4851    0.0641    0.0124
    0.2348   -0.7276   -0.0340    0.2852


RR =

         0   -0.8004         0         0
         0   -2.1461         0         0
         0   -1.8579         0         0
         0   -0.5122         0         0


SS =

   -1.1957   -1.1369   -0.0531    0.4457
   -1.9831   -3.0484   -0.1424    0.1950
   -0.5527   -2.6391   -0.1233    0.0346
    0.2348   -0.7276   -0.0340    0.2852


NN =

    0.6842         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0    0.7498
```

```
PP =

    0.4591         0         0
   -0.3415         0         0
   -0.5122         0         0


QQ =

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    0.2348   -0.7276   -0.0340    0.2852


NN =

    0.6842         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0    0.7498
```

Step 2 Writing the Solution in State Space Form

– Transition equation

$$\begin{pmatrix} s_t \\ z_t \end{pmatrix} = \begin{pmatrix} PP & QQ * N \\ 0 & N \end{pmatrix} \begin{pmatrix} s_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} Q \\ I \end{pmatrix} \varepsilon_t =$$

$$\begin{pmatrix} s_t \\ z_t \end{pmatrix} = F \begin{pmatrix} s_{t-1} \\ z_{t-1} \end{pmatrix} + G\varepsilon_t$$

– Measurement equation

$$s_t = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_t \\ z_t \end{pmatrix} = H \begin{pmatrix} s_t \\ z_t \end{pmatrix}$$

Step 3 Evaluating the Likelihood function using the Kalman Filter.

12 Summary

- Day 2: **Posterior** is proportional to **Likelihood** times **Prior**

$$\pi\left(\theta|Y^T\right) \propto f(Y^T|\theta)\pi\left(\theta\right)$$

How do we **Compute Posterior Moments**?

- Day 3: **Metropolis Hasting Algorithm** allows us to draw from $\pi\left(\theta|Y^T\right)$.

(a) Initialize the algorithm with an arbitrary value θ_0, N and set $j = 1$.

(b) Generate $\theta_j^* = \theta_{j-1} + \varepsilon \sim N\left(0, \Sigma_\varepsilon\right)$ and u from $Uniform[0,1]$.

(c) If $u \leq \alpha\left(\theta_{j-1}, \theta_j^*\right) = \min\left\{\frac{f(Y^T|\theta_j^*)\pi\left(\theta_j^*\right)}{f(Y^T|\theta_{j-1})\pi(\theta_{j-1})}, 1\right\}$ then $\theta_j = \theta_j^*$, $\theta_j = \theta_{j-1}$ otherwise.

(d) If $j \leq N$ then $j \rightsquigarrow j+1$ and got to 3.

How do we **Evaluate the Likelihood**?

- Day 4: Use the **Kalman Filter** to evaluate **Likelihood** $f(Y^T|\theta_j^*)$.

13 Priors for the Sticky Price Model

Let $\theta = (\sigma, \gamma, \theta_p, \rho_r, \gamma_\pi, \gamma_y, \rho_a, \rho_g, \sigma_a, \sigma_{ms}, \sigma_\mu, \sigma_g)$

Parameter		Mean	Std.Dev.
σ^{-1}	Gamma$(2, 1.25)$	2.5	1.76
$\frac{1}{1-\theta_p} - 1$	Gamma$(2, 1)$	2	1.42
γ_π	Normal$(1.5, 0.25)$	1.5	0.25
γ_y	Normal$(0.125, 0.125)$	0.125	0.125
γ	Normal$(1, 0.5)$	1	0.5
ρ_a, ρ_g, ρ_r	Uniform$[0, 1)$	0.5	0.28
$\sigma_a, \sigma_m, \sigma_\mu, \sigma_g$	Uniform$[0, 0.4)$	0.2	0.12
β	-	0.99	-
$\bar{\varepsilon}$	-	6	-
δ	-	0.36	-

14 Algorithm (gencoeffsggls.m)

Step 0 Read data (usadefl1d.txt)

Step 1 Intial value for θ_0, N and set $j = 1$.

Step 2 Evaluate $f(Y^T|\theta_0)$ and $\pi(\theta_0)$ and make sure $f(Y^T|\theta_0), \pi(\theta_0) > 0$

(a) Given θ_0 evaluate prior $\pi(\theta_0)$ (priorggls.m)

(b) Given θ_0: Uligh algorithm to solve the model (modelggls.m and solve2.m)

(c) Kalman Filter to evaluate $f(Y^T|\theta_0)$ (likeliggls.m)

Step 3 $\theta_j^* = \theta_{j-1} + \varepsilon \sim N(0, \Sigma_\varepsilon)$ and u from $Uniform[0, 1]$

(a) Given θ_j^* evaluate prior $\pi(\theta_j^*)$ (priorggls.m)

(b) Given θ_j^*: Uligh algorithm to solve the model (modelggls.m and solve2.m)

(c) Kalman Filter to evaluate $f(Y^T|\theta_j^*)$ (likeliggls.m)

Step 4 If $u \leq \alpha\left(\theta_{j-1}, \theta_j^*\right) = \min\left\{\frac{f(Y^T|\theta_j^*)\pi\left(\theta_j^*\right)}{f(Y^T|\theta_{j-1})\pi(\theta_{j-1})}, 1\right\}$ then $\theta_j = \theta_j^*$, $\theta_j = \theta_{j-1}$ otherwise.

Step 5 If $j \leq N$ then $j \rightsquigarrow j + 1$ and got to 3.