An Efficiency Rationale for Bundling of Public Goods*

Preliminary

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Abstract

This paper studies the role of bundling in the efficient provision of excludable public goods. Our main finding is that there is an efficiency rationale to provide unrelated public goods as a bundle. The main source of the result is that bundling reduces the variance in the distribution of valuations.

1 Introduction

This paper studies the role of bundling in the efficient provision of non-rival goods. The question is simple: is there an efficiency rationale to bundle unrelated public goods together rather than providing each public good separately? Our main finding is that the surplus maximizing mechanism will always be characterized by some degree of bundling, so while “pure bundling” isn’t necessarily the optimal solution there is a straightforward argument in favor of considering the provision of multiple public goods jointly.

We consider an environment with two excludable public goods and a numeraire private good.1 Every consumer in the economy is characterized by a valuation for each of the public goods, and

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1The term “excludable public goods” refers to a good which is fully non-rival, but where it is possible to costlessly exclude any consumer from usage.
the valuation for consuming both public goods is assumed to be the sum of the valuations for the individual goods. This assumption of “no complementarities in payoffs” is made in order to rule out bundling arising due to the value of the components being smaller than the value of the whole package. For the same reason, we assume that the cost of provision of each good is independent of whether the other good is provided or not, so there are no complementarities on the cost side either.

The separability assumptions on preferences and costs imply that the informationally unconstrained efficient provision rule is to provide public good \( j \) if and only if the sum of valuations for public good \( j \) exceeds the cost of provision for good \( j \). This is also the ex post efficient rule when goods are provided separately, so there is no role for bundling in our environment under perfect information. Conditional on a good being provided, no agent will be excluded from usage in this benchmark case.

In this paper, valuations are private information to the consumers and stochastically independent across individuals. Moreover, agents are free to choose whether to participate in the mechanism and that the mechanism needs to be self-financing. Taken together, these constraints imply that the (non-bundling) perfect information social optimum is no longer feasible.\(^2\)

Under our assumptions, exclusions of consumers are active in the constrained efficient mechanism (for non-trivial cases). Indeed, if the economy is large, use exclusions is essentially the only instrument that can provide any incentives for truthful revelation of preferences. As shown in Norman [7], the constrained efficient mechanism in the case of a single public good is then well approximated by standard third-degree price discrimination: for each agent the designer sets a fixed user fee and the agent is included if and only if she is willing to pay the fee. If all agents are ex ante identical the user fee is the same for all agents and the mechanism boils down to average cost pricing.

This characterization of the provision problem for a single good generates a simple intuition for the usefulness of bundling. For simplicity, assume that goods are symmetric and that valuations

\(^2\)All these restrictions are essential to the analysis. Removing either the voluntary participation or the self-financing constraint makes it possible to construct pivot-mechanisms that implement the first best. Assuming that there is correlation in types, an adaption of the analysis in Cremer and McLean [3] can be used to implement the efficient outcome.
are drawn from the same distribution for all agents. The best mechanism that does not use the bundling option is then approximately a fixed user fee, which is the same for both goods. The crucial observation is then that the valuation for the “average good” is less dispersed than the distribution of valuations for each separate good. Given that the price without bundling will be below the expected valuation (which is true under some natural restrictions on the distribution of valuations), intuition then suggests that fewer consumers are excluded from usage if the goods are sold only as a bundle at a price given by the sum of the unbundled prices. This intuition carries through in several parametric examples, but we have not been able to provide a useful characterization of distributions for when it is generally true that bundling at the sum of the non-bundled prices excludes fewer customers. However, what can be shown is that bundling a sufficient number of goods will always lead to an improvement. Indeed, the informational asymmetry essentially disappears when a sufficient number of goods are bundled together, and with a large number of goods and consumers bundling can be used to approximate first best.

There is a considerable literature on bundling of private goods. While there are many similarities at the technical level, there are significant qualitative differences between the private goods case and the case with public goods. In particular, with private goods it is simply impossible to generate any social role for bundling of goods. A profit maximizing seller with monopoly power may want to bundle a shirt and a tie, but from an efficiency viewpoint marginal cost pricing is impossible to beat.

There are quite a few real world examples that fit our setup reasonably well. A perfect example of an excludable public good is cable TV. Moreover, as far as we understand, there are no particular technological reasons for why the local cable company could not allow an individual customer to choose whatever channels he or she is willing to pay for without constraints. Nevertheless, this is not the way it works. While some premium channels are priced separately and some programming is provided on a pay-per-view basis, the basic pricing scheme usually consists of a very limited number of available bundles.

Another good example of an excludable public good is a digitally formatted media file. In the case of the sale of music the record industry seems reluctant to move away from the album format, and the internet sales of music is often using a flat fee for access and no per-download charge. Both

3Unless there are significant fixed costs, but these fixed costs can then be viewed as a public good.
these cases are rather extreme forms of bundling.

Finally, the example that was the initial motivation for this work is the casual observation that publicly provided goods are provided in bundles. Clearly, there are several layers of governments in most economies and some goods in the bundle can hardly be viewed as public goods. Still, there is no fundamental reason for why policing, highway maintenance, fire fighting, public schools etc. could not be run by separate institutions. There are alternative explanations, but our model provides a plausible efficiency rationale for an arrangement where everybody in a given municipality is entitled to a bundle of public services provided by the local government as long as she pays the property taxes, despite the fact that it seems reasonable to believe that several of the goods in the bundle are of no value at all that many residents.

2 The Model

The environment we consider is a simplified version of Norman (2002), except that we allow for multiple public goods. In essence, the problem is to characterize how much of two excludable public goods to provide in a world where valuations are private information, no resources from outside are available, and where agents can opt out from an agreement after learning their valuations for the projects.

There are two excludable public goods, labeled by \( j = 1, 2 \); and \( n \) agents, indexed by \( i \in I = \{1, \ldots, n\} \). Provision of either public good is a binary decision and the cost of providing good \( j \) is \( C^j(n) \). The nature of the public good is reflected in the assumption that the cost of provision is independent of the number of users. The reason for indexing the provision costs by the number is that we will derive large economy results, which is the most tractable way to characterize the solution to the design problem we analyze. To keep things interesting, it is therefore necessary to assume that \( C^j(n)/n \) is bounded away from zero; otherwise the provision problem is trivial in a large economy. We therefore assume that there exists \( c^j > 0 \) such that \( \lim_{n \to \infty} C^j(n)/n = c^j \).

It is not necessary to give the assumption that costs are increasing in the size of the economy any particular economic interpretation. Since the limiting economy is a good approximation of a large finite economy (in a sense to be detailed below) the assumption is best thought of as a

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4The simplifications are that all agents have identical distributions of valuations and that there is no quantity dimension in this paper.
normalization to ensure that the provision problem is “significant”.

Agent $i$’s valuation from good $j$ is denoted by $\theta_i^j$. Write $I_i^j$ as the dummy variable taking value 1 when agent $i$ consumes good $j$ and 0 otherwise, and $t_i$ as the quantity private goods (or the transfer of “money”) that $i$ sacrifices to consume the goods (if at all), the utility for agent $i$ of type $(\theta_i^1, \theta_i^2)$ is

$$I_i^1 \theta_i^1 + I_i^2 \theta_i^2 - t_i.$$  \hspace{1cm} (1)

A number of restrictions are imposed with this formulation. Besides the additive separability between “money” and the public good, it also rules out complementarities between the two public goods. Preferences over lotteries are given by the expected utility extension of (1).

Valuations are independently and identically distributed across agents. We will make more specific assumptions when actually solving the model, but since the setup is easier to understand and the exposition of some intermediate results is clearer with a generic distribution of valuations we will momentarily keep some generality.

Agent $i$’s valuations of public goods $(\theta_i^1, \theta_i^2)$ are drawn from a joint distribution $F$. The marginal distribution of good $j$’s valuation will be denoted by $F_j^j$ and the conditionals will be denoted by $F_j^1(\cdot | \theta_i^2)$ and $F_j^2(\cdot | \theta_i^1)$ respectively. The set of possible realizations of $(\theta_i^1, \theta_i^2)$ is denoted $\Theta_i = \Theta^1 \times \Theta^2$, where $\Theta^j$ is the set of possible values for $\theta_j$. For brevity, we write $\theta_i = (\theta_i^1, \theta_i^2) \in \Theta_i$, $\theta = (\theta_1, ..., \theta_n) \in \prod_{i \in I} \Theta_i \equiv \Theta$, $\theta_{-i} = (\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_n) \in \prod_{j \in I \setminus \{i\}} \Theta_j \equiv \Theta_{-i}$, and $\theta^j = (\theta_i^1, ..., \theta_i^j)$ \hspace{2mm} \in \hspace{2mm} [\Theta^j]^n$, $j = 1, 2$. By independence, $F$ is the prior distribution over agent $i$’s valuations $\theta_i$ for the mechanism designer and all the other agents when the revelation game is played.\footnote{For the case when $F^1$ and $F^2$ are continuous distributions we will need to impose a number of further conditions later on to be able to make any definite comparisons between mechanisms with bundling and the case where the provisions problems are separated.}

In general, the outcome of any mechanism must specify the following:

- Whether or not public good $j$ should be provided, for $j = 1, 2$;
- If public good $j$ is provided, which agents should be allowed access to it, for $j = 1, 2$;
- How the costs of public good provision should be shared among the agents.
The set of feasible pure outcomes is thus

\[
A = \underbrace{\{0,1\} \times \{0,1\}}_{\text{provision/no provision for goods 1 and 2}} \times \underbrace{\{0,1\}^n \times \{0,1\}^n \times R^n}_{\text{inclusion/no inclusion for good 1 and 2 for all agents and \text{"taxes"}}}.
\] (2)

By the revelation principle we will, without loss of generality, restrict attention to direct mechanisms for which truth-telling is a Bayesian Nash equilibrium. A pure direct mechanism is simply a map from \(\Theta\) to \(A\). The most direct way to think of a randomized mechanism would be as a map from \(\Theta\) to the set of probability distributions over \(A\). However, it turns out to be more convenient for our purposes to follow Aumann [1] and model a randomized mechanism as a measurable mapping \(g : \Theta \times X \to A\), where \(X = [0,1]\) and \(x \in X\) is to be thought of as the outcome of a lottery, where without loss of generality \(x\) is uniformly distributed and independent of \(\theta\).6

Transfers enter into quasi-linearly into the utility functions, so there are no gains possible from randomizing over these. We will therefore immediately impose that the transfer rule is pure. With respect to the “real” allocation rule a conceptual advantage of our modelling strategy is that we can break any mechanism up in an intuitive way. That is, we denote a mechanism by \(M = (\zeta^1, \zeta^2, \omega^1, \omega^2, \tau)\) where

\[
\zeta^j : \Theta \times X \to \{0,1\} \quad \text{for } j = 1, 2
\]

\[
\omega^j : \Theta \times X \to \{0,1\}^n \quad \text{for } j = 1, 2
\]

\[
\tau : \Theta \to R^n.
\]

We refer to \(\zeta^j\) as the provision rule for good \(j\), where \(E_X \zeta^j(\theta, x)\) is interpreted as the probability of provision given announcements \(\theta\). The rule \(\omega^j\) is referred to as the inclusion rule for good \(j\) and \(E_X \omega^j_i(\theta, x)\) is the probability that agent \(i\) gets access to good \(j\) when announcements are \(\theta\), conditional on good \(j\) being provided. Finally, \(\tau\) is referred to as the cost-sharing rule, where \(\tau_i(\theta)\) is the transfer from agent \(i\) to the mechanism (if positive).

Denoting by \(E_{-i}\) the expectation operator with respect to \((\theta_{-i}, x)\), we can write agent \(i\)'s expected payoff when the announcements are \(\theta\) given that her true valuations are given by \(\theta_i = \)

\[\text{the rationale for this way of modelling is usually to overcome measurability problems, but since } A \text{ is finite this is not the case for our model. Instead, the main rationale is that it gives the most convenient notation to work with: more convient also than either the “natural approach” or randomizations in analogy to “distributional strategies” popularized by Milgrom and Weber [5].}\]
\((\theta_1^i, \theta_2^i)\) as
\[
E_{-i} \left[ \sum_{j=1,2} \zeta_i^j(\theta, x) \omega_i^j(\theta, x) \theta_i^j - \tau_i(\theta) \right] \quad \forall i \in I, \theta_i \in \Theta_i.
\] (4)

From the formulation in (4) the reader may observe that we allow the mechanism designer to collect taxes also from consumers who do not use either of the public goods. Nothing in our analysis depends on this assumption: restricting the designer to tax only customers who use at least one of the goods is equivalent in terms of the set of feasible inclusion and provision rules.

Incentive compatibility, that is, the requirement that truth-telling is a Bayesian Nash equilibrium in the revelation game induced by \(M\), requires that
\[
E_{-i} \left[ \sum_{j=1,2} \zeta_i^j(\theta, x) \omega_i^j(\theta, x) \theta_i^j - \tau_i(\theta) \right] \geq E_{-i} \left[ \sum_{j=1,2} \zeta_i^j(\hat{\theta}_i, \theta_{-i}, x) \omega_i^j(\hat{\theta}_i, \theta_{-i}, x) \theta_i^j - \tau_i(\hat{\theta}_i, \theta_{-i}) \right] \quad \forall i \in I, \theta \in \Theta, \hat{\theta}_i \in \Theta_i.
\] (5)

Allocations also have to be feasible in the sense that the taxes collected are enough to pay for the costs of provision. The analytically most convenient and seemingly most flexible version of such a restriction is the \textit{ex ante resource constraint},
\[
E \left[ \sum_i \tau_i(\theta) - \sum_{j=1,2} \zeta_i^j(\theta, x) C^j(n) \right] \geq 0.
\] (6)

Literally, this would be the relevant constraint only if the designer can purchase insurance against budget deficits, but standard arguments (see Mailath and Postlewaite [4] and Cramton et al [2]) can be used to show that for any allocation that can be implemented using transfers satisfying the \textit{ex ante constraint}, there exists a transfer rule that satisfies the feasibility constraint for every \(\theta\) which implements the same allocation.

Finally, we assume that \textit{voluntary participation} or \textit{individual rationality} must be respected. Agents know their type when they decide whether to participate in the mechanism, but that they do so before the uncertainty is resolved. Hence, individual rationality is imposed at the interim stage as,
\[
E_{-i} \left[ \sum_{j=1,2} \zeta_i^j(\theta, x) \omega_i^j(\theta, x) \theta_i^j - \tau_i(\theta) \right] \geq 0 \quad \forall i \in I, \theta_i \in \Theta_i.
\] (7)

We will refer to mechanisms satisfying (5),(6) and (7) as \textit{incentive feasible and} will mainly be concerned with \textit{constrained efficient mechanisms}. In this environment with transferrable utility,
the objective function of a social planner is thus to maximize

$$\sum_{j=1,2} E \zeta_j^i(\theta, x) \left[ \sum_{i \in I} \omega_i^j(\theta, x) \theta_i^j - C^j(n) \right].$$  \hspace{1cm} (8)

The unconstrained ex post efficient rule is to provide good $j$ if and only if $\sum_i \theta_i^j \geq C^j(n)$ and never exclude any agent from its usage when it is provided. With more than two agents this rule is implementable only in trivial cases: either because the first best allocation is never to provide any public good, or because it suffices to charge the lowest possible valuation to finance the provision cost. In any other case we can apply Mailath and Postlewaites’ [4] adaption of the fundamental inefficiency in bargaining result from Myerson and Satterthwaite [6] to conclude that ex post efficiency is impossible to achieve.

### 2.1 A Lower Dimensional Class of Mechanisms

The problem we are interested in solving is to maximize (8) subject to the constraints in (5),(6) and (7). This is a problem where all variables enter linearly in both the constraints and the objective function. These linearities and the symmetry of the problem can be used to reduce the dimensionality of the problem drastically.

We will now consider a simplified class of mechanisms where a generic element is given by $M = (\rho^1, \rho^2, \eta^1, \eta^2, t)$ where

$$\rho^j : \Theta \rightarrow [0,1] \text{ for } j = 1, 2$$

$$\eta^j : \Theta_i \rightarrow [0,1] \text{ for } j = 1, 2$$

$$t : \Theta_i \rightarrow \mathbb{R}.$$  \hspace{1cm} (9)

Here $\rho^j$ is the provision rule, $\eta^j$ is the inclusion rule and $t$ is the transfer rule. There are a number of restrictions built into the specification in (9) relative to (3). First of all, conditional on the realization of $\theta$, the provision probabilities $\rho^1(\theta), \rho^2(\theta)$ are stochastically independent of the inclusion probabilities $\eta^1(\theta_1), \eta^2(\theta_1), \ldots, \eta^1(\theta_n), \eta^2(\theta_n)$. Secondly, the inclusion and transfer rules are the same for all agents. Finally, the inclusion and transfer rules for any agent $i$ are independent of the realization of $\theta_{-i}$.

Call a (simple) mechanism anonymous if $\rho^j(\theta) = \rho^j(\theta')$ for $j = 1, 2$ and every $\theta, \theta' \in \Theta$ where $\theta'$ is permutation of $\theta$. Since it is already built into (9) that inclusion and transfer schemes are
symmetric across agents this removes the only remaining possibility for the index to matter. Using the symmetry of the problem and the linearity in the payoffs we can show that mechanisms of the form in (9) are sufficient for our purposes. That is,

**Proposition 1** For any incentive feasible mechanism $M$ of the form in (3) there exists an anonymous incentive feasible mechanism $M$ of the form in (9) that generates the same value of the planners objective function as $M$.

As a consequence of this result, whenever we refer to a “mechanism” in the remainder of this paper it will be some $M$ of the form in (9). The proof is somewhat tedious, but straightforward, and relegated to the appendix. The first step of the argument is to notice that each individual and type only cares about their perceived probability of consuming each public good and the expected transfer. For this reason there is nothing to gain either by making transfers and inclusion probabilities functions of $\theta_{-i}$ or by making inclusion and provision rules conditionally dependent. Hence (also without symmetry) it is sufficient use a provision rule of the form $\rho^j : \Theta \to \mathbb{R}$ and $n$ inclusion rules of the form $\eta^j_i : \Theta_i \to [0, 1]$ for $j = 1, 2$.

To understand why all agents without loss of generality may be treated symmetrically it is useful to begin by considering why this would be true if customers could not be excluded from usage. Starting from an asymmetric mechanism one could then create a new mechanism that generates the same surplus by permuting the roles of the agents. Randomizing over all $n!$ permuted provision rules and averaging over the transfers one can then create a symmetric mechanism that generates the same surplus. Since incentive and participation constraints of the symmetric mechanism is an average over all the permuted incentive and participation constraint, and since neither the expected transfer revenue or the probability of provision is changed, incentive feasibility of the original mechanism implies incentive feasibility of the symmetric mechanism. The argument for inclusion probabilities different from one has the same flavor, but since inclusion probabilities and provision probabilities are potentially correlated (they both depend on $\Theta_i$) the construction of the symmetric inclusion rule involves averaging over perceived probabilities to consume the goods rather than averaging directly over the inclusion probabilities.

