

4 Groves-Ledyard Mechanism

4.1 The Mechanism

- Agent i sends a message to the PGB that says how much increment in the output of the public good that she desires, denoted by Δ_i ;

- PGB sets the level of the public good to be

$$\hat{G} = \sum_{i=1}^n \Delta_i.$$

- The board sets taxes according to the following

rule:

$$T_i = \underbrace{\frac{\hat{G}}{n}}_{\text{pre-arranged share}} + \frac{\gamma}{2} \underbrace{\left\{ \frac{n-1}{n} (\Delta_i - A_i)^2 - \sum_{j \neq i} \frac{1}{n-2} (\Delta_j - A_i)^2 \right\}}_{\text{adjustment factor}}$$

where

$$A_i = \frac{1}{n-1} \sum_{j \neq i} \Delta_j = \frac{1}{n-1} (\hat{G} - \Delta_i)$$

and γ is some positive number.

4.2 Remarks

- In the tax rule, \hat{G}/n represents an equal share of the cost of \hat{G} . It could be generalized to an arbitrary fixed tax share. That is, $1/n$ could be replaced with a fixed t_i such that

$$\sum_{i=1}^n t_i = 1.$$

- In the adjustment factor, the term A_i is the mean of the $n - 1$ other people's desired increments.

4.3 Nash Equilibrium

- Given $\Delta_j, j \neq i$, agent i chooses Δ_i to maximize

$$v_i \left(\Delta_i + \sum_{j \neq i} \Delta_j \right) - T_i$$

which is equivalent to

$$v_i \left(\Delta_i + \sum_{j \neq i} \Delta_j \right) - \frac{\Delta_i + \sum_{j \neq i} \Delta_j}{n} - \frac{\gamma}{2} \left\{ \frac{n-1}{n} (\Delta_i - A_i)^2 - \sum_{j \neq i} \frac{1}{n-2} (\Delta_j - A_i)^2 \right\}.$$

The first order condition is

$$v'_i \left(\Delta_i + \sum_{j \neq i} \Delta_j \right) = \frac{1}{n} + \frac{\gamma(n-1)}{n} (\Delta_i - A_i). \quad (*)$$

- A Nash equilibrium of the game is $\{\Delta_i\}_{i=1}^n$ such that for every i , Δ_i satisfies the above first order condition given $\{\Delta_j\}_{j \neq i}$.

4.4 Properties of the Nash Equilibrium:

- It satisfies the Samuelson Condition: To see this, we simply sum the first order conditions over i , and obtain

$$\sum_{i=1}^n v_i'(\hat{G}) = 1 + \frac{\gamma(n-1)}{n} \sum_{i=1}^n (\Delta_i - A_i) = 1,$$

which is the Samuelson efficiency condition.

- Balanced Budget:

$$T_i = \frac{\hat{G}}{n} + \frac{\gamma}{2n(n-2)} \left[n^2 \Delta_i^2 - n \sum_{j=1}^n \Delta_j^2 + 2\hat{G}^2 - 2n\hat{G}\Delta_i \right]$$
$$\sum_{i=1}^n T_i = \hat{G}.$$

4.5 Problems with Groves-Ledyard Mechanism:

- Implementation is in Nash. That is, agent i has to know the preferences of all the other agents, which is a very stringent condition;
- Agents may be worse off by participating in the mechanisms. That is, PGB may be coercive.

5 Asymptotic Impossibility Theorem when Individual Rationality is also Imposed

- What happens when we also impose individual rationality in the feasible mechanisms? Will there exist mechanisms to insure the efficient provision of public good? If not, what is the probability of provision of public goods under the optimal mechanism? The main reference is Mailath and Postlewaite (ReStud, 1990).

5.1 An Illustrative Example: two agent case

- 2 individuals, $i = 1, 2$
- v_i : i 's valuation for the public good, and assume that $v_i \in \{L, H\}$ and ex ante $v_i = L$ or $v_i = H$ occurs with equal probability; v_1 and v_2 are independent. Let $L = 0$, $H = 10$. v_i is agent i 's private information.
- Cost of providing the public good is 8.

- Ideally we would like to have a mechanism that has the following three properties:
 1. Efficiency: produce the public good if and only if it is efficient to do so. [In our example, to provide the public good unless both agents have valuation L]
 2. Individual rationality: participation must be voluntary;
 3. Incentive compatibility: each agent has incentive to truthfully report her privately known valuation.
 4. Balanced Budget.

