Standard Auctions with Financially Constrained
Bidders: Comment

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In an important paper, Che and Gale (1998, CG henceforth) developed a methodology for analyzing the revenue and efficiency performance of auctions when bidders have private information about their valuation and ability to pay. Their methodology allows one to compare the revenue and social surplus for a family of auction rules that satisfy the following four properties: (P1). The highest bidder receives the object, and the rules of the auction apply identically to all bidders; (P2). There exists a symmetric pure-strategy Bayes-Nash equilibrium in which an active bidder’s equilibrium bidding function is continuous in both her valuation $v$ and her budget $w$; (P3). The equilibrium bidding function is increasing in $(w,v)$ and strictly increasing if both $w$ and $v$ rise; (P4). For any equilibrium bid, there exists an unconstrained type with budget $\bar{w}$ that makes the same bid. They then apply this methodology to rank the revenue and social surplus for the first and second price auctions with absolute budget constraints. For this purpose, they need to verify that both auction formats satisfy the four required properties.

In this comment, we report an error in CG’s equilibrium characterization of the first price auction. Our finding suggests that the equilibrium bidding function in a first price auction may not be continuous in $w$ and $v$, thus violating P2. Therefore the main results of CG may not be applicable to rank the revenue and social surplus for some first and second price auctions.

The environment that CG study in their Section 3.B can be summarized as follows. $N \geq 2$ risk neutral bidders compete for a single object in a first price auction. A typical bidder values the object at $v$ and has a budget $w$. He can not bid more than his available budget. For each bidder,

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$w$ and $v$ are private information, and the $(w, v)$ pairs are independently and identically distributed across bidders with a joint density $f(\cdot, \cdot)$ that is positive and continuously differentiable in the interior of the support, $[w, \bar{w}] \times [v, \bar{v}]$, where $(w, v) \geq (0, 0)$. Let $r_f \geq v$ be the seller’s reserve price.

CG impose a technical assumption before characterizing the unique symmetric equilibrium of the first price auction in their Lemma 1. For completeness, they are reproduced here.

**CG’s Assumption A5.** $(N - 1)w + G(w, v)/G_1(w, v)$ is strictly increasing in $w$ for all $v \in (v, \bar{v})$, where

\[
G(w, v) = \Pr(\tilde{w} \leq w \text{ or } \tilde{v} \leq v) = 1 - \int_{w}^{\tilde{w}} \int_{v}^{\tilde{v}} f(w, v) \, dw \, dv, \quad \text{and}
\]

\[
G_1(w, v) = \frac{\partial G(w, v)}{\partial w}.
\]

**Lemma 1 (CG 1998).** Under assumption A5, there exists a unique, symmetric equilibrium in which bidders employ a bid function

\[
B_f(w, v) = \min\{w, b_f(v)\}
\]

where $b_f(v)$ satisfies

\[
b_f(v) = v - \frac{\int_{r_f}^{v} F_f(s)^{N-1} \, ds}{F_f(v)^{N-1}} \quad \text{for } v \geq r_f, \quad \text{where } F_f(s) = G(b_f(s), s)
\]

and is continuous and strictly increasing.

1. THE GENERAL PROBLEM

We first show that in general Assumption A5 does not ensure the existence of a symmetric equilibrium in the form of (1) for some continuous and strictly increasing function $b_f(\cdot)$. Let $N = 2$. Suppose that $\tilde{w}$ and $\tilde{v}$ are independently distributed so that $f(w, v) = h(w)k(v)$, and denote $H(\cdot)$ and $K(\cdot)$ as the respective CDFs of $\tilde{w}$ and $\tilde{v}$. Let $r_f = 0$. Suppose that bidder 2 follows the bidding strategy (1). Write the inverse of $b_f(\cdot)$ as $\psi(\cdot)$. The best response for an unconstrained bidder 1 with valuation $v$ is given by the solution to the following problem:

\[
\max_{b \geq 0} \Pi(b; v) = \{H(b) + [1 - H(b)]K(\psi(b))\} (v - b),
\]