A similar argument can be used to show that goods may be treated symmetrically as well if the provision costs and conditional distributions of the two goods are identical. To make this argument
we need to adjust the notation somewhat. Write \( \theta = (\theta^1, \theta^2) \), where \( \theta^j = (\theta^1_j, ..., \theta^n_j) \) for the vector of valuations. That is, we simply write the arguments in a different order in order to be able to switch the roles of the goods more easily. Given any \( (\theta^1, \theta^2) \) we then write \( (\theta^2, \theta^1) \) for the valuation vector \( \theta' \) where \( (\theta^1_i', \theta^2_i') = (\theta^2_i, \theta^1_i) \) for every \( i \in I \) and write \( (\theta^2_i, \theta^1_i) \) for the pair of agent \( i \) valuation \( \theta^i \) where \( (\theta^1_i, \theta^2_i) = (\theta^2_i, \theta^1_i) \). We can then show:

**Proposition 2** Suppose that \( \Theta^1 = \Theta^2, F^1(v|\cdot) = F^2(v|\cdot) \) for any \( v \in \Theta^1 = \Theta^2 \) and \( C^1(n) = C^2(n) \). Then, given any anonymous incentive feasible mechanism \( M \) there exists an incentive feasible mechanism \( \tilde{M} \) that generates the same social surplus where \( \rho^1(\theta^1, \theta^2) = \rho^2(\theta^2, \theta^1) \) and \( \eta^1(\theta^1_i, \theta^2_i) = \eta^2(\theta^2_i, \theta^1_i) \) for each \( \theta = (\theta^1, \theta^2) \in \Theta \).

The idea is very much the same as for Proposition 1. Reversing the roles of the goods one can construct a mechanism that generates the same surplus as the initial mechanism. The symmetric mechanism is then constructed by averaging the original and the reversed mechanism, where again the inclusion probabilities will not be raw averages, but constructed so that the perceived probability of consuming each good is the average of the perceived probabilities in the two mechanisms.

### 3 Pure Bundling Versus Separate Provision

For either the case of separate provision or pure bundling (meaning that consumers can either get the full bundle or nothing at all) the provision problem is a single-dimensional problem. This makes it possible to compare these two extremes by appealing to known results about provision of a single excludable public good.

#### 3.1 Constrained Optimal Mechanism in the Absence of Bundling

With a mechanism without bundling we mean a mechanism where the mechanism provider proceeds as if the other goods didn’t exist. Maybe the most straightforward way to think about it is to ask what would happen if there are two separate mechanism designers and where all agents recognize that whatever information they reveal to the designer providing good 1 will not be passed on to the designer in charge of good 2 and vice versa. Specifically, a non-bundling mechanism for provision of good \( j \) is a mechanism where:
1. the provision probability for good \( j \) depends only on announcements about valuations for good \( j \)

2. the inclusion probabilities for consumption of good \( j \) depend only on the announced valuation for good \( j \)

3. the financing of the two goods are separate.

Nothing in the proof of Proposition 1 depends on the number of public goods, so to find the best non-bundling mechanism it is sufficient to restrict attentions to mechanisms of the form

\[ M^j_S = (\rho^j_S, \eta^j_S, t^j_S), \]

where \( \rho^j_S : [\Theta^j]^n \to [0, 1] \) is the provision rule, \( \eta^j_S : \Theta^j \to [0, 1] \) is the inclusion rule and \( t^j_S : \Theta^j \to R \) is the transfer rule financing the provision of good \( j \). The problem to maximize social surplus over mechanisms of this form is a special case of the setup studied in Norman [7], which shows that:

Propositions 2 and 3) Let \( F^j \) denote the marginal distribution over \( \theta^j_i \) and assume that \( \theta^j_i - (1 - F^j(\theta^j_i)) / f^j(\theta^j_i) \) is strictly increasing. Define

\[ p^j_M = \arg \max_{p \in \Theta^j} p \left( 1 - F^j(p) \right). \] (10)

Then the ex ante probability of provision for the optimal mechanism converges to one as \( n \) goes to infinity if

\[ p^j_M \left( 1 - F^j(p^j_M) \right) > \lim_{n \to \infty} C^j(n) / n. \]

Moreover, the ex ante probability of provision in any feasible mechanism converges to zero as \( n \) goes to infinity if

\[ p^j_M \left( 1 - F^j(p^j_M) \right) < \lim_{n \to \infty} C^j(n) / n. \]

The interpretation of \( p^j_M \) is that it is the monopoly price that a profit maximizing provider would charge for access if it is taken for granted that the good is provided for sure. The basic intuition for the result is that since the (average) probability of being pivotal necessarily goes to zero as the number of agents goes to infinity it becomes impossible to price discriminate between agents who are included with the same probability. The restriction that virtual valuations are increasing ensures that the best inclusion rule is a threshold rule, where agents get access if an only if their type is above the threshold. Hence, all agents above the threshold must be charged.

\(^7\) Notice that we are making the assumption that all agents have valuations drawn from the same distribution. If, as in Norman [7], different agents have valuations drawn from different distributions, then the monopoly price becomes individual-specific. That is, while discrimination based on unobservables becomes a useless instrument in the limit, there are still gains from third degree price discrimination that will be exploited both by a profit maximizing provider and a welfare maximizing social planner.
(Proposition 4 and Lemma 3) Average cost pricing is asymptotically optimal in the parameter range where the provision probability converges to unity. Here the notion of “asymptotic optimality” is that the difference in per capita surplus between the best average cost pricing mechanism and the optimal mechanism can be made arbitrarily small by considering a sufficiently large economy. Average cost pricing in this context is simply to find the smallest \( p_j \) such that \( C^j(n)/n = p_j(1 - F^j(p_j)) \) and provide the good for sure.

### 3.2 Example 1: Bundling versus Non-Bundling with Uniformly Distributed and Independent Valuations

To fix ideas, assume that \( \theta_1^i \) and \( \theta_2^i \) are stochastically independent and that \( \theta_1^j \) is uniformly distributed on \([0, 1]\) for \( j = 1, 2 \). Moreover, assume that \( C^j(n) = cn \) for \( j = 1, 2 \), where we make the parametric restriction that \( c < \frac{1}{4} \). The reason for this restriction is that the “monopoly price” defined in (10) is \( p^j_M = \frac{1}{2} \). Hence, \( p^j_M \left(1 - F^j(p^j_M)\right) = \frac{1}{4} \), implying that the provision probability goes to zero whenever \( c > \frac{1}{4} \).

Under the assumption that \( c < \frac{1}{4} \) we know from the discussion in Section 3.1 that it is asymptotically optimal to provide for sure and use average cost pricing. Let \( p^j_* \) denote the price generated by the average cost pricing rule, that is, the smallest solution to

\[
c = p^j(1 - F^j(p^j)) = p^j(1 - p^j).
\]

Solving the quadratic, we find that \( p^j_* = \frac{1}{2} - \sqrt{\frac{1}{4} - c} < \frac{1}{2} = p^j_M \). That is, we get the obvious result that the social planner sets a lower price than a profit maximizing monopolist.

Suppose instead that the mechanism designer continues to provide both goods for sure, but charges \( 2p^{1*} = 2p^{2*} = 1 - 2\sqrt{\frac{1}{4} - c} \) for access to the bundle. That is, consumers have to decide whether to get both goods at a price given by twice the price in the unbundled case or nothing at all. Let \( \theta^b_i = \theta_1^i + \theta_2^i \) denote the valuation for the bundled good and let \( f^b \) denote the probability density function for \( \theta^b_i \), which has support \([0, 2]\) and is given by

\[
f^b(\theta^b_i) = \begin{cases} 
\theta^b_i & \text{if } \theta^b_i \leq 1 \\
2 - \theta^b_i & \text{if } \theta^b_i > 1.
\end{cases}
\]

In Figure 1 we plot the distribution of the average valuation \( \theta^b_i/2 \) and the distribution of valuations for a single good. In both the unbundled and the bundled case the utility loss compared to the
first best is given by the surplus lost due to exclusions. In the unbundled case, the agents that are excluded are represented by the rectangular area between 0 and $p^{1*}$ below the uniform density. In the bundled case, it is the triangular area between 0 and $p^{1*}$ below the density of average valuations. To realize from the geometry that the lost surplus is smaller in the bundled case one makes the observation that the lost surplus is the same if the price is $1/2$. Finally, a reduction in the price in the range $[1/4, 1/2]$ leads to a larger increase in surplus in the bundled case since more agents are included (that bundling dominates for prices on $[0, 1/4]$ is obvious).

Alternatively, we calculate the lost surplus in the unbundled case to be $2 \int_0^{p^{1*}} \theta_i^2 d\theta_i = (p^{1*})^2$. The lost surplus with bundling is $\int_0^{2p^{1*}} (\theta_i^2)^2 d\theta = 2(p^{1*})^3 < (p^{1*})^2$ since $p^{1*} < 1/2$. Hence, the pure bundling mechanism dominates the unbundled mechanism. Since there are fewer exclusions in the bundled mechanism there is a budget surplus with bundling, so further gains can be achieved by decreasing the price up to the point where the budget is exactly balanced.

Obviously, one shouldn’t take away too much from a single example, but what is interesting is that the reason for bundling to improve matters in the example is that bundling decreases the variation in willingness to pay for the good. Given that (which is the case in the example) the unbundled price is set to the left of the center of the distribution this reduction in the variance implies fewer exclusions when goods are bundled at the same (average) price as with separate
provision.

3.2.1 Difficulties of Generalization

If we maintain the assumption that \( \theta^*_1 \) and \( \theta^*_2 \) are identically and independently distributed random variables with cumulative density \( F \) we observe that the example generalizes to all problems where

\[
\Pr \left[ \frac{\theta^*_1 + \theta^*_2}{2} \leq p^*_j \right] < \Pr [\theta^*_2 \leq p^*_j] .
\]

(12)

Moreover, one can impose some restrictions on \( F \) that guarantees that the best price without bundling \( p^*_j \) is such that the ex ante probability of being excluded is less than 1/2. One set of restrictions that is sufficient is that \( F \) has a symmetric, logconcave density with support on non-negative values only. The problem is that, while (12) holds for many well-known parametric densities (such as the normal) for any \( p^*_j \) below the mean, we have not been able to find any sufficient conditions.

One example that nicely illustrates the difficulty is the distribution that is generated if \( \theta^*_j \) is uniform with probability \( \alpha \in (0, 1) \) and equal to the midpoint of the support with probability \( 1 - \alpha \).\(^8\) If we for simplicity assume that the support is \([0, 2]\), the cumulative of this distribution is

\[
F(\theta^*_j) = \begin{cases} \frac{\alpha}{2} \theta^*_j & \text{for } \theta^*_j < 1 \\ (1 - \alpha) + \frac{\alpha}{2} \theta^*_j & \theta^*_j \geq 1 \end{cases}.
\]

The cumulative density of the average will also be continuous except at 1, but the probability that the average is exactly 1 is \((1 - \alpha)^2 < (1 - \alpha)\). Letting \( \bar{F} \) denote the cumulative of the average we may use the symmetry to conclude that \( \lim_{y \to 1^-} \bar{F}(y) = \frac{1}{2} - \frac{(1-\alpha)^2}{2} > \frac{\alpha}{2} = \lim_{y \to 1^-} F(y) \). We therefore conclude that there exists \( \epsilon > 0 \) such that \( \bar{F}(y) > F(y) \) for any \( y \in (1 - \epsilon, 1) \), implying that bundling the two goods at a price in this interval at a price being twice the price for an individual good would increase the number of exclusions.

The example has a masspoint, but it should be clear that we can find smooth distributions with cumulative densities near enough the one above that would give the same result. Such a smooth distribution to be logconcave, but we have not been able to show that logconcavity implies that (12) holds for prices to the left of the mean.

\(^8\)We thank Larry Samuelson and Ted Bergstrom for this example.
3.3 Example 2: Bundling of a Large Number of Independent Goods

While it is hard to give conditions on the primitives for (12) to hold when two goods are bundled together the law of large numbers suggests that the logic of the uniform example should extend more or less trivially if a large number of goods are bundled together. This is true, but with a large number of goods and agents we can do even better. Indeed, with a sufficiently large number of agents and goods it is possible to approximate the ex post efficient provision rule.

Instead of two public goods, let \( \{j\}_{j=1}^{\infty} \) be an infinite sequence of “potential public goods”. Let \( \theta_j^i \) denote the \( i \)th agents valuation for public good \( j \) and assume that \( \theta_j^i \) is identically and independently distributed across agents. Moreover, assume that \( \theta_j^i \) and \( \theta_j^i' \) are stochastically independent for any pair \( j, j' \). Finally, assume that there exist uniform upper bounds for the means and variances. That is, there are finite numbers \( \mu \) and \( \sigma^2 \) such that \( E\theta_j^i \leq \mu \) for all \( j \) and \( \text{Var}\theta_j^i \leq \sigma^2 \) for every \( j \) (both these properties are implied by the assumption that there exists an interval \( [a, b] \) such that \( \Theta_j^i \subset [a, b] \) for every \( j \)). We will not impose symmetry in either costs or distribution of valuations, so Proposition 2 will not be used.

The ex post efficient rule is to provide good \( j \) if and only if \( \sum_{i=1}^{n} \theta_j^i \geq C^j(n) \) and exclude nobody from usage. First assume that \( E\theta_j^i > \lim_{n \to \infty} \frac{C^j(n)}{n} = c^j \). Then there exists \( N \) such that \( C^j(n) \leq nc^j + n(\text{E}\theta_j^i - c^j)/2 \) for every \( n \geq N \). Applying Chebyshev’s inequality we find that

\[
\Pr \left[ \sum_{i=1}^{n} \theta_j^i \leq C^j(n) \right] \leq \Pr \left[ \sum_{i=1}^{n} \theta_j^i + \frac{n(\text{E}\theta_j^i - c^j)}{2} \right] = \Pr \left[ \sum_{i=1}^{n} \theta_j^i - \frac{n(\text{E}\theta_j^i - c^j)}{2} \right] \leq \frac{4\text{Var} \left( \sum_{i=1}^{n} \theta_j^i \right)}{n^2(\text{E}\theta_j^i - c^j)^2} \leq \frac{4\sigma^2}{n^2(\text{E}\theta_j^i - c^j)^2}.
\]

Hence, for every \( \varepsilon > 0 \) we can find some \( N' \) such that the probability that the ex post efficient rule provides good \( j \) is at least \( 1 - \varepsilon \).

If \( E\theta_j^i < c^j \) a symmetric argument establishes that the first best provision probability converges to zero as the number of agents goes out of bounds. Only the first case is interesting, so we will now assume that there exists \( \delta > 0 \) such that \( E\theta_j^i - c^j \geq \delta \) for all \( j \) for the remainder of this section.\footnote{If \( E\theta_j^i > c^j \) for some goods, but the inequality is reversed for others the analysis still applies as long as there are sufficiently many goods that should be provided in a large economy according to the ex post efficient rule. Goods for which the first best probability of provision converges to zero may simply be dropped from the bundle and the rest of the analysis carries over.}
Let $\epsilon > 0$ and consider an anonymous mechanism where all $m$ public goods are provided for sure ($\rho^j(\theta) = 1$ for all $\theta \in \Theta$ and all $j$) and where the inclusion and transfer rules are given by,

$$
\eta^j(\theta) = \begin{cases} 
1 & \text{if } \sum_j \theta_i^j \geq \sum_j E\theta_i^j - \epsilon m \\
0 & \text{if } \sum_j \theta_i^j < \sum_j E\theta_i^j - \epsilon m 
\end{cases} \quad j \in \{1,\ldots,m\} (14)
$$

$$
t(\theta) = \begin{cases} 
\sum_j E\theta_i^j - \epsilon m & \text{if } \sum_j \theta_i^j \geq \sum_j E\theta_i^j - \epsilon m \\
0 & \text{if } \sum_j \theta_i^j < \sum_j E\theta_i^j - \epsilon m 
\end{cases}
$$

In words, whether an agent gets access to a good depends only on the valuation for the whole bundle. If that valuation for the whole bundle exceeds the threshold $\sum_j E\theta_i^j - \epsilon m$, then the agent gets access to all goods and pays a “user fee” for the bundle given by $\sum_j E\theta_i^j - \epsilon m$. If the valuation for the bundle is below the “price for the bundle” the agent gets nothing and pays nothing. Truth-telling is a dominant strategy for this mechanism and participation constraints are satisfied (also ex post). The only constraint that remains to be checked for the mechanism is therefore the feasibility constraint (6)

By the assumption of a uniform upper bound $\sigma^2$ for the variances of $\theta_i^1, \ldots, \theta_i^m$ we may appeal to Chebyshev’s inequality to conclude that

$$
\Pr \left[ \sum_j \theta_i^j - \sum_j E\theta_i^j < -\epsilon m \right] \leq \frac{\Var \left( \sum_j \theta_i^j \right)}{\epsilon^2 m^2} \leq \frac{m\sigma^2}{\epsilon^2 m^2} = \frac{\sigma^2}{\epsilon^2 m}. (15)
$$

Since the right hand side converges to zero as $m \to \infty$ it follows that for every $\epsilon > 0$ there exists $M < \infty$ such that

$$
\Pr \left[ \sum_{j=1}^m \theta_i^j - \sum_{j=1}^m E\theta_i^j < -\epsilon m \right] \leq \epsilon (16)
$$

for every $m \geq M$. The interpretation of (16) is that the probability that an agent is excluded to consume the bundle in mechanism (14) can be made arbitrarily small by increasing the number of goods. From (16) it follows that for every $\epsilon > 0$ there exists $M$ such that the expected transfer revenue collected in the bundling mechanism proposed satisfies

$$
\Ent(\theta) \geq n \left(1 - \epsilon \right) \left[ \sum_{j=1}^m E\theta_i^j - \epsilon m \right] (17)
$$

for $m \geq M$. The mechanism provides all goods with probability 1, so the costs of provision are $\sum_{j=1}^m C^j(n)$. Fix $N$ so that $C^j(n) \leq n(\rho_i^j + \epsilon^j)/2$ for every $j$. Mor
we observe that for every $\epsilon > 0$ there exists a finite $M$ such that (18) applies for $m \geq M$. Picking $\epsilon < \frac{\delta}{2M}$, where $\delta > 0$ is the uniform bound between $E\theta_i^j$ and $c^j$ ($E\theta_i^j - c^j \geq \delta$ for all $j$) and $\mu$ is the uniform bound for the expected valuations ($E\theta_i^j \leq \mu$ for all $j$).

\[
\text{Ent} (\theta) - \sum_{j=1}^{m} C^j (n) \geq n (1 - \epsilon) \left[ \sum_{j=1}^{m} E\theta_i^j - \epsilon m \right] - n \sum_{j=1}^{m} (E\theta_i^j + c^j)/2
\] (18)

It is easy to check that this implies that the feasibility constraint holds for the bundling mechanism given that $\epsilon$ is small enough and $\frac{\sum_{j=1}^{m} C^j (n)}{n}$ is sufficiently close to $\sum_{j=1}^{m} c^j$ (that is, if $C^j (n) = nc^j$ for each $n$ there is no need take the number of agents to infinity for budget balance). Incentive compatibility holds trivially and the participation constraints hold even ex post, so this means that the bundling mechanism is feasible if there are sufficiently many goods bundled together.

Moreover, by increasing the number of goods the probability that an agent is excluded can be made arbitrarily small (near the efficient inclusion rule “never exclude”) and for a large number of agents the rule “always provide” is near the efficient provision rule. We can thus conclude that the bundling mechanism can approximate the outcome of the first best efficient mechanism arbitrarily well by adding a sufficient number of goods and agents to the economy.

If bundling is not allowed, the problem collapses to a special case of a model considered in Norman [7], for which it is known that the probability of exclusion is bounded away from zero in the constrained optimal mechanism. The example thus illustrates that bundling the goods together may improve economic efficiency.

The intuition for what is going on is also rather straightforward. By selling usage of the good only as a bundle it becomes irrelevant what valuations for individual goods are. It is then only the average valuation that the mechanism designer needs to make the consumer reveal. But, the distribution over the average valuation collapses into a mass point as the number of goods in the bundle is taken to infinity, so the informational problem essentially disappears. Moreover, unlike the case with private goods, there is no cost in providing access to something the consumer doesn’t really value, so there are no inefficiencies associated with providing access to everything that is provided.