- Assume for now that the cost is shared if both agents announces H . Does there exist a mechanism with the above three properties?
- Suppose that agent 2 tells the truth, consider agent 1 with type H . Her payoff matrix if under such a mechanism with the efficiency property is as follows:

		Agent 2's type	
		L	H
Agent 1's announcement	\tilde{L}	0	10
	\tilde{H}	2	6

The expected payoff for agent 1 when announcing \tilde{L} is $0.5 \times 0 + 0.5 \times 10 = 5$ while announcing \tilde{H} is $0.5 \times 2 + 0.5 \times 6 = 4$. Hence truth telling is not a Bayesian Nash equilibrium.

This simple example illustrates that there may not exist a mechanism that simultaneously satisfy efficiency,

individual rationality and incentive compatibility. The intuition is simply the free riding problem: in cases when an individual's announcement will not change the probability of the public good being provided (due to the requirement of efficiency), he will have incentive to report a lower valuation so as to be taxed less. Now it is natural to ask the following question: what efficiency level can we achieve by mechanisms that satisfy IR and IC constraints? In this example, suppose that the public good provision probabilities are given by

$$\rho(L, L) = 0, \rho(H, L) = \rho(L, H) = \rho, \rho(H, H) = 1$$

where $\rho(H, L)$ denotes the probability of the public good being provided when agent 1 announces H and agent 2 announces L . The tax rule remains the same as before. Hence we have the following table:

Announ. Profile	Probability of providing PG	Taxes if PG is provided
(L, L)	0	n.a.
(L, H)	ρ	$(0, 8)$
(H, L)	ρ	$(8, 0)$
(H, H)	1	$(4, 4)$.

Suppose that the other agent tells the truth. We first verify the IC constraints for L type:

$$u(\tilde{L}|L) = 0 > u(\tilde{H}|L) = \frac{1}{2}(-8 - 4) = -6$$

hence the IC constraints for the low type is satisfied for all p . It is intuitive because if your valuation is 0, why would you lie upward?

Now we verify the IC constraints for H type:

$$u(\tilde{H}|H) = \frac{1}{2}[\rho(10 - 8) + (10 - 4)] = \rho + 3$$
$$u(\tilde{L}|H) = \frac{1}{2}(10\rho) = 5\rho$$

IC for type H hence requires that

$$u(\tilde{H}|H) \geq u(\tilde{L}|H) \Leftrightarrow \rho \leq \frac{3}{4}.$$

Hence IR, IC mechanism will have to sacrifice efficiency.

5.2 Impossibility Result for Large Economies

It is desirable to have a mechanism that is *ex ante* efficient, that is, that provides the public good whenever the cost was less than the sum of the benefits to the agents in the economy. When agents' valuations for the public good are privately known, this may be difficult, as we showed in the two agent example above: if agents' cost shares do not depend on their announced values, agents whose valuations exceed their cost shares will have incentive to overstate their valuations; on the other hand, if agents' cost shares *do* depend on their announcements, they may have an incentive to understate their valuations. If we impose an individual rationality constraint that prohibits taxing an agent more than his announced valuation for the public good, we will be in the latter case where agents may have an incentive to understate their valuations. It is intuitive that the understatement problem gets more severe when the number of agents get larger.

NOTATIONS:

- n agents, $i = 1, \dots, n$;
- v_i : i 's valuation for the public good, $v_i \in \{H, L\}$ which is private information, and ex ante $\Pr(v_i = H) = p \in (0, 1)$.
- Agent i 's utility when the public good is provided and he pays tax t is $v_i - t$;
- The cost of providing the public good in an n person economy is cn [i.e. per capita cost of the public good is constant];
- Assume that $H > c > L$.

DIRECT REVELATION MECHANISM:

In the direct revelation mechanism, the message space for agent i is $M_i = M = \{H, L\}$. The mechanism needs to specify an outcome function (ξ, ρ) where:

- $\xi : M^n \rightarrow R_+^n$ is the vector of taxes the agents pay as a function of the announced valuation profile *if* the public good is produced;
- $\rho : M^n \rightarrow [0, 1]$ is the probability that the public good is produced as a function of the agents' announced valuation profile.

We can further restrict (without loss of generality) that $\rho(\cdot)$ depends only on k , the number of agents announcing L , and ignore which specific agents announced L . It can also be shown easily that the tax scheme that provides the greatest incentive to agents with high value H to announce truthfully is

$$\xi_i(v) = \begin{cases} L & \text{if } v_i = L \text{ and } kL + (n - k)H \geq nc \\ \frac{nc - kL}{n - k} & \text{if } v_i = H \text{ and } kL + (n - k)H \geq nc. \end{cases}$$