where the term in the braces is bidder 1’s probability of winning the object if he bids $b$ when his opponent bids according to (1). Since we are assuming that $b_f(\cdot)$ is continuous and strictly
increasing, \( b'_f (\cdot) \) is well defined almost everywhere. In a symmetric equilibrium, the derivative of \( \Pi (\cdot; v) \) with respect to \( b \) evaluated at \( b_f (v) \) should equal zero almost everywhere, that is,

\[
\frac{k (v) \left[ 1 - H (b_f (v)) \right] [v - b_f (v)]}{b'_f (v)} = H (b_f (v)) + \left[ 1 - H (b_f (v)) \right] K (v) - h (b_f (v)) \left[ 1 - K (v) \right] [v - b_f (v)].
\]

We can express the above first order condition as a differential equation:

\[
b'_f (v) = \frac{k (v) \left[ 1 - H (b_f (v)) \right] [v - b_f (v)]}{K (v) + H (b_f (v)) \left[ 1 - K (v) \right] - h (b_f (v)) \left[ 1 - K (v) \right] [v - b_f (v)]}.
\]  

(2)

If for some \( v, b_f (v) < w \), then ODE (2) is reduced to

\[
b'_f (v) = \frac{k (v) [v - b_f (v)]}{K (v)},
\]

or equivalently,

\[
K (v) b'_f (v) + k (v) b_f (v) = v k (v).
\]  

(3)

The solution to (3) is given by

\[
b_f (v) = v - \int_v^w \frac{K (s)}{K (v)} ds,
\]  

(4)

which is exactly the equilibrium bidding function in the first price auction with no financial constraints. Denote \( v^* \) as bidder 1’s valuation type such that

\[
b_f (v^*) = v^* - \int_{v^*}^w \frac{K (s)}{K (v^*)} ds = w.
\]  

(5)

Note that \( v^* \) is uniquely determined because \( b'_f (\cdot) \) as defined in (4) is strictly increasing. Also note that \( v^* > v \) unless \( v^* = v = w \).

Now consider the case that \( b_f (v) \geq w \). The first derivative \( \partial \Pi (b; \cdot) / \partial b \) as a function of \( v \) will have a discontinuity at \( v^* \) because the budget density \( h (b_f (v)) \) start to kick in. Since in a first-price auction, \( v \geq b_f (v) \) in equilibrium for all \( v \), the sign of \( b'_f (v) \) will be the same as the sign of the denominator in (2). After dividing all terms in the denominator of (2) by \( 1 - K (v) \), we obtain

\[
b'_f (v) \geq 0 \text{ iff } \frac{K (v)}{1 - K (v)} + H (b_f (v)) - h (b_f (v)) [v - b_f (v)] \geq 0.
\]

Consider now the type \( v^* + \varepsilon \) for \( \varepsilon > 0 \) but arbitrarily small. The sign of \( b'_f (\cdot) \) evaluated at \( v^* + \varepsilon \) is the same as the sign of

\[
\frac{K (v^* + \varepsilon)}{1 - K (v^* + \varepsilon)} + H (b_f (v^* + \varepsilon)) - h (b_f (v^* + \varepsilon)) [v^* + \varepsilon - b_f (v^* + \varepsilon)].
\]
If \( b_f(\cdot) \) is continuous, then
\[
\lim_{\varepsilon \to 0} \left\{ \frac{K(v^* + \varepsilon)}{1 - K(v^* + \varepsilon)} + H(b_f(v^* + \varepsilon)) - h(b_f(v^* + \varepsilon))[v^* + \varepsilon - b_f(v^* + \varepsilon)] \right\} = \frac{K(v^*)}{1 - K(v^*)} - h(w)[v^* - w].
\] (6)

Note that if \( v^* - w > 0 \), then the above term need not be positive. That is, if
\[
h(w) > \frac{K(v^*)}{1 - K(v^*)}[v^* - w],
\] (7)
then in fact the derivative of \( b_f(\cdot) \) evaluated at points close to \( v^* \) and to the right of \( v^* \) will be negative, contradicting Lemma 1.