Obviously, the feature that the distribution over the average valuation becomes degenerate when the number of goods in the bundle is taken to infinity is extreme and unrealistic. Hence, the
example is only meant to be suggestive. In a very stark way the example illustrates that if the variance for the “average good” is smaller than the variance of each individual good, then bundling may be a good idea since the reduced variance allows fewer consumer to be excluded from usage. In the rest of this paper we will investigate this idea in a stylized version of the model.

4 The Model with Binary Valuations

We now impose some further simplifying assumptions to get a tractable programming problem. First of all we assume that the valuation of each public good is a binary random variable, where we for simplicity assume that the two public goods are symmetric in all respects. That is, the valuation for good j can either be “high” (\(\theta^j_i = h\)) or “low” (\(\theta^j_i = l\)), implying that the type space for an individual is \(\Theta_i = \{(h, h), (h, l), (l, h), (l, l)\}\). For notational brevity we will henceforth write \(\theta_i = hh\) instead of \((h, h)\), \(\theta_i = hl\) instead of \((h, l)\), and analogously for other valuations. In the baseline model we also assume that \(\alpha = \Pr[\theta^1_i = h] = \Pr[\theta^2_i = h] \in (0, 1)\), implying that the probability distribution \(F\) over \(\Theta_i\) is

\[
\left\{ \alpha^2, \alpha (1 - \alpha), \alpha (1 - \alpha), (1 - \alpha)^2 \right\}.
\]

While independence across agents is absolutely crucial for the analysis, independence across goods is not. Everything would go through with rather minor modifications also with a probability distribution of the form \(\left\{ \sigma(hh), \frac{\sigma(m)}{2}, \frac{\sigma(m)}{2}, \sigma(ll) \right\}\), where \(\sigma(m)\) is the probability of a “mixed type”. That is, as long as the symmetry is kept we can handle positive or negative correlation between the valuation for the goods rather easily. For simplicity of notation however, we stick to the case where valuations are independent for now and return to the implications of correlation later.

Finally, we assume that costs are given by \(C^1(n) = C^2(n) = cn\). The most important simplification here is that costs are the same for both goods, which together with the symmetry on the demand side will allow us to restrict attention to symmetric optimal mechanisms. The per capita costs are also kept constant, which does simplify the notation, but would be easy to relax.

An ex post optimal mechanism is to provide good j if and only if \(\sum_{i=1}^n \theta^j_i \geq C^j(n) = cn\). If \(h \leq c\) “never provide” is thus ex post optimal, which can be trivially implemented. Furthermore, if \(l \geq c\) “always provide” is ex post optimal and can be implemented by charging a constant tax.
equal to c. We therefore maintain the assumption that \( l < c < h \) in order to keep the problem interesting.

### 4.1 Benchmark: Asymptotic Provision Probabilities Without Bundling

In this section, we establish, as a benchmark, the asymptotic provision probabilities of the two public goods when the provision problem for each public good is considered in isolation, i.e., when no bundling is allowed. As we argued in Section 3.1 the inclusion probability may without loss be assumed to depend on the type of the individual agent and the provision rule may be assumed to treat all agents equally.

To emphasize that the solution depends on the size of the economy we denote a non-bundling mechanism for the provision of good \( j \) in an economy of size \( n \) by \((\rho_n^j, \eta_n^j, t_n^j)\), where \( \rho_n^j : \{1, \ldots, n\} \to [0, 1] \) and \( \rho_n^j(m) \) denotes the probability of provision if \( m \) agents announce a high valuation for good \( j \); \( \eta_n^j \in [0, 1] \) is the inclusion probability for type \( l \) and \( t_n^j = (t_n^j(h), t_n^j(l)) \) are the transfers. In principle it is also possible to exclude agents of type \( h \), but this tightens the downwards incentive constraint for type \( h \) and is an option that will never be used, so we immediately build that in to the mechanism to simplify notation.

Arguments similar to Propositions 2 and 3 in Norman [7] can be used to get a tight characterization of the asymptotic provision and inclusion rules in the model where the provision problems must be solved independently.

**Proposition 3** Consider a sequence of economies of size \( \{n\}_n=1^\infty \). Then,

1. \( \lim_{n \to \infty} E\rho_n^j(m) = 0 \) for any sequence of feasible mechanisms \( \{\rho_n, \eta_n, t_n\} \) if \( ah < c \).

2. \( \lim_{n \to \infty} E\rho_n^j(m) = 1 \) for any sequence of constrained optimal mechanisms \( \{\rho_n, \eta_n, t_n\} \) if \( ah > c \). Moreover, in this case the inclusion probabilities and transfers for a sequence of optimal mechanisms satisfy

\[
\lim_{n \to \infty} \eta_n^j = \frac{ah - c}{ah - l},
\]

\[
\lim_{n \to \infty} t_n^j(l) = \frac{ah - c}{ah - l},
\]

\[
\lim_{n \to \infty} t_n^j(h) = \left[1 - \frac{ah - c}{ah - l}\right] h + \frac{ah - c}{ah - l} l.
\]
We have omitted the formal proof, but since the idea can be conveyed without the technical details we will here provide an heuristic explanation for the result.\textsuperscript{10} Since the effect on the provision probability from any individual announcement is negligible in a large economy, the incentive constraint for a type-\(h\) agent is roughly that

\[ E \hat{\rho}^j_n (m) h - \hat{\rho}^j_n (l) \geq E \hat{\rho}^j_n (m) \hat{\eta}^j_n h - \hat{\eta}^j_n (l). \tag{19} \]

and the participation constraint for the low type dictates that \(\hat{\rho}^j_n (l) = E \hat{\rho}^j_n (m) \hat{\eta}^j_n l\). Because the incentive constraint for the high type binds in the optimal mechanism, budget balance then requires that, approximately,

\[
E \hat{\rho}^j_n (m) c = \alpha \hat{\rho}^j_n (h) + (1 - \alpha) \hat{\rho}^j_n (l) \approx \alpha \left[ \hat{\rho}^j_n (l) + E \hat{\rho}^j_n (m) h (1 - \hat{\eta}^j_n) \right] + (1 - \alpha) \hat{\rho}^j_n (l) \\
= \hat{\rho}^j_n (l) + \alpha E \hat{\rho}^j_n (m) h (1 - \hat{\eta}^j_n) \\
= E \hat{\rho}^j_n (m) \hat{\eta}^j_n l + E \hat{\rho}^j_n (m) \alpha h (1 - \hat{\eta}^j_n).
\tag{20} \]

Hence, \(\hat{\eta}^j_n \approx \frac{\alpha h - c}{\alpha h - l}\) follows from (??), given that we believe that it is valid to ignore the effects from being pivotal in the decision.

Note that, by inspecting (20), if \(\alpha h < c\), (since by assumption, \(l < c\) as well), then \(\lim_{n \to \infty} E \hat{\rho}^j_n (m) = 0\). Otherwise the budget balance constraint must be violated for large \(n\). On the other hand, if instead \(\alpha h > c\), it is feasible to provide for sure (for any \(n\)) with the transfers specified in Proposition 3, and inclusion probability \(\hat{\eta}^j_n = (\alpha h - c) / (\alpha h - l)\). Conditional on this inclusion probability, the ex post efficient rule is to provide whenever

\[
\frac{mh + (n - m) \hat{\eta}^j_n l}{n} \geq c,
\tag{21}
\]

\[
\iff \frac{m}{n} h + \frac{n - m}{n} \hat{\eta}^j_n l \geq c
\]

An application of Chebyshevs inequality guarantees that

\[
\text{plim} \left( \frac{m}{n} h + \frac{n - m}{n} \hat{\eta}^j_n l \right) = \alpha h + (1 - \alpha) \frac{\alpha h - c}{\alpha h - l} > \alpha h > c.
\tag{22}
\]

Thus, the ex post efficient rule conditional on the given inclusion probability converges towards “always provide”. Hence \(\lim_{n \to \infty} E \hat{\rho}^j_n (m) = 1\) in the optimal mechanism. The limits for the transfers can then be obtained by substituting \(\lim_{n \to \infty} E \hat{\rho}^j_n (m) = 1\) back into the incentive and participation constraints.

\textsuperscript{10}Details available on request from the authors.
4.2 Example 3: Improvement when Bundling is Allowed

Asymptotically, the optimal mechanisms for a single public good in isolation characterized in Propositions 3 may not be efficient. First of all, the asymptotic provision probability is zero when \( \alpha h < c \) while efficiency requires that the public good be provided whenever \( \alpha h + (1 - \alpha) l > c \); second, when \( \alpha h > c \), there is still inefficiency due to positive probability of exclusion of low valuation agents, even though the public good is provided asymptotically with probability 1. Before we characterize the optimal provision mechanism with bundling, we first provide an example to demonstrate that improvement can be indeed be achieved via bundling by showing a particular incentive compatible, balanced-budget voluntary mechanism (that may not necessarily be optimal) can improve upon the mechanisms without bundling.

Consider the following provision mechanism with bundling:

- \( t_{hh} = t_{hl} = t_{lh} = 2c/\left(2\alpha - \alpha^2\right), t_{ll} = 0; \)
- \( \eta_{hh} = \eta_{hl} = \eta_{lh} = 1, \eta_{ll} = 0. \)
- \( \rho^1(x) = \rho^2(x) = 1 \) for all \( x \in X \), i.e. always provide the public goods.

Notice that the (ex ante) feasibility constraint holds for any choice of \( n \) by construction of the mechanism since

\[
\alpha^2 t_{hh} + (1 - \alpha) \alpha t_{hl} + \alpha(1 - \alpha)t_{lh} + (1 - \alpha)^2 t_{ll} = \left[\alpha^2 + (1 - \alpha) \alpha + \alpha(1 - \alpha)\right] \frac{2c}{2\alpha - \alpha^2} = 2c
\]

Ex post there will typically be either a surplus or a deficit, but since we know that an ex post budget balancing mechanism can be constructed given any ex ante budget balancing mechanism (for example by letting one of the agents provide “fair insurance” to the mechanism designer) this is no real problem.

The only question is thus whether the incentive constraints for the higher types hold. Under the proposed mechanism, this is the case whenever an agent of type \( hl \) or \( lh \) does not have incentive to mimic \( ll \), i.e., when \( h + l - \frac{2c}{2\alpha - \alpha^2} \geq 0 \). Given that this inequality is satisfied the mechanism is incentive feasible for any \( n \).
Now, suppose that the valuations of each public good satisfies that \( l < c \), and \( \alpha h < c \), then by Proposition 3, we know that the provision probability without bundling converges to zero for each good. However, if \( h + l \geq 2c/ (2\alpha - \alpha^2) \), the mechanism proposed above will provide both public goods with probability one. It is also easy to show that there exists configurations of \( l, h, c \) such that \( l < c, \alpha h < c \) and \( h + l \geq 2c/ (2\alpha - \alpha^2) \). We summarize the above discussions as follows:

**Claim.** Fix any \( c > 0, \alpha \in (0, 1) \). There exists \( h > l \) such that

1. the provision probability under the optimal mechanism is zero when each public good is considered in isolation; but
2. both public goods are provided with probability one under the proposed mechanism with bundling.

The range of values of \( h \) and \( l \) for any \( c > 0 \), and \( \alpha \in (0, 1) \) for which the above-proposed bundling mechanism outperforms the optimal mechanism without bundling is depicted in Figure 2. The intuition for the improvement of bundling mechanism is as follows. In the revenue maximizing mechanism without bundling, only high valuation types are included in the public good provision, thus a fraction \( \alpha^2 \) of the agents are included in both goods, and a fraction \( 2\alpha (1 - \alpha) \) agents are included on one and only one good, and the remainder agents are excluded from both goods. In the proposed bundling mechanism, all agents are included in both public goods except that type-\( ll \) agents are excluded from both. Thus in the bundling mechanism, more agents contribute since only \( (1 - \alpha)^2 \) consumers are excluded, even though the contribution is smaller per agent.

### 5 The Full Design Problem with Binary Valuations

We will now solve the design problem to maximize social surplus (8) subject to the incentive compatibility constraints in (5), the feasibility constraint (6) and the participation constraints (7). Appealing to Proposition 1 we may without loss of generality only consider mechanisms whose provision rule depends only the number of agents who and where transfers and inclusion probabilities for agent \( i \) depends only on \( \theta_i \). That is, it is without loss of generality to consider mechanisms of the form

\[
M = \left( \{ \rho^j_i, \eta^j_i \}_{j=1,2}, t \right),
\]

(24)
Figure 2: The Bundling Mechanism Outperforms Optimal Non-bundling Mechanism in the Shaded Region of \((h, l)\).

where \(\eta^j = (\eta^j_{hh}, \eta^j_{hl}, \eta^j_{lh}, \eta^j_{ll}) \in [0,1]^4\) for \(j = 1, 2\), \(t = (t_{hh}, t_{hl}, t_{lh}, t_{ll}) \in \mathbb{R}^4\), \(\rho^j : X \rightarrow [0,1]\), and where a generic element of the set \(X\) is a vector \(x \equiv (x_{hh}, x_{hl}, x_{lh}, x_{ll})\). For each \(\theta_i \in \Theta_i = \{hh, hl, lh, ll\}\) the interpretation of \(x_{\theta_i}\) is the number of agents announcing type \(\theta_i\), and the set \(X\) may thus be expressed as

\[
X = \left\{ x \in \{0, ..., n\}^4 : x_{hh} + x_{hl} + x_{lh} + x_{ll} = n \right\}.
\]  

To avoid notational clutter we define

\[
s^3(x, \eta) = (\eta^1_{hh} x_{hh} + \eta^1_{hl} x_{hl}) h + (\eta^1_{lh} x_{lh} + \eta^1_{ll} x_{ll}) l - cn  
\]

\[
s^2(x, \eta) = (\eta^2_{hh} x_{hh} + \eta^2_{hl} x_{hl}) h + (\eta^2_{lh} x_{lh} + \eta^2_{ll} x_{ll}) l - cn.
\]

That is, \(s^j(x, \eta)\) is the surplus generated if good \(j = 1, 2\) is provided in state \(x\) (given truth-telling) if the inclusion probabilities are given \(\eta = (\eta^1, \eta^2) \in [0,1]^8\). It is also useful to define

\[
\rho^j_i(\theta_i) = E \left[ \rho^j(x) \mid \theta_i \right]
\]

for each \(\theta_i \in \{hh, hl, lh, ll\}\). In words \(\rho^j_i(\theta_i)\) is the perceived probability of good \(j\) being provided for an agent with type \(\theta_i \in \{hh, hl, lh, ll\}\). The probability that an agent with type \(\theta_i\) consumes the good in a truth-telling equilibrium is thus \(\eta^j\rho^j_i(\theta_i)\).
In principle there are 12 incentive constraints that needs to be satisfied. However, types are naturally ordered as \( hh \) being the “highest type”, \( hl \) and \( lh \) being “middle types” and \( ll \) being the “lowest type”. We therefore conjecture that only downwards incentive constraints are relevant and will therefore ignore all upwards constraints as well as the constraints between type \( hl \) and \( lh \). Once the solution to the relaxed problem is fully characterized we will verify that these constraints are satisfied. Finally, it is easy to check that if \( hh \) is better off announcing her true type than \( lh \) and \( hl \) is better off announcing her true type than \( ll \), then there are no incentives for \( hh \) to announce \( ll \). The incentive constraints we will consider in the relaxed program are thus, using the notation (27) for brevity,

\[
\begin{align*}
\eta_{hh}^1 \rho_1^1 (hh) h + \eta_{hh}^2 \rho_1^2 (hh) h - t_{hh} & \geq \eta_{hh}^1 \rho_1^1 (hl) h + \eta_{hh}^2 \rho_1^2 (hl) h - t_{hl} \\
\eta_{hl}^1 \rho_1^1 (hh) h + \eta_{hl}^2 \rho_1^2 (hh) h - t_{hh} & \geq \eta_{hl}^1 \rho_1^1 (lh) h + \eta_{hl}^2 \rho_1^2 (lh) h - t_{lh} \\
\eta_{ll}^1 \rho_1^1 (hl) h + \eta_{ll}^2 \rho_2^2 (hl) l - t_{ll} & \geq \eta_{ll}^1 \rho_1^1 (ll) l + \eta_{ll}^2 \rho_2^2 (ll) l - t_{ll}.
\end{align*}
\]  

(28)  

(29)  

(30)  

Next, given that all downward incentive constraints and the participation constraint for type \( ll \) are fulfilled it follows by a standard argument that the participation constraints for types \( hh, hl \) and \( lh \) are also fulfilled.\(^ {11} \) Hence, the only relevant participation constraint is

\[
\eta_{ll}^1 \rho_1^1 (ll) l + \eta_{ll}^2 \rho_2^2 (ll) l h - t_{ll} \geq 0.
\]  

(32)

Finally, the budget balance constraint can be simplified considerably due to the simple transfer schemes and the constant per capita costs. That is, we have that

\[
E \left( \sum_i \tau_i (\theta) - \sum_{j=1,2} \rho_j (\theta) C_j (n) \right) = \sum_i E \tau_i (\theta) - \sum_{j=1,2} E \rho_j (\theta) C_j (n)
\]

\[
= n \left[ \alpha^2 t_{hh} + \alpha (1 - \alpha) [t_{hl} + t_{lh}] + (1 - \alpha)^2 t_{ll} \right] - \sum_{j=1,2} E \rho_j (\theta) cn
\]

\[
= n \left[ \alpha^2 t_{hh} + \alpha (1 - \alpha) [t_{hl} + t_{lh}] + (1 - \alpha)^2 t_{ll} - \sum_{j=1,2} E \rho_j (x) c \right].
\]

\(^ {11} \)The argument is that, by incentive compatibility, all higher types are better off than pretending to be type \( ll \). Since the payoff from pretending to be \( ll \) is higher than the payoff for a (truth-telling) type \( ll \), interim individual rationality follows for any other type.
We may thus express the budget balance constraint (6) in per capita form as

\[ \alpha^2 t_{hh} + \alpha (1 - \alpha) [t_{hl} + t_{lh}] + (1 - \alpha)^2 t_{ll} - \sum_{j=1,2} \mathbb{E} \rho^j (x) c \geq 0. \]  

(34)

Letting $\mathcal{M}$ denote the set of all mechanisms of the form in (24) we can thus express the relaxed programming problem as

\[ \max_{M \in \mathcal{M}} \mathbb{E} \sum_{j=1,2} \rho^j (x) s^j (x, \eta) \]  

(35)

\[ \text{s.t. } \eta^1_{hh} \rho^1_i (hh) h + \eta^2_{hh} \rho^2_i (hh) h - t_{hh} \geq \eta^1_{hl} \rho^1_i (hl) h + \eta^2_{hl} \rho^2_i (hl) h - t_{hl} \]

\[ \eta^1_{lh} \rho^1_i (lh) l + \eta^2_{lh} \rho^2_i (lh) l - t_{lh} \geq \eta^1_{ll} \rho^1_i (ll) l + \eta^2_{ll} \rho^2_i (ll) l - t_{ll} \]

\[ \eta^1_{hl} \rho^1_i (hl) l + \eta^2_{hl} \rho^2_i (hl) l - t_{hl} \geq \eta^1_{lh} \rho^1_i (lh) h - t_{lh} \]

\[ \eta^1_{ll} \rho^1_i (ll) l + \eta^2_{ll} \rho^2_i (ll) l - t_{ll} \geq 0. \]

\[ \alpha^2 t_{hh} + \alpha (1 - \alpha) [t_{hl} + t_{lh}] + (1 - \alpha)^2 t_{ll} - \sum_{j=1,2} \mathbb{E} \rho^j (x) c \geq 0. \]

Lemma 1 There exists at least one optimal solution to (35)

A proof is in the appendix for completeness. The basic idea is to show that there is no loss to bound the transfers from below and above. If any of the taxes is sufficiently high, the incentive constraints and the remaining participation constraint cannot be all satisfied. If any of the agents receive too large a transfer, it is impossible to satisfy both the feasibility constraint and the incentive constraints. Hence, the other constraints in the problem implies that the transfers stay within a bounded set. We may thus add a condition that $t_{\theta_i} \in [\underline{t}, \overline{t}]$ for every $\theta_i$ and appropriately chosen $\underline{t}$ and $\overline{t}$ without changing the feasible set. We conclude that the constraint set is compact.