It is trivial to see that the incentive compatibility constraint for L is satisfied with this tax rule. The incen-

tive constraint for H is

$$\begin{aligned} & \sum_{k=0}^{n-1} p(k) \rho(k) \left(H - \frac{nc - kL}{n - k} \right) \\ & \geq \sum_{k=0}^{n-1} p(k) \rho(k + 1) (H - L) \end{aligned}$$

where $p(\cdot)$ is the probability that there are exactly k agents of type L among the remaining $n - 1$ agents [note that the number of type L agents among the $n - 1$ remaining agents have a Binomial distribution with parameters $n - 1$ and p] [convince yourself that the left (respectively, right) hand side is the agent's expected utility of announcing H (respectively announcing L) assuming all other agents tell truth. This can be rewritten as

$$\begin{aligned} & \sum_{k=0}^{n-1} p(k) [\rho(k) - \rho(k + 1)] [H - L] \\ & \geq \sum_{k=0}^{n-1} p(k) \rho(k) \left(\frac{nc - nL}{n - k} \right). \end{aligned}$$

We will show that the left hand side must go to zero as n goes to infinity; and that the probability of provision

of the public good in the n person problem must be smaller than the right hand side.

STEP 1: Show that the right hand side goes to zero as n goes to infinity. Because $p(\cdot)$ is the binomial density, it is maximized at an integer $t \in ((n+1)p - 1, (n+1)p]$. For this t , $p(k-1) < p(k)$ for $k < t$ and $p(k-1) > p(k)$ for $k > t$ and $p(t) \geq p(t-1)$.

CLAIM 1: $\sum_{k=0}^{n-1} p(k) [\rho(k) - \rho(k+1)]$ is maximized with $\rho(k) = 1$ for $k \leq t$ and $\rho(k) = 0$ for $k > t$.

Proof. Let $\rho^*(\cdot)$ be the maximizer of the expression. First we show that $\rho^*(t) = 1$ and $\rho^*(t+1) = 0$. Suppose not, let $\rho'(\cdot) = \rho^*(\cdot)$ except let $\rho'(t) = 1$ and $\rho'(t+1) = 0$. Then

$$\begin{aligned} & \sum_{k=0}^{n-1} p(k) [\rho'(k) - \rho'(k+1)] \\ & - \sum_{k=0}^{n-1} p(k) [\rho^*(k) - \rho^*(k+1)] \\ = & [1 - \rho^*(t)] [p(t) - p(t-1)] - \\ & \rho^*(t+1) [p(t+1) - p(t)] > 0. \end{aligned}$$

A contradiction. Now we can claim that $\rho^*(k) = 1$ for all $k \leq t - 1$. Suppose not, then there exist $k \leq t - 1$ with $\rho(k) < \rho(k + 1)$. Suppose that the smallest such k is k' . Let $\rho'(\cdot) = \rho^*(\cdot)$ except at this k' and let $\rho'(k') = 1$. Then

$$\begin{aligned} & \sum_{k=0}^{n-1} p(k) [\rho'(k) - \rho'(k + 1)] \\ & - \sum_{k=0}^{n-1} p(k) [\rho^*(k) - \rho^*(k + 1)] \\ & = [1 - \rho^*(k')] [p(k') - p(k' - 1)] > 0 \end{aligned}$$

a contradiction. Similarly it can be shown that $\rho(k) = 0$ for all $k > t$. ■

The implication of this claim is that

$$\begin{aligned} & \sum_{k=0}^{n-1} p(k) [\rho(k) - \rho(k + 1)] [H - L] \\ & \leq p(t) (H - L) \\ & = (H - L) \frac{(n - 1)!}{(n - 1 - t)! t!} p^t (1 - p)^{n-1-t} \end{aligned}$$

Using Sterling's formula which states that

$$\lim_{n \rightarrow \infty} \frac{n!}{\sqrt{2\pi n} n^{n+1/2} e^{-n}} = 1$$

and using the approximation that $p \approx t/(n-1)$ when n is large, we get

$$\begin{aligned} & \frac{(n-1)!}{(n-1-t)! t!} p^t (1-p)^{n-1-t} \\ \rightarrow & \frac{1}{\sqrt{2\pi (n-1) p (1-p)}} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence the left hand side of the incentive constraint goes to zero.

Now we examine the RHS of the incentive constraint:

$$\begin{aligned} & \sum_{k=0}^{n-1} p(k) \rho(k) \left(\frac{nc - nL}{n-k} \right) \\ = & (c-L) \sum_{k=0}^{n-1} p(k) \rho(k) \frac{n}{n-k} \\ \geq & (c-L) \sum_{k=0}^{n-1} p(k) \rho(k). \end{aligned}$$

Since the LHS goes to zero as n goes to infinity, the right hand side goes to zero as well. Hence

$$\sum_{k=0}^{n-1} p(k) \rho(k) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

But $\sum_{k=0}^{n-1} p(k) \rho(k)$ is simply the expected probability of the public good being provided.