Does CG’s Assumption A5 rule out the possibility of (7)? The answer is no. The following proposition shows the exact bite of Assumption A5 in our special setting:

**Proposition 1** If \( f(w,v) = h(w)k(v) \), then CG’s Assumption A5 holds if and only if \( h(w) \) is non-increasing for all \( w \in (\mathbf{w}, \mathbf{\bar{w}}) \).

**Proof:** When \( f(w,v) = h(w)k(v) \), we have \( G(w,v) = K(v) + [1 - K(v)]H(w) \), and \( G_1(w,v) = [1 - K(v)]h(w) \). Hence Assumption A5 requires that
\[
(N - 1)w + \frac{K(v)}{1 - K(v)}h(w) + \frac{H(w)}{h(w)}
\] (8)
be strictly increasing in \( w \) for all \( v \in (\mathbf{v}, \mathbf{\bar{v}}) \).

Taking derivative of (8) with respect to \( w \), we obtain
\[
(N - 1) - \frac{K(v)}{1 - K(v)} \frac{h'(w)}{h(w)^2} + \frac{h(w)^2 - h'(w)H(w)}{h(w)^2}.
\] (9)
If \( h(w) \) is non-increasing for all \( w \), i.e. if \( h'(\cdot) \leq 0 \), then (9) is obviously positive.

On the other hand, if \( h(\cdot) \) is strictly increasing for some \( \mathbf{\tilde{w}} \in (\mathbf{w}, \mathbf{\bar{w}}) \), then as \( v \) gets arbitrarily close to \( \mathbf{\bar{v}} \), the second term in (9) is negative and dominates the first and the third terms. That is, for any \( \mathbf{\tilde{w}} \in (\mathbf{w}, \mathbf{\bar{w}}) \) such that \( h'(\mathbf{\tilde{w}}) > 0 \), we have:
\[
\lim_{v \to \mathbf{\bar{v}}} \left[ (N - 1) - \frac{K(v)}{1 - K(v)} \frac{h'(\mathbf{\tilde{w}})}{h(\mathbf{\tilde{w}})^2} + \frac{h(\mathbf{\tilde{w}})^2 - h'(\mathbf{\tilde{w}})H(\mathbf{\tilde{w}})}{h(\mathbf{\tilde{w}})^2} \right] = -\infty,
\]
contradicting Assumption A5.

Proposition 1 tells us that when \( \mathbf{\tilde{w}} \) and \( \mathbf{\bar{v}} \) are independent, the only restriction imposed by Assumption A5 is that the budget density must be non-increasing, which implies that \( h(w) \) must be strictly positive. Therefore CG’s Assumption A5 does not rule out the possibility of (7). It is
also interesting to note that when \( \tilde{w} \) and \( \tilde{v} \) are independent, common budget distributions such as log-normal, Weibull, Gamma and Beta do not satisfy CG’s assumption, while uniform distribution does. CG provide a correct example in which \( \tilde{w} \) and \( \tilde{v} \) are independent uniform distributions from \([0,1]\).\(^1\) Notice, however, a special feature of the uniform example is that \( v^* = \underline{w} = \underline{v} = 0 \), hence the possibility of (7) will not arise and therefore \( b'_f (\cdot) \) is positive almost everywhere.

2. AN EXPLICIT COUNTER-EXAMPLE

The previous section shows that in general CG’s Assumption A5 does not ensure their Lemma 1. Here we provide an explicit counter-example. Let \( N = 2 \). Suppose that \( \tilde{v} \) is uniformly distributed on \([0,1]\) and \( \tilde{w} \) is distributed on \([\underline{w}, 1]\) where \( \underline{w} < 1/2 \), with the following density:

\[
h(w) = \frac{2 \exp(-2w)}{\exp(-2w) - \exp(-2)}.
\]

Note that \( h \) is strictly decreasing in \( w \), therefore, by Proposition 1, CG’s Assumption 5 is satisfied.

Now we show that condition (7) will also be satisfied for sufficiently small \( \underline{w} \). First, Equation (5) in this example can be rewritten as:

\[
\underline{w} = \int_0^{v^*} \left[ 1 - \frac{s}{v^*} \right] ds = \frac{v^*}{2},
\]

therefore, \( v^* = 2\underline{w} \).