The final step in simplifying the programming problem is the observation that Lemma 2 implies that we may restrict attention to mechanisms where the goods are treated symmetrically. That is,

\[ \rho^1 (x_{hh}, x_{hl}, x_{lh}, x_{ll}) = \rho^2 (x_{hh}, x_{lh}, x_{hl}, x_{ll}) \]

(36)

\[ \eta^1_{hh} = \eta^2_{hh}, \eta^1_{hl} = \eta^2_{hl}, \eta^1_{lh} = \eta^2_{lh}, \eta^1_{ll} = \eta^2_{ll} \]

(37)

\[ t_{hl} = t_{lh} \]

(38)
From (36) and (37) it is intuitively clear and easy to show that

\[ \rho_1^1 (hh) = \rho_2^2 (hh), \rho_1^1 (hl) = \rho_2^2 (hl), \rho_1^1 (lh) = \rho_2^2 (lh) \quad (39) \]

\[ E \rho_1^1 (x) s_1^1 (x, \eta) = E \rho_2^2 (x) s_2 (x, \eta), \quad (40) \]

implying that (35) may be replaced by

\[ \max_{\rho^1 (x), \eta^1, \rho_1^1 (hh), \eta_1^1, t_{hh}, \rho_1^1 (hl), \eta_1^1, t_{hl}, \rho_1^1 (ll), \eta_1^1, t_{ll}} 2E \rho_1^1 (x) s_1^1 (x, \eta) \quad (41) \]

s.t. \[ 2\eta_1^1 \rho_1^1 (hh) h - t_{hh} \geq \eta_1^1 \rho_1^1 (hl) h + \eta_1^1 \rho_1^1 (lh) h - t_{hl} \]

\[ \eta_1^1 \rho_1^1 (hl) h + \eta_1^1 \rho_1^1 (lh) l - t_{hl} \geq \eta_1^1 \rho_1^1 (ll) (h + l) - t_{ll} \]

\[ 2\eta_1^1 \rho_1^1 (ll) l - t_{ll} \geq 0. \]

\[ \alpha^2 t_{hh} + \alpha (1 - \alpha) 2t_{hl} + (1 - \alpha)^2 t_{ll} - 2E \rho_1^1 (x) c \geq 0. \]

The multiplicative constant 2 in the objective is obviously redundant, but we will nevertheless keep it through the analysis since it aids interpretations to keep the units in the objective function and the constraints comparable.

6 The Solution

It is relatively straightforward to show that Slatters’ condition for constraint qualification holds, so the Kuhn-Tucker conditions are necessary for an optimum. Since we know that a solution to (41) exists (and that a solution to (41) solves (35)) these first order conditions therefore provide a characterization of the optimal mechanism, provided that we can demonstrate that the relaxed program satisfies the constraints that we ignored when formulating (35).

6.1 Relationship Between Multipliers

Taxes enter linearly into all constraints and there are no boundary constraints to worry about. It is therefore convenient to begin the analysis by taking first order conditions with respect to \( t_\theta \). This allows us to express the multiplier of any other constraint as a linear scaling of the multiplier of the feasibility constraint. For notation we let \( \lambda_{hh} \) denote the multiplier on the first incentive constraint, \( \lambda_{hl} \) the multiplier on second incentive constraint, \( \lambda_{ll} \) the multiplier on the participation constraint.
constraint for the lowest type, and \( \Lambda \) the multiplier on the feasibility constraint.\(^{12} \) The first order conditions with respect to \( t = (t_{hh}, t_{hl}, t_{ll}) \) are,

\[
\begin{align*}
(\text{w.r.t. } t_{hh}) & \quad -\lambda_{hh} + \Lambda \alpha^2 = 0 \\
(\text{w.r.t. } t_{hl}) & \quad \lambda_{hh} + \lambda_{hl} + \Lambda 2\alpha (1 - \alpha) = 0, \\
(\text{w.r.t. } t_{ll}) & \quad \lambda_{hl} - \lambda_{ll} + \Lambda (1 - \alpha)^2 = 0
\end{align*}
\] (42)

and from (42) we can immediately conclude that

**Lemma 2** In any solution to (41) the multipliers \((\lambda_{hh}, \lambda_{hl}, \lambda_{ll}, \Lambda)\) satisfy: \(\lambda_{hh} = \alpha^2 \Lambda, \lambda_{hl} = (2\alpha - \alpha^2) \Lambda, \) and \(\lambda_{ll} = \Lambda.\)

### 6.2 Optimal Inclusion Rules

In order to evaluate the optimal inclusion rules we need to be more explicit about the probability distribution over \( x = (x_{hh}, x_{hl}, x_{lh}, x_{ll}) \) than in the analysis up to this point. We will let \( a(x) \) denote the multinomially distributed probability for any \( x \in X, \) which has probability density

\[
a(x) = \frac{n!}{x_{hh}! x_{hl}! x_{lh}! x_{ll}!} (\alpha^2)^{x_{hh}} [\alpha (1 - \alpha)]^{x_{hl}} [\alpha (1 - \alpha)]^{x_{lh}} [(1 - \alpha)^2]^{x_{ll}}.
\] (43)

The incentive and participation constraints are also expectations with respect to a multinomially distributed random variable. However, the expectations in these constraints are over \( \theta_{-1} \) rather than \( \theta. \) In order to express these constraints in a simple way we write

\[
a_{-1}(x) = \frac{n - 1!}{x_{hh}! x_{hl}! x_{lh}! x_{ll}!} (\alpha^2)^{x_{hh}} [\alpha (1 - \alpha)]^{x_{hl}} [\alpha (1 - \alpha)]^{x_{lh}} [(1 - \alpha)^2]^{x_{ll}}
\] (44)

for every \( x \in X_{-1}, \) where \( X_{-1} = \{ x \in \{0, \ldots, n - 1\}^4 : x_{hh} + x_{hl} + x_{lh} + x_{ll} = n - 1 \}. \) In words, \( a_{-1}(x) \) probability of a particular number of each type if one individual is ignored. Using the

\(^{12}\text{Recall that, with the symmetry, the first incentive constraint is the constraint that type } hh \text{ shouldn’t have any incentive to mimic either type } hl \text{ or type } lh. \text{ The constraint } \lambda_{hl} \text{ is the constraint that neither type } hl \text{ or } lh \text{ should have incentives to announce type } ll.\)
Lemma 3 Let \( M = (\rho^1, \rho^2, \eta^1, \eta^2, t) \) be a symmetric solution to (35) and let \( \Phi = \frac{\Lambda}{1+\Lambda} \), where \( \Lambda \) is the associated multiplier on the resource constraint. Also, suppose that \( \rho^1_i(\theta_i) > 0 \) for all \( \theta_i \in \Theta_i \) and \( j = 1, 2 \). Then,

1. \( \eta^1_{hh} = \eta^2_{hh} = \eta^1_{hl} = \eta^2_{lh} = 1 \)

As before we let \( \theta = \Theta \), denote the multipliers for constraints (46),(47),(48), and (49) respectively. We also let \( \gamma_{\theta_i} \) and \( \phi_{\theta_i} \) be the multipliers corresponding with the constraints \( \eta^1_{\theta_i} \geq 0 \) and \( 1 - \eta^1_{\theta_i} \geq 0 \) respectively.

For ease of stating the result that characterizes the optimal inclusion rules we define two linear functions \( G : [0,1] \to R \) and \( H : [0,1] \to R \) as

\[
G(\Phi) \equiv (1-\Phi)2l + \Phi \left[ \frac{2\alpha - \alpha^2}{\alpha(1-\alpha)} - \frac{\alpha^2}{\alpha(1-\alpha)^2} \right] h
\]

\[
H(\Phi) \equiv (1-\Phi)2l + \Phi \left[ \frac{\alpha^2}{(1-\alpha)^2} - \frac{2\alpha - \alpha^2}{(1-\alpha)^2} \right] \theta(h + l)
\]

The result is:
Proof. CASE 1: Consider first the optimality conditions with respect to \( \eta^1_{hh} \), which are

\[
0 = 2 \sum_{x \in X} a(x) \rho^1(x) \frac{x_{hh}^i}{n} + \lambda_{hh} \sum_{x \in X_{-i}} a_{-1}(x) \rho^1(x_{hh}, x_{hl}, 1, x_{ll}) h + \gamma_{hh} - \phi_{hh} \tag{52}
\]

All terms except \( \gamma_{hh} - \phi_{hh} \) in the first order condition are strictly positive, so \( \gamma_{hh} - \phi_{hh} < 0 \). The only possibility for this is that \( \phi_{hh} > 0 \), which requires that \( \eta^1_{hh} = 1 \) for the complementary slackness constraint to be fulfilled. \( \eta^2_{hh} = 1 \) follows from proposition 2.

CASE 2: The first order condition with respect to \( \eta^1_{hl} \) reads

\[
2 \sum_{x \in X} a(x) \rho^1(x) h \frac{x_{hl}^i}{n} - \lambda_{hh} \sum_{x \in X_{-i}} a_{-1}(x) \rho^1(x_{hh}, x_{hl}, 1, x_{ll}) h + \lambda_{hl} \sum_{x \in X_{-i}} a_{-1}(x) \rho^1(x_{hh}, x_{hl}, 1, x_{ll}) h + \gamma_{hl} - \phi_{hl} = 0.
\]  

To make this condition more transparent we use (43) and (44) to observe that the identity

\[
a(x) = \frac{n!}{x_{hh}!x_{hl}!x_{ll}!} (\alpha^2)^{x_{hh}} [\alpha (1 - \alpha)]^{x_{hl}} [\alpha (1 - \alpha)]^{x_{ll}} [(1 - \alpha)^2]^{x_{ll}}
\]

\[
= \frac{n}{x_{hl}} \alpha (1 - \alpha) \frac{(n-1)!}{x_{hh}(x_{hh}-1)!x_{hl}!x_{ll}!} (\alpha^2)^{x_{hh}-1} [\alpha (1 - \alpha)]^{x_{hl}-1} [\alpha (1 - \alpha)]^{x_{ll}} [(1 - \alpha)^2]^{x_{ll}}
\]

\[
= \frac{n}{x_{hl}} \alpha (1 - \alpha) a_{-1}(x_{hh}, x_{hl} - 1, x_{ll}, x_{ll}),
\]  

holds for any \( x \) such that \( x_{hl} \geq 1 \). Hence

\[
\sum_{x \in X} a(x) \rho^1(x) h \frac{x_{hl}^i}{n} = \sum_{x \in X | x_{hl} \geq 0} a(x) \rho^1(x) h \frac{x_{hl}^i}{n}
\]

\[
= \sum_{x \in X | x_{hl} \geq 1} \frac{n}{x_{hl}} \alpha (1 - \alpha) a_{-1}(x_{hh}, x_{hl} - 1, x_{ll}, x_{ll}) \rho^1(x) h \frac{x_{hl}^i}{n}
\]

\[
= \alpha (1 - \alpha) h \sum_{x \in X | x_{hh} \geq 1} a_{-1}(x_{hh}, x_{hl} - 1, x_{ll}, x_{ll}) \rho^1(x)
\]

\[
= \alpha (1 - \alpha) h \sum_{x \in X_{+}} a_{-1}(x) \rho^1(x_{hh}, x_{hl} + 1, x_{ll}, x_{ll}).
\]
Substituting from (55) into (53) we obtain the condition

\[
2\alpha (1 - \alpha) h - \lambda_{hh} h + \lambda_{hl} h + \gamma_{hl} - \phi_{hl} = 0, \tag{56}
\]

where

\[
\gamma_{hl} = \sum_{x \in X^-} a_{-1}(x) \rho^1(x_{hh}, x_{hl} + 1, x_{lh}, x_{ll}) \tag{57}
\]

\[
\phi_{hl} = \sum_{x \in X^-} a_{-1}(x) \rho^1(x_{hh}, x_{hl} + 1, x_{lh}, x_{ll}).
\]

By hypothesis of the Lemma, \(\sum_{x \in X^-} a_{-1}(x) \rho^1(x_{hh}, x_{hl} + 1, x_{lh}, x_{ll}) = \rho^1(hl) > 0\), so the “rescaled multipliers” are well-defined, weakly positive, and equal to zero if and only if the “original multiplier” is equal to zero. From Lemma 2, we know that \(\lambda_{hh} = \alpha^2 \Lambda\) and \(\lambda_{hl}^H = (2\alpha - \alpha^2) \Lambda\), where \(\Lambda\) is the multiplier on the feasibility constraint. Hence the condition (56) can be rewritten as

\[
2\alpha (1 - \alpha) h - \alpha^2 \Lambda h + \alpha(2 - \alpha) \Lambda h + \gamma_{hl} - \phi_{hl} = 2\alpha (1 - \alpha) h + 2\alpha \Lambda h + \gamma_{hl} - \phi_{hl} = 0. \tag{58}
\]

Since \(2\alpha (1 - \alpha) h + 2\alpha \Lambda h > 0\), we conclude that \(\phi_{hl} > 0\). Hence \(\eta_{hl}^1 = 1\) for all \(x\) by the complementarity slackness condition and by. By Proposition 2 \(\eta_{lh}^2 = 1\) follows. Taking **CASE 1** and **CASE 2** together proves the first part of the lemma.

**CASE 3**: Next we consider the optimality conditions for \(\eta_{lh}^1\) and to economize on derivations we immediately observe that

\[
\sum_{x \in X^-} a_{-1}(x) \rho^1(x_{hh}, x_{hl}, x_{lh} + 1, x_{ll}) = \sum_{x \in X^+} a_{-1}(x_{hh}, x_{hl}, x_{lh} - 1, x_{ll}) \rho^1(x). \tag{59}
\]

The optimality condition can thus be written as

\[
2 \sum_{x \in X} a(x) \rho^1(x) \frac{x_{lh}}{n} - \lambda_{hh} \sum_{x \in X^+} a_{-1}(x_{hh}, x_{hl}, x_{lh} - 1, x_{ll}) \rho^1(x) h + \lambda_{hl} \sum_{x \in X^+} a_{-1}(x_{hh}, x_{hl}, x_{lh} - 1, x_{ll}) \rho^1(x) h + \gamma_{lh} - \phi_{hl} = 0. \tag{60}
\]

A calculation along the same lines as (54) shows that

\[
a(x) = \frac{n}{x_{lh}} \alpha (1 - \alpha) a_{-1}(x_{hh}, x_{hl}, x_{lh} - 1, x_{ll}). \tag{61}
\]
for any $x$ such that $x_{lh} \geq 1$ and since $a(x) \rho_1(x) \frac{x_{lh}}{n} = 0$ when $x_{lh} = 0$ we have that

\[
\sum_{x \in X} a(x) \rho_1(x) \frac{x_{lh}}{n} = \sum_{x \in X \mid x_{lh} \geq 1} a(x) \rho_1(x) \frac{x_{lh}}{n} = \sum_{x \in X \mid x_{lh} \geq 1} \frac{n}{x_{lh}} \alpha (1 - \alpha) a_{-1}(x_{hh}, x_{hl}, x_{lh}-1, x_{ll}) \rho_1(x) \frac{x_{lh}}{n} = \alpha (1 - \alpha) l \sum_{x \in X \mid x_{lh} \geq 1} a_{-1}(x_{hh}, x_{hl}, x_{lh}-1, x_{ll}) \rho_1(x).
\]

We may thus write (60) as

\[
0 = 2\alpha (1 - \alpha) l - \lambda_{hh} h + \lambda_{hl} l + \gamma_{lh} - \phi_{lh}
\]  

(63)

where $\gamma_{lh}^1(x)$ and $\phi_{lh}^1(x)$ are respectively scalings of $\gamma_{lh}(x)$ and $\phi_{lh}(x)$ through multiplication of $\frac{1}{\rho_1(x)} > 0$. We conclude that $G(\Phi) > 0 \Rightarrow \phi_{lh} > 0 \Rightarrow \eta_{lh} = 1$. Symmetrically, $G(\Phi) < 0 \Rightarrow \gamma_{lh} > 0 \Rightarrow \eta_{lh} = 0$, whereas if $G(\Phi) = 0$ the value of both multipliers must be zero, which imposes no restrictions on $\eta_{lh}$. Proposition 2 implies that $\eta_{hh}^2 = \eta_{lh}^1$, which completes the proof of the second part of the result.

**CASE 4:** Finally, we consider the optimality condition for $\eta_{ll}^1$, which, using the same kind of identity as (59) and the fact that $a(x) \rho_1(x) \frac{x_{ll}}{n} = 0$ for each $x$ such that $x_{ll} = 0$, may be written as

\[
2 \sum_{x \in X \mid x_{ll} \geq 1} a(x) \rho_1(x) \frac{x_{ll}}{n} - \lambda_{hl} \sum_{x \in X \mid x_{lh} \geq 1} a_{-1}(x_{hh}, x_{hl}, x_{lh}, x_{ll}-1) \rho_1(x) (h + l)
\]  

\[+ \lambda_{ll} \sum_{x \in X \mid x_{lh} \geq 1} a_{-1}(x_{hh}, x_{hl}, x_{lh}, x_{ll}-1) \rho_1(x) 2l + \gamma_{ll} - \phi_{ll}
\]

A calculation similar to (54) shows that $a(x) = \frac{a}{x_{ll}} (1 - \alpha)^2 a_{-1}(x_{hh}, x_{hl}, x_{lh}, x_{ll} - 1)$ and again we can eliminate the terms with the perceived probabilities of provision and expected average welfare.
to get
\[
0 = (1 - \alpha)^2 2l + \Lambda \left[ 2l - (2\alpha - \alpha^2) (h + l) \right] + \hat{\gamma}_{ll} - \hat{\phi}_{ll}
\]
\[
= (1 - \alpha)^2 (1 + \Lambda) \left( \frac{1}{1 + \Lambda} 2l + \frac{\Lambda}{1 + \Lambda} \left[ \frac{2}{(1 - \alpha)^2} l - \frac{(2\alpha - \alpha^2)}{(1 - \alpha)^2} (h + l) \right] + \frac{\hat{\gamma}_{ll} - \phi}{(1 - \alpha) 2 (1 + \Lambda)} \right)
\]
\[
= (1 - \alpha)^2 (1 + \Lambda) \left( H(\Phi) + \frac{\hat{\gamma}_{ll} - \phi}{(1 - \alpha) 2 (1 + \Lambda)} \right).
\]
Arguing as in the previous case completes the proof. ■

6.3 Interpretations

Notice that
\[
G(\Phi) \geq 0 \iff (1 - \Phi) \alpha (1 - \alpha) 2l + \Phi \left[ \alpha (2 - \alpha) l - \alpha^2 h \right] \geq 0
\] (66)

Recall that \( \Phi \) is the multiplier normalized to be a number in between 0 and 1. It is immediate that 
\( G(\Phi) \geq 0 \) for all \( \Phi \in [0, 1] \) if \( \alpha (2 - \alpha) l \geq \alpha^2 h \) (with strict equality if \( \Phi < 1 \)). This has an intuitive interpretation. We have seen in the proof of Lemma 3 that the optimal inclusion rule depends only on the multiplier on the resource constraint and not directly on the optimal provision rule. Hence, we can think of the optimal inclusions as if both public goods are always provided. One candidate solution to the problem is to set \( \eta^1_{lh} = \eta^2_{hl} = \eta^2_{ll} = 0 \). That is, give agent \( i \) access to good \( j \) if and only if \( \theta_i^j = h \). High valuation agents are willing to pay \( h \) for access to a good, so such a rule could generate an expected revenue of at most \( \alpha 2h \) from each agent.

Next, consider the case where \( \eta^1_{lh} = \eta^2_{hl} = 1 \) and \( \eta^1_{ll} = \eta^2_{ll} = 0 \). It is then possible to charge all agents who get access to both goods \( h + l \) and the probability that an agent will be of a type that gets access (any other type than \( ll \)) is \( \alpha (2 - \alpha) \). The expected revenue per agent is thus \( \alpha (2 - \alpha) (h + l) \) and
\[
\alpha (2 - \alpha) (h + l) - \alpha 2h = \alpha (2 - \alpha) l - \alpha^2 h.
\]
We conclude that \( G(\Phi) \geq 0 \) for all \( \Phi \) precisely when allowing the mixed types access to their low valuation good and charging a price \( h + l \) gives a higher expected revenue than if only high valuation consumers are given access and the price is \( h \) for each good.

Returning to the expression (66) we also realize that \( \alpha (1 - \alpha) 2l \) is the marginal increase in per capita surplus from increasing the inclusion probability \( \eta^1_{lh} \) while \( \alpha (2 - \alpha) l - \alpha^2 h \) is the marginal
gain or loss in transfer income. The condition may thus be viewed as a weighted average of the optimality conditions for an unconstrained social planner and a profit maximizing provider.

For the case where \( \alpha (2 - \alpha) l < \alpha^2 h \) some algebra on the condition in (66) shows that

\[
G(\Phi) \geq 0 \iff \Phi \leq \Phi^*_h = \frac{(1 - \alpha) 2l}{\alpha (h - l)}
\]

Clearly, \( \Phi^*_h > 0 \), and \( \Phi^*_h < 0 \iff \alpha (2 - \alpha) l < \alpha^2 h \). We conclude that for the case where the profit maximizing rule is to include only the high valuation consumers there exists a critical value \( \Phi^*_h \in (0, 1) \) for the multiplier such that \( G(\Phi) \geq 0 \) if and only if \( \Phi \leq \Phi^*_h \).

For the inclusion rule for type \( ll \) we see that

\[
H(\Phi) \geq 0 \iff (1 - \Phi) (1 - \alpha)^2 2l + \Phi (2l - \alpha (2 - \alpha) (h + l)) \geq 0.
\]

Again the condition has the interpretation as a weighted average of surplus and profit maximization and again the condition for when \( H(\Phi) \geq 0 \) for all \( \Phi \in [0, 1] \) has the interpretation as the (not so interesting case) when the revenue is higher if all types are given access to both goods (revenue of \( 2l \)) than when \( hh \) and mixed types (but not \( ll \)) are given access to both goods (revenue \( \alpha (2 - \alpha) (h + l) \)).

Fix \( h \) and let \( l^*_h \) be the critical value for \( l \) such that \( G(1) = 0 \) (which implies that \( G(\Phi) > 0 \) for all \( \Phi < 1 \)). Symmetrically let \( l^*_l \) be the critical value so that \( H(1) = 0 \) (in the same fashion, this guarantees that \( H(\Phi) > 1 \) for \( \Phi < 1 \)). We have that

\[
l^*_l = \frac{\alpha (2 - \alpha) h}{2 - \alpha (2 - \alpha)} > \alpha h > \frac{\alpha h}{2 - \alpha} = l^*_h,
\]

since \( 2 - \alpha (2 - \alpha) < 2 - \alpha \) and \( 2 - \alpha > 1 \). We can thus conclude that \( H(\Phi) \geq 0 \) for all \( \Phi \) implies that \( G(\Phi) > 0 \) for all \( \Phi \). If \( 2l - \alpha (2 - \alpha) (h + l) < 0 \) we have that

\[
H(\Phi) \geq 0 \iff l + \Phi \leq \Phi^*_l = \frac{(1 - \alpha)^2 2l}{\alpha (2 - \alpha) (h - l)}.
\]

Intuitively, the mixed types should be more valuable to the designer than the agents of type \( ll \). Indeed, when taking the ratio of the critical values we see that \( \frac{\Phi^*_l}{\Phi^*_h} = \frac{2 - \alpha}{\alpha} > 1 \), implying that the mixed types are always “first in line” to get access to the good for which their valuation is low.

This discussion can be summed up as follows:

\footnote{The reader may observe that the exercise imposes that \( \eta^2_{hl} \) increases at the same time, which is the heuristic explanation for the number 2 in \( \alpha (1 - \alpha) 2l \).}
Lemma 4 Assume that $\rho_j^i (\theta_i) > 0$ for all $\theta_i \in \Theta_i$ and $j = 1, 2$ and let $\Lambda$ be the multiplier associated with the balanced-budget constraint. The optimal inclusion rule in any symmetric optimal solution to the social planner's problem satisfies the following:

1. All agents with a high valuation for good $j$ is included with probability one for using good $j$ if it is provided;

2. If $\frac{\Lambda}{1 - \Lambda} < \Phi_{ih}^* < \Phi_{lh}^*$, then all agents get access to both public goods.

3. If $\frac{\Lambda}{1 - \Lambda} = \Phi_{ih}^* < \Phi_{lh}^*$, then $\eta_{lh}^1 = \eta_{lh}^2 = 0$ and $\eta_{ih}^1 = \eta_{ih}^2 = 1$.

4. If $\Phi_{ih}^* < \frac{\Lambda}{1 - \Lambda} < \Phi_{lh}^*$, then $\eta_{ih}^1 = \eta_{ih}^2 = 0$ and $\eta_{lh}^1 = \eta_{lh}^2 = 1$.

5. If $\Phi_{ih}^* < \Phi_{lh}^* < \frac{\Lambda}{1 - \Lambda}$, then $\eta_{ih}^1 = \eta_{ih}^2 = 1 = \eta_{lh}^1 = \eta_{lh}^2 = 0$.

6. If $\Phi_{ih}^* < \Phi_{lh}^*$, then $\eta_{ih}^1 = \eta_{ih}^2 = 0 = \eta_{lh}^1 = \eta_{lh}^2 = 1$.

While we still have not determined the value of $\Lambda$ in the optimal mechanism we have a rather simple characterization of the optimal inclusions as a function of the still unknown multiplier on the resource constraint. We have shown that there is a pair of threshold values for the normalized multiplier on the resource constraint such that $\Phi_{ih}^* < \Phi_{lh}^*$ (where either both or $\Phi_{lh}^*$ only are possible larger than 1 in which case the multiplier is irrelevant for the inclusion decision) where type $\theta_i$ agents are included if and only if $\frac{\Lambda}{1 - \Lambda} < \Phi_{ih}^*$. An immediate consequence of this characterization is that mixed types have a “higher priority” for to their low valuation good than type ll in the sense that it is only if the demand for the good from the mixed type is “satiated” that the ll type is given access. That is

Corollary 1 If $\eta_{ih}^1 = \eta_{ih}^2 > 0$, then $\eta_{lh}^1 = \eta_{lh}^2 = 1$. Symmetrically, if $\eta_{ih}^1 = \eta_{ih}^2 < 1$, then $\eta_{lh}^1 = \eta_{lh}^2 = 0$.

6.4 Optimal Provision Rules

In order to characterize the optimal provision rules we again use the formulation of the optimization problem in (45). We use the same notation as in section (45) for the multipliers to the incentive, resource and feasibility constraints. The multipliers for the boundary constraints on the provision probabilities are denoted $\gamma (x)$ (for constraint $\rho^1 (x) \geq 0$) and $\phi (x)$ (for constraint $1 - \rho^1 (x) \geq 0$).
After rearranging the summations in the constraints in accordance with (59) express the first order condition as

\[ 0 = 2 a(x) \left[ \left( \eta_{hh} x_{hh} + \eta_{hl} x_{hl} + \eta_{ll} x_{ll} \right) \frac{h}{n} + \left( \eta_{hh} x_{lh} + \eta_{hl} x_{ll} \right) \frac{l}{n} - c \right] \]  

(67)

\[ + \lambda_{hh} \left[ 2 \eta_{hh} a_{-1} (x_{hh} - 1, x_{hl}, x_{ll}) \right] - \lambda_{hh} \left[ \eta_{hl} a_{-1} (x_{hh}, x_{hl} - 1, x_{lh}, x_{ll}) \right] h - \eta_{hl} a_{-1} (x_{hh}, x_{hl}, x_{lh}, x_{ll}) h] + \lambda_{hl} \left[ \eta_{hl} a_{-1} (x_{hh}, x_{hl}, x_{lh}, x_{ll}) ] + \lambda_{ll} \left[ \eta_{ll} a_{-1} (x_{hh}, x_{hl}, x_{lh}, x_{ll}) \right] \right) \]

(68)

In order to make (67) generally valid we adopt the convention that if \( x_{hh} = 0 \), then \( a_{-1} (x_{hh} - 1, x_{hl}, x_{lh}, x_{ll}) = 0 \). The other types are treated in the same way, which mean that we don‘t have to treat \( x \) with \( x_{\theta_i} = 0 \) for some \( \theta_i \in \Theta_i \) asymmetrically. Derivations using the fact that the multinomial probability with \( n \) draws can be expressed in terms of the multinomial probability with \( n - 1 \) draws along the lines of the calculation in (54) show that

\[ a_{-1} (x_{hh} - 1, x_{hl}, x_{lh}, x_{ll}) = \frac{a(x) x_{hh}(1 - \alpha^2)}{\alpha^2 n} \]

(68)

\[ a_{-1} (x_{hh}, x_{hl} - 1, x_{lh}, x_{ll}) = \frac{a(x) x_{hl}}{\alpha (1 - \alpha) n} \]

\[ a_{-1} (x_{hh}, x_{hl}, x_{lh} - 1, x_{ll}) = \frac{a(x) x_{lh}}{\alpha (1 - \alpha) n} \]

\[ a_{-1} (x_{hh}, x_{hl}, x_{lh}, x_{ll} - 1) = \frac{1}{(1 - \alpha)^2} \frac{x_{ll}}{n} \]

where we observe that (68) hold also for \( x \) with some \( x_{\theta_i} = 0 \). Substituting from (68), using the relationships between multipliers in Lemma 2 and using that \( \eta_{hh} = \eta_{hl} = 1 \) (Lemma 3), we can simplify (67) to

\[ 0 = 2 \left[ \frac{(x_{hh} + x_{hl}) h + (\eta_{hh} x_{ll} + \eta_{ll} x_{ll}) l}{n} - c \right] (1 - \Phi) \]

(69)

\[ + \alpha^2 \Phi \left[ 2 \frac{x_{hh}}{\alpha^2 n} h - \frac{1}{\alpha} \frac{x_{hl}}{n} \right] + \eta_{hl} \frac{1}{\alpha (1 - \alpha) n} x_{lh} h - \eta_{hl} \frac{1}{\alpha (1 - \alpha) n} x_{lh} h] + \alpha (2 - \alpha) \Phi \left[ \frac{1}{\alpha} \frac{x_{hh}}{n} h + \eta_{hh} \frac{1}{\alpha (1 - \alpha) n} x_{ll} l - \eta_{hh} \frac{1}{\alpha (1 - \alpha) n} x_{ll} l (h + l)] + \Phi \frac{1}{(1 - \alpha)^2} \frac{x_{ll} l}{n} - \Phi 2c + \frac{\gamma (x) - \phi (x)}{a(x)} \]
where $\Phi = \Lambda / (1 + \Lambda)$. This condition can be seen as a weighted average of surplus and profit maximization. For an intuitive understanding of this think of a large economy and (as will happen with arbitrarily high probability) realizations where $x_{\theta_i}/n$ is near its expectation for every $\theta_i \in \Theta_i$. The first bracketed expression is then near $h(1 - \eta_{lh}^1) = 2h - (h + \eta_{lh}^1h)$ reflecting the price difference between what the monopolist can charge to type $hh$ and what can be charged from type $hl$ and still make it incentive compatible for agents of type $hh$ to report truthfully. The second bracketed term has the same interpretation (where it is type $hl$ versus type $ll$) and $2\eta_{lh}^1$ is what type $ll$ is maximally willing to pay. A profit maximizer wants to maximize the sum of these three terms, which is done by providing the good when the sum of the terms are positive. This is exactly what (67) says if $\Phi = 1$. A more direct way to see the connection with profit maximization is to simply compare with the optimality conditions for a profit maximizing provider (include in draft?).

We next define

$$Q^1 \left( \frac{x}{n}, \Phi \right) = \frac{x_{hh}}{n} h + \frac{x_{hl}}{n} h + \frac{x_{lh}}{n} \max \left\{ 0, G(\Phi) \right\} + \frac{x_{hl}}{n} \max \left\{ 0, H(\Phi) \right\} - c. \quad (70)$$

and collect terms in (69) to get

$$0 = 2 \frac{x_{hh}}{n} h + 2 \frac{x_{hl}}{n} h + \frac{x_{lh}}{n} \eta_{lh}^1 \left( (1 - \Phi) 2l + \Phi \left[ \frac{\alpha (2 - \alpha)}{\alpha (1 - \alpha)} l - \frac{\alpha^2}{\alpha (1 - \alpha)} h \right] + \frac{\gamma (x) - \phi (x)}{a (x)} \right) \quad (71)$$

$$= 2 Q^1 \left( \frac{x}{n}, \Phi \right) + \frac{\gamma (x) - \phi (x)}{a (x)}$$

By using the complementary slackness conditions we may then conclude:

**Lemma 5** Let $M = (\rho^1, \eta^1, t)$ be an optimal solution to (45) and $\Lambda$ be the value of the multiplier associated with the constraint (49) at the optimal solution. Let $\Phi = \Lambda / (1 + \Lambda)$. Then,

1. $\rho^1 (x) = 1$ whenever $Q^1 (x/n, \Phi) > 0$,

2. $\rho^1 (x) = 0$ whenever $Q^1 (x/n, \Phi) < 0$.

Analogously, given the multiplier $\Lambda$ associated with the balanced-budget constraint at the op-
timal solution, the optimal provision rule for good 2 is determined by the value of a function

\[ Q^2 \left( \frac{x}{n}, \Phi \right) \equiv \frac{x_{hh}}{n} h + \frac{x_{hl}}{n} h + \frac{x_{lh}}{n} h + \frac{x_{ll}}{n} \max \{0, G(\Phi)\} + \frac{x_{hh}}{n} \max \{0, H(\Phi)\} - c, \]

where \( \Phi = \Lambda/(1 + \Lambda) \), such that \( \rho^2(x) = 1 \) whenever \( Q^2(x/n, \Phi) > 0 \) and \( \rho^2(x) = 0 \) whenever \( Q^2(x/n, \Phi) < 0 \).

To summarize: up to now we characterized the optimal inclusion and provision rules for any given value of the Lagrange multiplier \( \Lambda \) associated with the balanced-budget constraint. Such characterization provides some partial information regarding the asymptotic provision probability in the optimal mechanism with bundling. For example, the above characterization tells us that \( \alpha h > c \) is a sufficient but not necessary condition for the provision probability to converge to one, in contrast to the model without bundling in which \( \alpha h > c \) is the necessary and sufficient for asymptotic probability one provision. (Recall that in the example in Section 4.2, the proposed bundling mechanism achieves provision with probability one for cases when \( \alpha h < c \).) To see this, write

\[ \mu = \left( \alpha^2, \alpha (1 - \alpha), \alpha (1 - \alpha), (1 - \alpha)^2 \right) \]

as the asymptotic proportions of agents with different valuation combinations \( hh, hl, lh, \) and \( ll \); and write \( \Phi_n = \Lambda_n/(1 + \Lambda_n) \) where \( \Lambda_n \) is the associated multiplier on the resource constraint in the optimal solution when the number of agents in the economy is \( n \). Note that

\[ \lim_{n \to \infty} Q^1 \left( \frac{x}{n}, \Phi_n \right) = Q^1(\mu, \Phi) \]

\[ = \alpha h + \alpha (1 - \alpha) \max \{0, G(\Phi)\} + (1 - \alpha)^2 \max \{0, H(\Phi)\} - c \quad (72) \]

where \( \Phi = \lim_{n \to \infty} \Phi_n \). Thus a sufficient condition for \( Q^1(\mu, \Phi) > 0 \) (and hence probability one provision of public good 1 asymptotically) is \( \alpha h > c \). A similar conclusion can be obtained for public good 2.

### 6.5 Asymptotic Results

In this section, we provide a full characterization of the asymptotic provision probability in an optimal bundling mechanism. We proceed with a few useful lemmas.

**Lemma 6** For any \( \epsilon > 0 \) there exists \( N \) such that \( \Pr \left( \left| Q^1 \left( \frac{x}{n}, \Phi_n \right) - Q^1(\mu, \Phi_n) \right| \geq \epsilon \right) \leq \epsilon \) for every \( n \geq N \).
Proof. Fix an arbitrary $\epsilon > 0$. Let $Y_i(\theta_i; \Phi_n)$ be a transformation of the random variable $\theta_i$ given by

$$Y_i(\theta_i; \Phi_n) = \begin{cases} 
  h - c & \text{if } \theta_i \in \{hh, hl\} \\
  \max\{0, G(\Phi_n)\} - c & \text{if } \theta_i = lh \\
  \max\{0, H(\Phi_n)\} - c & \text{if } \theta_i = ll
\end{cases}$$

(73)

Since $Y_i(\theta_i; \Phi_n)$ has bounded support, there exists $\sigma^2 < \infty$ such that the variance of $Y_i(\theta_i; \Phi_n)$ is less than $\sigma^2$ for any $\Phi_n \in [0, 1]$. Moreover, $\{Y(\theta_i; \Phi_n)\}_{i=1}^n$ is a sequence of i.i.d. random variables and

$$E_{\theta_i} Y_i(\theta_i; \Phi_n) = \alpha h + \alpha (1 - \alpha) \max\{0, G(\Phi_n)\} + (1 - \alpha)^2 \max\{0, H(\Phi_n)\} - c = Q^1(\mu, \Phi_n).$$

(74)

Since for any sequence of realizations $\{y_i(\theta_i; \Phi_n)\}_{i=1}^n$

$$\sum_{i=1}^n \frac{y_i(\theta_i; \Phi_n)}{n} = \frac{x_{hh}}{n} h + \frac{x_{hl}}{n} h + \frac{x_{lh}}{n} \max\{0, G(\Phi_n)\} + \frac{x_{ll}}{n} \max\{0, H(\Phi_n)\} - c = Q^1\left(\frac{x}{n}, \Phi_n\right),$$

(75)

we can apply Chebyshev’s inequality to obtain

$$\Pr\left(|Q^1\left(\frac{x}{n}, \Phi_n\right) - Q^1(\mu, \Phi_n)| \geq \epsilon\right) = \Pr\left(\left|\sum_{i=1}^n \frac{y_i(\theta_i; \Phi_n)}{n} - E_{\theta_i} Y_i(\theta_i; \Phi_n)\right| \geq \epsilon\right) \leq \frac{\text{Var}[Y_i(\theta_i; \Phi_n)]}{\epsilon^2} \leq \frac{\sigma^2}{\epsilon^2}.$$

(76)

Hence, $\Pr(|Q^1\left(\frac{x}{n}, \Phi_n\right) - Q^1(\mu, \Phi_n)| \geq \epsilon) \leq \epsilon$ for all $n \geq N = \sigma^2/\epsilon^2 < \infty$.