Evaluating (6) at \( v^* = 2\underline{w} \), we obtain

\[
\frac{K(v^*)}{1-K(v^*)} - h(\underline{w}) [v^* - \underline{w}] = \frac{2\underline{w}}{1 - 2\underline{w}} - \frac{2\underline{w}}{1 - \exp[-2(1 - \underline{w})]} = \frac{2\underline{w} [2\underline{w} - \exp(-2(1 - \underline{w}))]}{(1 - 2\underline{w}) [1 - \exp(-2(1 - \underline{w}))]}
\]

Thus if \( 2\underline{w} - \exp(-2(1 - \underline{w})) < 0 \), the above term will be negative. More explicitly, if \( \underline{w} < 0.079297 \), then \( b'_f (v^* + \varepsilon) < 0 \) for small \( \varepsilon > 0 \).

3. FURTHER REMARKS

\(^1\)It should be noted that there is a typo in the differential equation for the uniform example on Page 11 of CG. It should read:

\[
b'_f (v) = \frac{[1 - b_f (v)] [v - b_f (v)]}{1 - (1 - v) [v + 1 - 2b_f (v)]}.
\]
We have shown that CG’s Assumption 5 is not sufficient for their Lemma 1. In general, the equilibrium bidding strategy will not take the form of (1). In the unpublished proof of Lemma 1 of CG (see Appendix A of Che and Gale 1995), a logical gap in Lemma A1 led to an incorrect claim of $B(\cdot, v)$ being continuous in $w$. Now we provide some intuition of why the equilibrium bidding function may not be continuous in $w$. Suppose that bidders 2, ..., $N$ play according to a strategy $B(\cdot, \cdot)$. Imagine an unconstrained bidder 1 with valuation $v$. To characterize bidder 1’s best response, we write $Q(b; B)$ as bidder 1’s probability of winning when he bids $b$ and other bidders follow $B$. In principle, knowing $B$, bidder 1 can explicitly calculate $Q(b; B)$. Bidder 1’s expected surplus function is then given by

$$S(b; v, B) = (v - b) Q(b; B).$$

Denote $α(v)$ (and $\bar{α}(v)$, respectively) as the maximizer of $S(b; v, B)$ when $b$ is restricted to be smaller (larger, respectively) than $w$. Of course, when $v$ is large enough, $α(v) = w$; when $v$ is small enough, $S(α(v); v, B) > S(\bar{α}(v); v, B)$. However, for intermediate values of $v$, the surplus function may be double-peaked. Note that the derivative of $S(\cdot; v, B)$ changes sign at $w$ because the budget density starts to be positive at $w$. Figure 1 depicts bidder 1’s expected surplus as a function of its bid $b$ when his valuation is at the intermediate level. Let $w^*(v)$ be the bid higher than $w$ that yields the same expected surplus as the low bid of $α(v)$. The best response of such

![Figure 1: Bidder 1’s expected surplus as a function of $b$ for intermediate values of $v$.](image-url)
valuation types, denoted by $R(w,v;B)$, will be

$$R(w,v;B) = \begin{cases} 
\alpha(v) & \text{if } w < w^*(v) \\
\alpha(v) \text{ or } w^*(v) & \text{if } w = w^*(v) \\
\min\{w, \bar{\alpha}(v)\} & \text{if } w > w^*(v).
\end{cases}$$

Obviously type $v$ bidder 1’s best response is discontinuous in $w$ at $w^*(v)$. A symmetric equilibrium is of course a bidding strategy $B^*$ such that $R(w,v;B^*) = B^*(w,v)$ for all $w,v$. It seems entirely possible that the equilibrating process will not necessarily result in a continuous $B^*$. It is an important question to investigate since if the equilibrium bidding function $B(w,v)$ is not continuous in $w$ or in $v$, then P2 of the mechanisms considered in CG in their main theorems will be violated for some first price auctions, and hence their methodology may not be applicable to rank the revenue and social surplus for some first and second price auctions.

References