The second lemma is an application of the Stirling’s Lemma:

**Lemma 7** For any $\epsilon > 0$ and $p \in (0, 1)$ there exists $N < \infty$ such that the binomial distribution with parameters $p, n$ satisfies

$$\Pr(Y = y) = \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y} \leq \epsilon$$

(77)

for every $n \geq N$ and $y \in \{0, ..., n\}$.

Now, recall that $\rho^j_i(\theta_i)$ is agent $i$’s perceived provision probability of public good $j$ in a truth-telling mechanism when her own type is $\theta_i \in \Theta = \{hh, hl, lh, ll\}$ (defined in (27)). It should be rather obvious that these perceived probabilities converge to the same number as $n \rightarrow \infty$. This is what our next lemma says:
Lemma 8 For every $\epsilon > 0$ there exists $N$ such that $|\rho^1_i(\theta_i) - \rho^1_i(\theta'_i)| \leq \epsilon$ for every $\theta_i, \theta'_i \in \{hh, hl, lh, ll\}$ in any truth-telling mechanism for any economy where $n \geq N$.

The basic intuition is that the probability of being pivotal in a yes/no decision is necessarily small if there are many voters. Indeed, the proof in the appendix uses the fact that the probability of being pivotal goes to zero as $n$ goes to infinity by observing that, for any two types $\theta_i$ and $\theta'_i$, conditional on the number of agents with the remaining two types the difference between $\rho^1_i(\theta_i)$ and $\rho^1_i(\theta'_i)$ is maximized by a threshold rule which provides if at least a certain number of agents announce $\theta_i$. Conditional on any number of agents that are of the remaining two types the probability distribution of $\theta_i$ is a binomial so Lemma 7 ensures that for any given $\epsilon > 0$ there exists $N$ such that the probability of being pivotal is at least $\epsilon/2$ if the number of agents of type $\theta_i$ or $\theta'_i$ is above $N$. The fraction of agents who draw the remaining two types converges in probability to its expectation so for large enough $n$ the probability that the number of agents with type $\theta_i$ or $\theta'_i$ is below $N$ is less than $\epsilon/2$, implying that the difference between $\rho^1_i(\theta_i)$ and $\rho^1_i(\theta'_i)$ is less than $\epsilon$.

It follows from Lemma 8 that the perceived probability of provision must be near the ex ante probability of providing the good, that is:

Lemma 9 For every $\epsilon > 0$, there exists $N$ such that, for all $n \geq N$, $|E[\rho^1(x) - \rho^1(\theta_i)]| \leq \epsilon$ for all $\theta_i \in \{hh, hl, lh, ll\}$ in any truth-telling mechanism.

Proof. Fix $\epsilon > 0$ arbitrarily. Let $N$ be such that $|\rho^1_i(\theta_i) - \rho^1_i(\theta'_i)| \leq \epsilon$ for every $n \geq N, \theta_i, \theta'_i \in \{hh, hl, lh, ll\}$. Then

\[
|E[\rho^1(x) - \rho^1(\theta_i)]| = |\alpha^2 \rho^1_i(hh) + \alpha(1 - \alpha) \rho^1_i(hl) + \alpha(1 - \alpha) \rho^1_i(lh) + (1 - \alpha^2) \rho^1_i(ll) - \rho^1_i(\theta_i)| \\
\leq \alpha^2 |\rho^1_i(hh) - \rho^1_i(\theta'_i)| + \alpha(1 - \alpha) |\rho^1_i(hl) - \rho^1_i(\theta_i)| \\
+ \alpha(1 - \alpha) |\rho^1_i(lh) - \rho^1_i(\theta_i)| + (1 - \alpha^2) |\rho^1_i(ll) - \rho^1_i(\theta_i)| \\
\leq \alpha^2 \epsilon + \alpha(1 - \alpha) \epsilon + \alpha(1 - \alpha) \epsilon + (1 - \alpha)^2 \epsilon = \epsilon.
\]

Since $\theta_i \leq c$ is necessary to make the problem non-trivial we will make this assumption.

To state the result we now need to index the mechanisms by the size of the economy, so we will write $(\rho^j_n, \eta^j_n)_{j=1}^2$, where $\rho^j_n : X(n) \to [0, 1]$ is the provision rule for good $j$ and $X(n) =$
\[ \{ x \in \{0, \ldots, n\}^4 : x_{hl} + x_{hd} + x_{lh} + x_{ll} = n \}, \eta_h = (\eta_n^1(lh), \eta_n^1(ll)) \] are the probabilities that types \( lh \) and \( ll \) are allowed access to good 1 conditional on provision, and \( \eta_n^2 = (\eta_n^2(hl), \eta_n^2(ll)) \) are the probabilities that types \( hl \) and \( ll \) are allowed access to good 2 conditional on provision (other types are included with probability 1 in any optimal mechanism).

**Proposition 4** Let \( (\rho_n^j, \eta_n^j)_{j=1}^2 \) be a sequence of optimal mechanism. Then, the following holds:

1. If \( \max\{2ah, \alpha(2-\alpha)(h+l)\} > 2c \), then \( \lim_{n\to\infty} E\rho_n^j(x) \to 1 \) for \( j = 1, 2 \);

2. If \( \max\{2ah, \alpha(2-\alpha)(h+l)\} < 2c \), then \( \lim_{n\to\infty} E\rho_n^j(x) \to 0 \) for \( j = 1, 2 \);

3. If \( \alpha(2-\alpha)(h+l) > 2c \), then there exists \( N < \infty \) such that \( \eta_n^1(lh) = \eta_n^2(hl) = 1 \) for every \( n \geq N \), \( \eta_n^1(ll) = \eta_n^2(ll) \) for every \( n \) and
\[
\lim_{n\to\infty} \eta_n^1(ll) = \lim_{n\to\infty} \eta_n^2(ll) = \eta^*, \tag{79}
\]
where
\[
\eta^* = \frac{\alpha(2-\alpha)[h+l] - 2c}{\alpha(2-\alpha)[h+l] - 2l} \in (0, 1) \tag{80}
\]

4. If \( 2ah > 2c > \alpha(2-\alpha)(h+l) \), then there exists \( N < \infty \) such that \( \eta_n^1(ll) = \eta_n^2(hl) = 0 \) for all \( n \geq N \) and \( \eta_n^1(lh) = \eta_n^2(hl) \) for every \( n \) and
\[
\lim_{n\to\infty} \eta_n^1(lh) = \lim_{n\to\infty} \eta_n^2(hl) = \eta^{**} \tag{81}
\]
where
\[
\eta^{**} = \frac{\alpha^22h - 2c}{\alpha^22h - \alpha(2-\alpha)(h+l)} \in (0, 1) \tag{82}
\]

**Proof. (Part 1)** We first prove part 1. Note from (72), we know that \( Q^1(\mu, \Phi_n) \geq \alpha h - c \) for any \( \Phi_n \in [0, 1] \), hence \( \lim_{n\to\infty} Q^1(\mu, \Phi_n) \geq \alpha h - c \). Thus if \( \alpha h > c \), part 1 of the proposition immediately follows from Lemmas 5 and 6. Suppose instead that \( \alpha(2-\alpha)(h+l) > 2c \geq 2ah \). Then,
\[
Q^1(\mu, \Phi_n) = \alpha h + \alpha(1-\alpha) \max\{0, G(\Phi_n)\} + (1-\alpha)^2 \max\{0, H(\Phi_n)\} - c \tag{83}
\]
\[
\geq \alpha h - c + \alpha(1-\alpha) G(\Phi_n)
\]
\[
= \alpha h - c + \alpha(1-\alpha) \left\{ l(1-\Phi_n) + \Phi_n \left[ \frac{2\alpha - \alpha^2}{2\alpha(1-\alpha)} l - \frac{\alpha^2}{2\alpha(1-\alpha)} h \right] \right\}
\]
\[
= (1-\Phi_n) [\alpha h + \alpha(1-\alpha) l - c] + \Phi_n \left[ \alpha (2-\alpha)(l+h) - c \right].
\]
Observe that

$$\alpha h + \alpha (1 - \alpha) l = \frac{\alpha (2 - \alpha) (l + h)}{2} + \frac{\alpha^2}{2} (h - l) > \frac{\alpha (2 - \alpha) (l + h)}{2}. \quad (84)$$

Hence, $Q^1 (\mu, \Phi_n) \geq c > 0$ if $\alpha (2 - \alpha) (h + l) > 2c$, then for all $\Phi_n \in [0, 1]$, implying that $\lim_{n \to \infty} Q^1 (\mu, \Phi_n) > 0$. Thus by Lemmas 5 and 6, $\lim_{n \to \infty} E\rho_n^j (x) \to 1$ for $j = 1, 2$. This proves part 1.

**Part 2**

We now prove part 2. Suppose to the contrary that there exists a (sub) sequence of optimal incentive compatible, balanced-budget voluntary mechanism with provision rules for public good 1, $\rho_1^j (x)$, such that $\lim_{n \to \infty} E\rho_n^1 (x) = \rho > 0$. Combining the incentive constraints (30),(31) with the participation constraint (32) we obtain

$$\rho_1^1 (hl) h + \rho_1^2 (hl) \eta_{nl}h - t_{hl} + \rho_1^1 (lh) \eta_{hl}l + \rho_1^2 (lh) h - t_{lh} \quad (85)$$

$$\geq \rho_1^1 (ll) \eta_{hl}h + \rho_1^2 (ll) \eta_{hl}l + \rho_1^1 (ll) \eta_{hl}l + \rho_1^2 (ll) \eta_{hl}h - 2t_{ll}$$

$$\geq \rho_1^1 (ll) \eta_{hl}l (h - l) + \rho_1^2 (ll) \eta_{hl}^2 (h - l).$$

Pick an arbitrary $\epsilon > 0$. Then, by Lemma 9, there exists finite $N$ such that for every $n \geq N$ and each $\theta_i \in \Theta_i$, for $j = 1, 2$,

$$\left| \rho_i^j (\theta_i) - E\rho_n^j (x) \right| < \epsilon_1 \equiv \frac{\epsilon}{6h}. \quad (86)$$

Substituting (86) into (85) we obtain that for all $n \geq N$,

$$[E\rho_n^1 (x) + \epsilon_1] (h + \eta_{nl}l) + [E\rho_n^2 (x) + \epsilon_1] (h + \eta_{hl}l) - (t_{hl} + t_{lh}) \quad (87)$$

$$\geq [E\rho_n^1 (x) - \epsilon_1] \eta_{hl} (h - l) + [E\rho_n^2 (x) - \epsilon_1] \eta_{hl}^2 (h - l).$$

Re-arranging terms, we have

$$t_{hl} + t_{lh} \leq E\rho_n^1 (x) \left[ h \left( 1 - \eta_{hl} \right) + \eta_{hl} \eta_{hl}l \right] + E\rho_n^2 (x) \left[ h \left( 1 - \eta_{hl}^2 \right) + \eta_{hl} \eta_{hl}^2 \right]$$

$$+ \epsilon_1 \left[ h + \eta_{hl}l + h + \eta_{hl}^2l + \eta_{hl} (h - l) + \eta_{hl}^2 (h - l) \right]$$

$$< E\rho_n^1 (x) \left[ h \left( 1 - \eta_{hl} \right) + \eta_{hl} \eta_{hl}l \right] + E\rho_n^2 (x) \left[ h \left( 1 - \eta_{hl}^2 \right) + \eta_{hl} \eta_{hl}^2 \right] + 6\theta_h \epsilon_1.$$

Similarly, we from incentive constraints (28) and (29) we obtain,

$$2t_{hh} \leq 2 \left[ \rho_1^1 (hh) + \rho_2^2 (hh) \right] h - \left[ \rho_1^1 (hl) + \rho_1^1 (lh) \eta_{hl} \right] h - \left[ \rho_2^2 (lh) + \rho_2^2 (hl) \eta_{hl}^2 \right] h + t_{lh} + t_{hl}. \quad (89)$$
Again, by Lemma 9, we know that there exist $N_2$ such that for all $n > N_2$, we have

$$2t_{hh} < 2 \left[ E\rho_n^1(x) + E\rho_n^2(x) \right] h - E\rho_n^1(x) (1 + \eta_{th}) h - E\rho_n^2(x) (1 + \eta_{hl}) h + t_{lh} + t_{hl} + \epsilon \quad (90)$$

$$= E\rho_n^1(x) (1 - \eta_{th}) h + E\rho_n^2(x) (1 - \eta_{hl}) h + t_{lh} + t_{hl} + \epsilon$$

$$< E\rho_n^1(x) (1 - \eta_{th}) h + E\rho_n^2(x) (1 - \eta_{hl}) h$$

$$+ E\rho_n^1(x) [h (1 - \eta_{th}) + (\eta_{th} + \eta_{lh}) l] + E\rho_n^2(x) [h (1 - \eta_{hl}) + (\eta_{hl} + \eta_{lh}) l] + 2\epsilon$$

$$= E\rho_n^1(x) [(2 - \eta_{lh}) h + (\eta_{lh} + \eta_{lh}) l] + E\rho_n^2(x) [(2 - \eta_{hl}) h + (\eta_{hl} + \eta_{lh}) l] + 2\epsilon.$$

Finally, from the participation constraint (32) we know that there exists $N_3$ such that for all $n > N_3$, we have

$$t_{lh} < \left[ E\rho_n^1(x) \eta_{lh}^1 + E\rho_n^2(x) \eta_{lh}^2 \right] l + \epsilon. \quad (91)$$

Under a symmetric mechanism, we have $E\rho_n^1(x) = E\rho_n^2(x)$, and $\eta_{th}^1, \eta_{lh}^1 = \eta_{hl}^2$. Now consider two cases:

**CASE 1:** $\eta_{th}^1 = \eta_{lh}^2 = 0$ and $\eta_{th}^1 = \eta_{hl}^2 = \eta_m \in (0, 1)$. In this case, we have $t_{lh} = 0$ from type-$ll'$ participation constraint. Hence

$$\alpha^2 t_{hh} + \alpha (1 - \alpha) (t_{hl} + t_{lh}) + (1 - \alpha)^2 t_{ll} \quad (92)$$

$$< \alpha^2 \left\{ E\rho_n(x) [(2 - \eta_m) h + \eta_m l] + \epsilon \right\} + 2\alpha (1 - \alpha) \left\{ E\rho_n(x) (h + \eta_m l) + \epsilon \right\}$$

$$= E\rho_n(x) \left\{ [\alpha^2 (2 - \eta_m) + 2\alpha (1 - \alpha)] h + [\alpha^2 + 2\alpha (1 - \alpha)] \eta_m l \right\} + \epsilon'$$

$$= E\rho_n(x) \left\{ [\alpha^2 (2 - \eta_m) + 2\alpha (1 - \alpha)] h + \alpha (2 - \alpha) \eta_m l \right\} + \epsilon'$$

$$\equiv Z_1(\eta_m)$$

Note that

$$\frac{\partial Z_1(\eta_m)}{\partial \eta_m} = \alpha (2 - \alpha) l - \alpha^2 h = [\alpha (2 - \alpha) (h + l)] - 2\alpha h. \quad (93)$$

Therefore,

$$Z_1(\eta_m) \begin{cases} Z(1) = \alpha (2 - \alpha) (h + l) & \text{if } 2\alpha h \leq \alpha (2 - \alpha) (h + l) \\ Z(0) = 2\alpha h & \text{if } 2\alpha h > \alpha (2 - \alpha) (h + l) \end{cases} \quad (94)$$

Thus

$$\alpha^2 t_{hh} + \alpha (1 - \alpha) (t_{hl} + t_{lh}) + (1 - \alpha)^2 t_{ll} < E\rho_n(x) \max \left\{ 2\alpha h, \alpha (2 - \alpha) (h + l) \right\} + \epsilon'. \quad (95)$$

Thus if $\max \{2\alpha h, \alpha (2 - \alpha) (h + l)\} < 2\epsilon$, then the budget balance condition can not be satisfied when $n$ is sufficiently large.
CASE 2: \( \eta^1_n = \eta^2_n = \eta_l \in (0, 1), \eta^1_{th} = \eta^2_{th} = 1 \). In this case,

\[
\alpha^2 t_{th} + \alpha (1 - \alpha) (t_{hl} + t_{lh}) + (1 - \alpha)^2 t_{ll} < \alpha^2 \{E \rho_n (x) [(1 - \eta_l) h + (1 + \eta_l) l] + \epsilon \} + \alpha (1 - \alpha) \{2E \rho_n (x) [h (1 - \eta_l) + (1 + \eta_l) l] + \epsilon \} + (1 - \alpha)^2 [2E \rho_n (x) \eta_l + \epsilon] = E \rho_n (x) \left\{ [\alpha^2 + 2\alpha (1 - \alpha)] (1 - \eta_l) h + [\alpha^2 + 2\alpha (1 - \alpha)] (1 + \eta_l) l + 2(1 - \alpha)^2 \eta_l l \right\} + \epsilon
\]

Note that \( Z_2 (0) = \alpha (2 - \alpha) (h + l) \) and \( Z_2 (1) = 2\alpha (2 - \alpha) l + 2(1 - \alpha)^2 l = 2l \). Since \( Z_2 (\eta_l) \) is linear in \( \eta_l \), we have

\[
Z_2 (\eta_l) \leq \max \{Z_2 (0), Z_2 (1)\} = \max \{\alpha (2 - \alpha) (h + l), 2l\}. \tag{97}
\]

If \( \max \{2ah, \alpha (2 - \alpha) (h + l)\} < 2c \), then \( \max \{\alpha (2 - \alpha) (h + l), 2l\} < 2c \) since by assumption \( l < c \). Therefore there exists \( N' \) such that for all \( n > N' \), the budget balance condition will not be satisfied under any incentive compatible voluntary mechanism.

**(Part 3)** Suppose it is not the case that there exists \( N \) such that \( \eta^1_n (lh) = \eta^2_n (hl) = 1 \) for every \( n \geq N \). Then, taking a subsequence if necessary, we have that \( \eta^1_n (lh) = \eta^2_n (hl) < 1 \) for all \( n \), which by use of Lemma 4 implies that \( \eta^1_n (ll) = 0 \) for all \( n \). The per capita surplus generated by the optimal mechanism in the \( n \)th economy in the sequence is then

\[
S (M_n) = \frac{E \rho^1_n (x) [(x_{hh} + x_{hl}) h + (n_{lh} (lh) x_{lh} + n_{ll} (ll) x_{ll}) l - cn]}{n} + \frac{E \rho^2_n (x) [(x_{hh} + x_{hl}) h + (n_{lh} (lh) x_{lh} + n_{ll} (ll) x_{ll}) l - cn]}{n} \leq \frac{E [(x_{hh} + x_{hl}) h + x_{lh} l - \rho^1_n (x) cn]}{n} + \frac{E [(x_{hh} + x_{hl}) h + x_{hl} l - \rho^2_n (x) cn]}{n} = 2[\alpha h + \alpha (1 - \alpha) l - \rho^1_n (x) + \rho^2_n (x)] c
\]

Since \( E \rho^1_n (x) \rightarrow 1 \) and \( E \rho^2_n (x) \rightarrow 1 \) as \( n \rightarrow \infty \) if follows that for each \( \epsilon > 0 \) there exists \( N \) such that

\[
S (M_n) \leq 2[\alpha h + \alpha (1 - \alpha) l - c] + \epsilon \tag{99}
\]
Consider an alternative sequence of mechanisms \( \{ \tilde{M}_n \}_{n=1}^{\infty} \), where for each \( n \)

\[
\begin{align*}
\tilde{\eta}^1_n (lh) &= \tilde{\eta}^2_n (hl) = 1 \\
\tilde{\eta}^1_n (ll) &= \tilde{\eta}^2_n (ll) = \eta^* = \frac{\alpha (2 - \alpha) [h + l] - 2c}{\alpha (2 - \alpha) [h + l] - 2l} \\
\bar{t}_n (hh) &= \bar{t}_n (hl) = \bar{t}_n (lh) = (1 - \eta^*) (h + l) + \eta^* 2l \\
\bar{t}_n (ll) &= 2 \eta^* l \\
\tilde{\rho}^j_n (x) &= 1 \text{ for all } x \in X (n) \text{ and all } n
\end{align*}
\]

We observe that the participation constraint for type \( ll \) holds with equality since

\[
E_{-i} \sum_{j=1,2} \tilde{\rho}^j_n (x) \bar{\eta}^j_n (ll) l - \bar{t}_n (ll) = 2 \eta^* l - 2 \eta^* l = 0
\]  

(101)

The downward incentive constraint for type \( hl \) also holds with equality since

\[
E_{-i} \left[ \tilde{\rho}^1_n (x) \bar{\eta}^1_n (hl) h + \tilde{\rho}^2_n (x) \bar{\eta}^2_n (hl) l - \bar{t}_n (hl) \right] = h + l - \tilde{t}_n (hl)
\]

\[
= h + l - [(1 - \eta^*) (h + l) + \eta^* 2l] = \eta^* (h + l) - \eta^* 2l
\]

\[
= E_{-i} \left[ \tilde{\rho}^1_n (x) \bar{\eta}^1_n (hl) h + \tilde{\rho}^2_n (x) \bar{\eta}^2_n (hl) l - \bar{t}_n (hl) \right] \theta_i = ll
\]

(102)

The calculation for type \( lh \) is symmetric and type \( hh \) gets payoff \( 2h - [(1 - \eta^*) (h + l) + \eta^* 2l] \) if telling the truth and \( \eta^* 2h - \eta^* 2l \) if pretending to be type \( ll \), and

\[
2h - [(1 - \eta^*) (h + l) + \eta^* 2l] - [\eta^* 2h - \eta^* 2l] = (h - l) (1 - \eta^*) > 0,
\]

(103)

so \( hh \) has a strict incentive not to announce \( ll \). It is easy to check that the payoff from announcing \( hh, lh \) or \( hl \) for type \( ll \) is \( (l - h) (1 - \eta^*) < 0 \) so all incentive constraints hold for type \( ll \). Finally, \( hh, lh \) or \( hl \) are indifferent over announcements \( hh, lh \) or \( hl \) , so all incentive constraints hold for all these types as well, and participation constraints are also satisfied since announcing \( ll \) leads to a strictly positive payoff for all the higher types. Hence, all (IC) and (IR) constraints are satisfied.
for mechanism $\tilde{M}_n$. Finally,

$$\begin{aligned}
\mathbb{E} \left( \sum_i \tilde{t}_n (\theta_i) - \sum_{j=1,2} \tilde{p}_n^j (x) c_n \right) \\
= n \left[ \alpha^2 \tilde{t}_n (hh) + \alpha (1 - \alpha) (\tilde{t}_n (hl) + \tilde{t}_n (lh)) + (1 - \alpha)^2 \tilde{t}_n (ll) - 2c \right] \\
= n \left[ \alpha (2 - \alpha) [(1 - \eta^*) (h + l)] + \eta^* 2l \right] + (1 - \alpha)^2 \eta^* 2l - 2c \\
= n \left[ \alpha (2 - \alpha) [(1 - \eta^*) (h + l)] + \eta^* 2l - 2c \right] \\
= n \left[ \alpha (2 - \alpha) (h + l) - 2c + \eta^* (2l - \alpha (2 - \alpha) (h + l)) \right] \\
= n \left[ \alpha (2 - \alpha) (h + l) - 2c + \alpha (2 - \alpha) [h + l] - 2c \frac{(2l - \alpha (2 - \alpha) (h + l))}{h + l} \right] \\
= n \left[ \alpha (2 - \alpha) (h + l) - 2c - (\alpha (2 - \alpha) (h + l) - 2c) \right] = 0,
\end{aligned}$$

so the feasibility constraint holds with equality for any $n$. Hence, $\tilde{M}_n$ is feasible for any $n$. The expected per capita surplus generates by $\tilde{M}_n$ is

$$\begin{aligned}
\frac{S (\tilde{M}_n)}{n} &= \alpha^2 2h + 2\alpha (1 - \alpha) (h + l) + (1 - \alpha)^2 \eta^* 2l - 2c \\
&= 2 [\alpha h + \alpha (1 - \alpha) l - c] + (1 - \alpha)^2 \eta^* 2l
\end{aligned}$$

Let $\varepsilon = (1 - \alpha)^2 \eta^* l > 0$ and pick $N < \infty$ such that $\frac{S(M_n)}{n} \leq 2 [\alpha h + \alpha (1 - \alpha) l - c] - \varepsilon$ for all $n \geq N$ and we can conclude that

$$\begin{aligned}
\frac{S (\tilde{M}_n)}{n} - \frac{S (M_n)}{n} \geq (1 - \alpha)^2 \eta^* 2l - \varepsilon = (1 - \alpha)^2 \eta^* 2l > 0
\end{aligned}$$

for all $n \geq N$, implying that mechanism $M_n$ could not be optimal for $n \geq N$.

Hence, we know that there exists $N$ such that $\eta^1_n (lh) = \eta^2_n (hl) = 1$ for every $n \geq N$. Next suppose that $\eta^1_n (ll)$ does not converge to $\eta^*$. Suppose first that there exists a subsequence such that $\eta^1_n (ll) \to \eta' < \eta^*$. An argument as the one above lets us conclude that for every $\varepsilon > 0$ there exists $N < \infty$ such that

$$\begin{aligned}
\frac{S (M_n)}{n} \leq 2 [\alpha h + \alpha (1 - \alpha) l + (1 - \alpha)^2 \eta' l - c] + \varepsilon,
\end{aligned}$$

so by picking $\varepsilon = (1 - \alpha)^2 (\eta^* - \eta') l$ we find that

$$\begin{aligned}
\frac{S (\tilde{M}_n)}{n} - \frac{S (M_n)}{n} \geq \alpha^2 2h + 2\alpha (1 - \alpha) (h + l) + (1 - \alpha)^2 \eta^* 2l - 2c \\
- 2 [\alpha h + \alpha (1 - \alpha) l + (1 - \alpha)^2 \eta' l - c] - \varepsilon \\
= (1 - \alpha)^2 (\eta^* - \eta') 2l - \varepsilon = (1 - \alpha)^2 (\eta^* - \eta') l > 0,
\end{aligned}$$
so again the mechanism $\tilde{M}_n$ is better for $n$ large enough. Finally, suppose there is a subsequence such that $\eta_n^j (ll) \to \eta' > \eta^*$ and let
\[
\varepsilon = \frac{(\eta' - \eta^*) (2 - \alpha) (h + l) - 2l}{4 (1 + \alpha)} > 0.
\] (109)

Then, since $\eta_n^1 (ll) + \eta_n^2 (ll) \to 2\eta'$ it follows that to satisfy the participation constraint for type $ll$ for all $n$ there must be some $N_1$ such that
\[
t_n (ll) \leq \sum_{j=1,2} \rho_n^j (ll) \eta_n^j (ll) l \leq (\eta_n^1 (ll) + \eta_n^2 (ll)) l \leq 2\eta' l + \varepsilon.
\] (110)

for all $n \geq N_1$. Moreover, there exists $N_2$ such that $\eta_n^1 lh = \eta_n^2 hl = 1$ for $n \geq N_2$, so the incentive constraint for type $hl$ (and $lh$) reduces to
\[
\rho_n^1 (hl) h + \rho_n^2 (hl) l - t_n (hl) \geq \rho_n^1 (ll) \eta_n^1 (ll) h + \rho_n^2 (ll) \eta_n^2 (ll) l - t_n (ll)
\]
\[
\iff
t_n (hl) \leq \ t_n (ll) + \left( \rho_n^1 (hl) - \rho_n^1 (ll) \eta_n^1 (ll) \right) h + \left( \rho_n^2 (hl) - \rho_n^2 (ll) \eta_n^2 (ll) \right) l
\]

We have that $\lim_{n \to \infty} \rho_n^j (hl) = \lim_{n \to \infty} \rho_n^j (hl) = 1$ and $\lim_{n \to \infty} \eta_n^1 (ll) = \eta'$, so there exists $N_3$ such that
\[
t_n (hl) = t_n (lh) \leq t_n (ll) + (1 - \eta') (h + l) + \varepsilon.
\] (112)

Finally, the incentive constraint that type $hh$ should not want to announce $hl$ may be written as
\[
[\rho_n^1 (hh) h + \rho_n^2 (hh) ] h - t_n (hh) \geq \rho_n^1 (hl) h + \rho_n^2 (hl) h - t_n (hl) \leftrightarrow
\]
\[
t_n (hh) \leq \ t (hl) + 2h \left[ \rho_n^1 (hh) h + \rho_n^2 (hh) - \rho_n^1 (hl) h - \rho_n^2 (hl) \right],
\]

where we note that since $\lim_{n \to \infty} \rho_n^j (hl) = \lim_{n \to \infty} \rho_n^j (hl) = 1$ we have that $t_h (hh) \leq t (hl) + \varepsilon$.

The expected per capita revenue of the mechanism therefore satisfies
\[
\alpha^2 t_n (hh) + 2\alpha (1 - \alpha) t_n (hl) + (1 - \alpha) t_n (hl)
\]
\[
\leq \ \left[ \alpha^2 + 2\alpha (1 - \alpha) \right] t_n (hl) + \alpha^2 \varepsilon + (1 - \alpha) t_n (hl)
\]
\[
\leq \ \left[ \alpha^2 + 2\alpha (1 - \alpha) \right] \left[ t_n (ll) + (1 - \eta') (h + l) + \varepsilon \right] + (1 - \alpha) t_n (hl) + \alpha^2 \varepsilon
\]
\[
= \ t_n (ll) + \left[ \alpha^2 + 2\alpha (1 - \alpha) \right] (1 - \eta') (h + l) + 2\alpha \varepsilon
\]
\[
\leq \ 2\eta' l + \varepsilon + \left[ \alpha^2 + 2\alpha (1 - \alpha) \right] (1 - \eta') (h + l) + 2\alpha \varepsilon
\]
\[
= \ \eta' 2l + (1 - \eta') \alpha (2 - \alpha) (h + l) + \varepsilon (1 + 2\alpha)
\]
and since there exists $N_4$ such that $E \left( \rho_n^1 (x) + \rho_n^2 (x) \right) c \geq 2c - \varepsilon$ we have that

$$
\alpha^2 t_n (hh) + 2\alpha (1 - \alpha) t_n (hl) + (1 - \alpha) t_n (hl) - E \left( \rho_n^1 (x) + \rho_n^2 (x) \right) c \\
\leq \eta' 2l + (1 - \eta') \alpha (2 - \alpha) (h + l) - 2c + 2\varepsilon (1 + \alpha) \\
= \eta' 2l + (1 - \eta') \alpha (2 - \alpha) (h + l) - \eta^* 2l + (1 - \eta^*) \alpha (2 - \alpha) (h + l) + 2\varepsilon (1 + \alpha) \\
= (\eta' - \eta^*) (2l - \alpha (2 - \alpha) (h + l)) + 2\varepsilon (1 + \alpha) \\
= (\eta' - \eta^*) (2l - \alpha (2 - \alpha) (h + l)) + \frac{(\eta' - \eta^*) (\alpha (2 - \alpha) (h + l) - 2l)}{2} \\
= - \frac{(\eta' - \eta^*) (\alpha (2 - \alpha) (h + l) - 2l)}{2} < 0.
$$

Hence, the mechanism must violate the feasibility constraint for $n \geq \max \{N_1, N_2, N_3, N_4\}$. We conclude that there can be no subsequence of optimal mechanisms such that $\eta_n^i (ll) \rightarrow \eta' \neq \eta^*$, proving the claim.

**Part 4** We first note that if there is no $N$ such that $\eta_n^1 (ll) = \eta_n^2 (ll) = 0$ for all $n \geq N$, then there exists a subsequence where $\eta_n^1 (ll) = \eta_n^2 (ll) > 0$ along the sequence, which from the optimal inclusion rules implies that $\eta_n^1 (lh) = \eta_n^2 (hl) = 1$ for all $n$ along the subsequence. Hence

$$
\lim_{n \rightarrow \infty} \eta_n^1 (lh) = \lim_{n \rightarrow \infty} \eta_n^2 (hl) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \eta_n^1 (ll) = \lim_{n \rightarrow \infty} \eta_n^2 (ll) = \eta' \geq 0.
$$

Let $\varepsilon = \frac{2c - \alpha (2 - \alpha) (h + l)}{4 (1 + \alpha)} > 0$. We can then use the same calculations as in Part 3 to conclude that there exists $N < \infty$ such that the revenues collected satisfy

$$
\alpha^2 t_n (hh) + 2\alpha (1 - \alpha) t_n (hl) + (1 - \alpha) t_n (hl) < \eta' 2l + (1 - \eta') \alpha (2 - \alpha) (h + l) + \varepsilon (1 + 2\alpha) \\
\leq \alpha (2 - \alpha) (h + l) + \varepsilon (1 + 2\alpha)
$$

Moreover, since there exists $N_2$ such that $E \left( \rho_n^1 (x) + \rho_n^2 (x) \right) c \geq 2c - \varepsilon$ we have that

$$
\alpha^2 t_n (hh) + 2\alpha (1 - \alpha) t_n (hl) + (1 - \alpha) t_n (hl) - E \left( \rho_n^1 (x) + \rho_n^2 (x) \right) c \\
\leq \alpha (2 - \alpha) (h + l) + \varepsilon (2 + 2\alpha) - 2c = \alpha (2 - \alpha) (h + l) + \frac{4c - \alpha (2 - \alpha) (h + l)}{4 (1 + \alpha)} (2 + 2\alpha) - 2c \\
= - \frac{2c - \alpha (2 - \alpha) (h + l)}{2} < 0,
$$

so, assuming participation and incentive constraints hold, the feasibility constraint must be violated. Establishing that $\lim_{n \rightarrow \infty} \eta_n^1 (lh) = \lim_{n \rightarrow \infty} \eta_n^2 (hl) = \eta^*$ is done by arguments similar to arguments in Part 3.
7 Conclusion

This paper studies the role of bundling in the optimal provision of multiple excludable public goods in large economies. We showed in a simple parametric example that allowing for bundling in the provision mechanism can improve the provision probability of efficient public goods in large economies. More explicitly, we consider an environment with \( n \) agents and two public goods. Each agent’s valuations for the public goods are independent and take values \( h \) and \( l \) with probability \( \alpha \) and \( 1 - \alpha \) respectively. Suppose that the per capita provision cost of each public good is \( c \in (l, h) \).

The following table summarizes the results:

<table>
<thead>
<tr>
<th>Bundling \ Exclusion</th>
<th>No Exclusion</th>
<th>Exclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Bundling</td>
<td>( E\rho_n^j * \to 0 )</td>
<td>( E\rho_n^j * \to 0 ), if ( \alpha h &lt; c ) ( E\rho_n^j * \to 1 ), if ( \alpha h &gt; c ).</td>
</tr>
<tr>
<td>Bundling Allowed</td>
<td>( E\rho_n^j * \to 0 )</td>
<td>( E\rho_n^j * \to 0 ), if ( \max{2\alpha h, \alpha (2 - \alpha)(h + l)} &lt; 2c ) ( E\rho_n^j * \to 1 ), if ( \max{2\alpha h, \alpha (2 - \alpha)(h + l)} &gt; 2c ).</td>
</tr>
</tbody>
</table>

Table 1: The Asymptotic Provision Probability under Different Bundling and Exclusion Possibilities.

A Appendix: Proofs

Proof of Proposition 1

We prove the result by proving two claims:

Claim 1 For any incentive feasible mechanism \( M \) of the form in (3) there exist an incentive feasible mechanism

\[
M = \left( \left( \rho_j, \eta_i^j \right)_{j=1,2}, t_1, \ldots, t_n \right)
\]

(A1)

that generates the same social surplus, where \( \rho_j : \Theta \to [0, 1] \) is the provision rule for good \( j \), \( \eta_i^j : \Theta_i \to [0, 1] \) is the inclusion rule for agent \( i \) and good \( j \), and \( t_i : \Theta_i \to R \) is the transfer rule for agent \( i \)

Proof. Pick an arbitrary mechanism \( M = (\zeta^1, \zeta^2, \omega^1, \omega^2, \tau) \) and let \( k \in [0, 1] \) and for \( j = 1, 2 \) and
i \in I \text{ define the functions } \rho^i : \Theta \to [0, 1], \eta^i : \Theta_i \to [0, 1], t_i : \Theta_i \to R \text{ by }

\begin{align*}
\rho^i (\theta) &= E_X \zeta (\theta, x) = \int_0^1 \zeta (\theta, x) \, dx \\
\eta^i (\theta_i) &= \begin{cases} \\
\frac{E_{-i} \omega^i (\theta_i, \zeta (\theta, x))}{E_{-i} \zeta (\theta, x)} &= \frac{\int_{\theta_{-i}} \int_0^1 \omega^i (\theta_i, \zeta (\theta, x)) \, dx \, d\theta_{-i}}{\int_{\theta_{-i}} \int_0^1 \zeta (\theta, x) \, dx \, d\theta_{-i}} & \text{if } \int_{\theta_{-i}} \int_0^1 \zeta (\theta, x) \, dx \, d\theta_{-i} > 0 \\
\eta^i (\theta_i) &= \text{otherwise} \\
\end{cases} \\
t_i (\theta_i) &= E_{-i} \tau (\theta) = \int_{\theta_{-i}} \tau (\theta) \, d\theta_{-i}
\end{align*}

Pick an arbitrary agent \( i \in I \) who makes the announcement \( \theta'_i \in \Theta_i \). Assuming that \( \int_{\theta_{-i}} \int_0^1 \zeta (\theta', x) \, dx \, d\theta_{-i} = 0 \) we immediately see that the payoff is \( -t_i (\theta'_i) = - \int_{\theta_{-i}} \tau (\theta) \, d\theta_{-i} \) for both the original and the simplified mechanism, whereas if \( \int_{\theta_{-i}} \int_0^1 \zeta (\theta', x) \, dx \, d\theta_{-i} > 0 \) the utility given the mechanism \( \left( \rho^i, \eta^i_1, ..., \eta^i_n \right) \) is

\begin{align*}
E_{-i} \left[ \sum_{j=1,2} \rho^j (\theta'_i, \theta_{-i}) \eta^j_i (\theta'_i) \, \theta_i - t_i (\theta'_i) \right] \\
= \sum_{j=1,2} E_{-i} \zeta (\theta'_i, \theta_{-i}, x) \frac{E_{-i} \omega^j_i (\theta'_i, \theta_{-i}, x) \zeta (\theta'_i, \theta_{-i}, x)}{E_{-i} \zeta (\theta'_i, \theta_{-i}, x)} \theta_i - E_{-i} \tau (\theta'_i, \theta_{-i}) \\
= \sum_{j=1,2} E_{-i} \omega^j_i (\theta'_i, \theta_{-i}, x) \zeta (\theta'_i, \theta_{-i}, x) \theta_i - E_{-i} \tau (\theta'_i, \theta_{-i}) \\
= E_{-i} \sum_{j=1,2} \omega^j_i (\theta'_i, \theta_{-i}, x) \zeta (\theta'_i, \theta_{-i}, x) \theta_i - \tau (\theta'_i, \theta_{-i}) ,
\end{align*}

which is the utility that agent \( i \) of type \( \theta_i \) gets if announcing type \( \theta'_i \) given mechanism \( M = (\zeta^1, \zeta^2, \omega^1, \omega^2, \tau) \). Hence, all incentive and participation constraints continue to hold if changing from \( M \) to the simplified mechanism. Moreover, since

\begin{align*}
E_{-i} \left[ \sum_{j=1,2} \rho^j (\theta) \eta^i_j (\theta_i) \theta_i \right] = E_{-i} \sum_{j=1,2} \zeta (\theta, x) \omega^j_i (\theta, x) \theta_i \text{ for every } \theta_i
\end{align*}

It follows by integration over \( \Theta_i \) and summation over \( i \) that

\begin{align*}
E \left[ \sum_{i \in I} \sum_{j=1,2} \rho^j (\theta) \eta^i_j (\theta_i) \theta_i \right] = E \sum_{i \in I} \sum_{j=1,2} \zeta (\theta, x) \omega^j_i (\theta, x) \theta_i .
\end{align*}

By construction we also have that \( \rho^j (\theta) = E_X \zeta (\theta, x) \) for every \( \theta \), so \( E \rho^j (\theta) C^j (n) = E \zeta (\theta, x) C^j (n) \), implying that

\begin{align*}
\sum_{j=1,2} E \rho^j (\theta) \left[ \sum_{i \in I} \eta^j_i (\theta_i) \theta_i - C^j (n) \right] = \sum_{j=1,2} E \zeta (\theta, x) \left[ \sum_{i \in I} \omega^j_i (\theta, x) \theta_i - C^j (n) \right] .
\end{align*}
Hence, the original and the simplified mechanisms generate the same social surplus. The final observation is that \( t_i (\theta) = E_{-i} \tau (\theta) \) implies that \( \sum_{i \in I} E t_i (\theta) = \sum_{i \in I} E_{-i} t_i (\theta) \) and that \( E \rho^j (\theta) = E \zeta (\theta, x) \). Taken together this means that the resource constraint is unaffected. We conclude that the simplified mechanism generates the same social surplus and satisfies the constraints if and only if the original mechanism satisfies the constraints. ■

Claim 2 For every incentive feasible mechanism \( M \) of the form in (A1) there exists an anonymous incentive feasible mechanism \( \tilde{M} \) of the form in (9) that generates the same social surplus.

Proof. Suppose \( M = (\rho^j, \eta^i_j, \eta^i_n)_{j=1,2,t_1,...,t_n} \) where \( \rho^j : \Theta \to [0,1] \), \( \eta^j_i : \Theta_i \to [0,1] \), \( t_i : \Theta_i \to R \) is incentive feasible. For any given \( \theta \), let \( P_k (\theta) \in \Theta \) denote a permutation of \( \theta \) and let \( \mathcal{P} (\theta) = \{ P_1 (\theta),...,P_n! (\theta) \} \) be the set of all possible permutations. Furthermore, for \( i \in I \), let \( P_k^i (\theta) \) denote the type of agent \( i \) for perturbation \( k \) of type vector \( \theta \). For each \( k = 1,...,n! \) let \( M_k = (\rho^j_k, \eta^i_{k1},...,\eta^i_{kn})_{j=1,2,t_{k1},...,t_{kn}} \) be given by

\[
\rho^j_k (\theta) = \rho^j (P_k (\theta)) \quad \forall \theta \in \Theta, \ j = 1,2
\]

\[
\eta^j_{ki} (\theta_i) = \eta^j_i (P_k^i (\theta)) \quad \forall \theta \in \Theta, \ j = 1,2, i \in I
\]

\[
t_{ki} (\theta_i) = t_i (P_k^i (\theta)) \quad \forall \theta \in \Theta, i \in I
\]

Now, construct an alternative mechanism \( \tilde{M} = (\tilde{\rho}^j, \tilde{\eta}^i_j, \tilde{\eta}^i_n)_{j=1,2,\tilde{t}_1,...,\tilde{t}_n} \), where

\[
\tilde{\rho}^j (\theta) = \frac{1}{n!} \sum_{k=1}^{n!} \rho^j_k (\theta) \quad \forall \theta \in \Theta, \ j = 1,2
\]  

\[
\tilde{\eta}^j_i (\theta_i) = \frac{1}{E_{-i} [\tilde{\rho}^j (\theta)]} \frac{1}{n!} \sum_{k=1}^{n!} E_{-i} \left[ \rho^j_k (\theta) \right] \eta^i_{ki} (\theta_i) = \frac{1}{n!} \sum_{k=1}^{n!} E_{-i} \left[ \rho^j_k (\theta) \right] \eta^i_{ki} (\theta_i) \quad \forall \theta \in \Theta, i \in I, \ j = 1,2
\]

\[
\tilde{t}_i (\theta_i) = \frac{1}{n!} \sum_{k=1}^{n!} t_{ki} (\theta_i)
\]

This mechanism is symmetric by construction. That is, permuting the roles of the agents for the “initial \( \Theta \)” doesn’t affect the set of permutations, so \( \tilde{\rho}^j (\theta) = \tilde{\rho}^j (\theta') \) if \( \theta' \) is a permutation of \( \theta \). Similarly, if \( \theta \) is such that \( \theta_i = \theta_i' \) then the sets \( \left\{ E_{-i} \left[ \rho^j_k (\theta) \right] \eta^i_{ki} (\theta_i) \right\}_{k=1}^{n!} \) and \( \left\{ E_{-i'} \left[ \rho^j_k (\theta) \right] \eta^i_{k'i'} (\theta_i') \right\}_{k=1}^{n!} \) are the same and \( E_{-i} \left[ \tilde{\rho}^j (\theta) \right] = E_{-i'} \left[ \tilde{\rho}^j (\theta) \right] \), so \( \tilde{\eta}^i_j (\theta_i) = \tilde{\eta}^i_j (\theta_i') \). Finally, it should be obvious
that \( \tilde{t}_i(\theta_i) = \tilde{t}_{i'}(\theta_{i'}) \) if \( \theta_i = \theta_{i'} \). Since this is true for any \( \theta \in \Theta, i, i' \in I \) we have that there exists \( \tilde{\eta}^j(\theta_i), \tilde{t} \) such that \( \left( \tilde{\eta}^j(\cdot), \tilde{t}(\cdot) \right) = \left( \tilde{\eta}^j(\cdot), \tilde{t}(\cdot) \right) \), so the mechanism \( \tilde{M} \) is of the form in (9).

Now, since \( M \) and \( M_k \) are identical except that the role of the agents have been permuted we have that

\[
\sum_{j=1,2} E \left[ \rho^j_k(\theta) \left[ \sum_{i \in I} \eta^j_{ki}(\theta_i) \theta_i - C^j(n) \right] \right] = \sum_{j=1,2} E \left[ \rho^j(\theta) \left[ \sum_{i \in I} \eta^j_i(\theta_i) \theta_i - C^j(n) \right] \right] \quad (A9)
\]

for \( k = 1, ..., n! \). Hence

\[
\sum_{j=1,2} E \left[ \tilde{\rho}^j(\theta) \left[ \sum_{i \in I} \tilde{\eta}^j(\theta_i) \theta_i - C^j(n) \right] \right] = \sum_{j=1,2} E \left[ \frac{1}{n!} \sum_{k=1}^{n!} E_{\theta_i} \left[ \frac{1}{n!} \sum_{k=1}^{n!} E_{-i} \left[ \rho^j_k(\theta) \eta^j_{ki}(\theta_i) \theta_i - C^j(n) \right] \right] \right] \quad (A10)
\]

\[
= \sum_{j=1,2} E_{\theta_i} \left[ \frac{1}{n!} \sum_{k=1}^{n!} E_{-i} \left[ \rho^j_k(\theta) \eta^j_{ki}(\theta_i) \theta_i \right] - E \left[ \frac{1}{n!} \sum_{k=1}^{n!} \rho^j_k(\theta) \right] C^j(n) \right]
\]

\[
= \sum_{j=1,2} \sum_{i \in I} E_{\theta_i} \left[ \frac{1}{n!} \sum_{k=1}^{n!} \rho^j_k(\theta) \eta^j_{ki}(\theta_i) \theta_i \right] - \sum_{j=1,2} \sum_{i \in I} E_{\theta_i} \left[ \frac{1}{n!} \sum_{k=1}^{n!} \rho^j_k(\theta) \right] C^j(n)
\]

\[
= \sum_{j=1,2} \sum_{i \in I} E_{\theta_i} \left[ \frac{1}{n!} \sum_{k=1}^{n!} \rho^j_k(\theta) \eta^j_{ki}(\theta_i) \theta_i \right] - \sum_{j=1,2} \sum_{i \in I} E_{\theta_i} \left[ \frac{1}{n!} \sum_{k=1}^{n!} \rho^j_k(\theta) \right] C^j(n)
\]

\[
= \frac{1}{n!} \sum_{j=1,2} \sum_{i \in I} E_{\theta_i} \left[ \rho^j_k(\theta) \left[ \sum_{i \in I} \eta^j_{ki}(\theta_i) \theta_i - C^j(n) \right] \right] \]

by virtue of (A9). Hence the social surplus generated by the symmetric mechanism \( \tilde{M} \) is the same as the social surplus generated by the original mechanism \( M \). Next, since \( M_k \) and \( M \) only differ in that the roles of the agents have been switched we have that \( E \sum_{i \in I} t_{ki}(\theta_i) = E \sum_{i \in I} t_i(\theta_i) \) and \( E \rho^j_k(\theta) = E \rho^j(\theta) \) for each \( k \). Hence the feasibility constraint is satisfied for mechanism \( \tilde{M} \) provided it is satisfied for mechanism \( M \). Moreover, incentive compatibility holds for any permuted mechanism, so

\[
E_{-i} \left[ \sum_{j=1,2} \rho^j_k(\theta) \eta^j_{ki}(\theta_i) \theta_i - t_{ki}(\theta_i) \right] \geq E_{-i} \sum_{j=1,2} \rho^j_k(\tilde{\theta}_i, \theta_{-i}) \eta^j_{ki}(\tilde{\theta}_i, \theta_{-i}) \theta_i - t_{ki}(\tilde{\theta}_i, \theta_{-i}) \quad (A11)
\]

\[ \forall i \in I, \theta_i, \theta'_i \in \Theta_i. \]
surplus is the same as the original mechanism, which proves the claim.

Proof. Consider a given anonymous mechanism \( M \) and let \( \tilde{M} \) be the "mirror image" in the sense that the roles of goods 1 and 2 are reversed. That is, for every \( \theta = (\theta^1, \theta^2) \) let

\[
\tilde{\rho}^1(\theta) = \rho^2(\theta^2, \theta^1) \quad \text{and} \quad \tilde{\rho}^2(\theta) = \rho^1(\theta^1, \theta^2) \quad (A13)
\]

\[
\tilde{\eta}^1(i, \theta^1) = \eta^2(\theta^2, i) \quad \text{and} \quad \tilde{\eta}^2(i, \theta^2) = \eta^1(i, \theta^1)
\]

\[
\tilde{t}(i, \theta^1) = t(\theta^2, i)
\]

Hence,

\[
E_{-i} \sum_{j=1,2} \tilde{\rho}^j(\theta) \tilde{\eta}^j(i, \theta) \theta^j_i - \tilde{t}(\theta_i) \quad (A12)
\]

\[
equiv E_{-i} \sum_{j=1,2} \left[ \frac{1}{n!} \sum_{k=1}^{n!} \tilde{\rho}^j_k(\theta) \right] \frac{1}{n!} \sum_{k=1}^{n!} E_{-i} \left[ \tilde{\rho}^j_k(\theta) \right] \tilde{\eta}^j_k(i, \theta) \theta^j_i - \frac{1}{n!} \sum_{k=1}^{n!} t_k(i, \theta_i)
\]

\[
equiv E_{-i} \sum_{j=1,2} \left[ \frac{1}{n!} \sum_{k=1}^{n!} \tilde{\rho}^j_k(\theta) \tilde{\eta}^j_k(i, \theta) \theta^j_i - t_k(i, \theta_i) \right]
\]

/ by (A11) / \equiv \frac{1}{n!} \sum_{k=1}^{n!} \left[ E_{-i} \sum_{j=1,2} \tilde{\rho}^j_k(\theta, \theta_i) \tilde{\eta}^j_k(\theta, \theta_i) \theta^j_i - t_k(\theta_i, \theta_i) \right]

\[
equiv E_{-i} \sum_{j=1,2} \frac{1}{n!} \sum_{k=1}^{n!} \tilde{\rho}^j_k(\theta, \theta_i) \tilde{\eta}^j_k(\theta, \theta_i) \theta^j_i - \frac{1}{n!} \sum_{k=1}^{n!} t_k(\theta_i, \theta_i)
\]

\[
equiv \sum_{j=1,2} E_{-i} \tilde{\rho}^j_k(\theta, \theta_i) \tilde{\eta}^j_k(\theta, \theta_i) \theta^j_i - \tilde{t}(\theta_i),
\]

so incentive compatibility holds. To check that all participation constraints hold we simply notice that each term on the third line in the calculation above is positive since the participation constraints must all hold for every permutation. Hence, all constraints are satisfied by \( \tilde{M} \) and the surplus is the same as the original mechanism, which proves the claim. \( \blacksquare \)

Proposition 1 follows by combining Claim 1 and Claim 2.

**Proof of Proposition 2**

**Proof.** Consider a given anonymous mechanism \( M \) that is incentive feasible and let \( \tilde{M} \) be the "mirror image" in the sense that the roles of goods 1 and 2 are reversed. That is, for every \( \theta = (\theta^1, \theta^2) \) let
We notice that

\[ \mathbb{E}_\rho^1 (\theta^1, \theta^2) \bar{\eta}^1(\theta^1, \theta^2) \theta^1_{i} = \mathbb{E}_\rho^2 (\theta^2, \theta^1) \eta^2(\theta^2, \theta^1) \theta^1_{i} \]  
(A14)

\[ = \int_{\theta^1 \in (\Theta^1)^n} \left[ \int_{\theta^2 \in (\Theta^2)^n} \rho^2 (\theta^2, \theta^1) \eta^2(\theta^2, \theta^1) \theta^1_{i} \Pi_{k=1}^n dF^2(\theta^2_k | \theta^1_k) \right] \Pi_{k=1}^n dF^1(\theta^1_k) \]

\[ = \int_{\theta^2 \in (\Theta^2)^n} \left[ \int_{\theta^1 \in (\Theta^1)^n} \rho^2 (\theta^1, \theta^2) \eta^2(\theta^1, \theta^2) \theta^2_{i} \Pi_{k=1}^n dF^2(\theta^2_k | \theta^2_k) \right] \Pi_{k=1}^n dF^1(\theta^1_k) \]

\[ = \int_{\theta^2 \in (\Theta^2)^n} \left[ \int_{\theta^1 \in (\Theta^1)^n} \rho^2 (\theta^1, \theta^2) \eta^2(\theta^1, \theta^2) \theta^2_{i} \Pi_{k=1}^n dF^2(\theta^2_k | \theta^1_k) \right] \Pi_{k=1}^n dF^1(\theta^2_k) \]

\[ = \mathbb{E}_\rho^2 (\theta^1, \theta^2) \eta^2(\theta^1, \theta^2) \theta^2_{i}, \]

where the third equality uses that the labeling of the variables being integrated is arbitrary and the fourth the assumption that \( F^1(\cdot | v) = F^2(\cdot | v) \) (which also implies that the marginals coincide). Symmetric calculations show that

\[ \mathbb{E}_\hat{\rho}^2 (\theta^1, \theta^2) \hat{\eta}^2(\theta^1, \theta^2) \theta^2_{i} = \mathbb{E}_\rho^1 (\theta^1, \theta^2) \eta^1(\theta^1, \theta^2) \theta^1_{i} \]

(A15)

\[ \mathbb{E}\hat{\ell}(\theta^1, \theta^2) = \mathbb{E}\ell(\theta^1, \theta^2) \]

Together (A14) and (A15) imply that the ex ante utility for each \( i \in I \) is the same given mechanism \( \hat{M} \) as for mechanism \( M \). Moreover, since \( C^1(n) = C^2(n) \) and \( \mathbb{E}_\rho^1 (\theta^1, \theta^2) = \mathbb{E}_\rho^2 (\theta^1, \theta^2) \) and \( \mathbb{E}_\hat{\rho}^2 (\theta^1, \theta^2) = \mathbb{E}_\rho^1 (\theta^1, \theta^2) \) the provision costs are the same, so \( \hat{M} \) and \( M \) generate the same social surplus. Since provision costs are the same and \( \mathbb{E}\hat{\ell}(\theta^1, \theta^2) = \mathbb{E}\ell(\theta^1, \theta^2) \) it follows directly that \( \hat{M} \) satisfies the resource constraint (6). Finally, a calculation in the same spirit as (A14) shows that

\[ \mathbb{E}_{-i}\rho^1 (\theta^1_{i}, \theta^2_{i}, \theta^2_{\neg i}) = \mathbb{E}_{-i}\rho^2 (\theta^2_{\neg i}, \theta^1_{i}, \theta^1_{\neg i}) \]

(A16)

\[ = \int_{\theta^1_{i}} \left[ \int_{\theta^1_{\neg i}} \rho^2 (\theta^2_{\neg i}, \theta^1_{i}, \theta^1_{\neg i}) \Pi_{k \neq i} dF^2(\theta^2_k | \theta^1_k) \right] \Pi_{k \neq i} dF^1(\theta^1_k) \]

\[ = \int_{\theta^2_{i}} \left[ \int_{\theta^1_{i}} \rho^2 (\theta^1_{i}, \theta^2_{\neg i}) \Pi_{k \neq i} dF^2(\theta^2_k | \theta^1_k) \right] \Pi_{k \neq i} dF^1(\theta^1_k) \]

\[ = \int_{\theta^2_{i}} \left[ \int_{\theta^1_{i}} \rho^2 (\theta^1_{i}, \theta^2_{\neg i}) \Pi_{k \neq i} dF^1(\theta^2_k | \theta^1_k) \right] \Pi_{k \neq i} dF^2(\theta^2_k) \]

\[ = \mathbb{E}_{-i}\rho^2 (\theta^1_{i}, \theta^2_{i}, \theta^2_{\neg i}), \]

where the third equality uses the arbitrariness of the labeling under the integration and the forth that \( F^1(\cdot | v) = F^2(\cdot | v) \). Symmetrically

\[ \mathbb{E}_{-i}\hat{\rho}^2 (\theta^1_{i}, \theta^2_{i}, \theta^2_{\neg i}) = \mathbb{E}_{-i}\rho^1 (\theta^2_{\neg i}, \theta^1_{i}, \theta^1_{\neg i}) \]

(A17)
For simplicity of notation, write $U(\theta_i, \theta^*_i; M)$ for the expected utility of type $\theta_i$ if announcing $\theta^*_i$. We then have that

$$U \left( \theta^1_i, \theta^2_i, \theta^3_i, \theta^4_i; \hat{M} \right) = E_{-i} \sum_{j=1,2} \hat{\rho}^j \left( \theta^1_i \theta^2_i, \theta^2_i \right) \hat{\eta}^j \left( \theta^3_i, \theta^4_i \right) \theta^2_i - \tilde{t}(\theta^1_i, \theta^2_i) \quad (A18)$$

$$= E_{-i} \hat{\rho}^1 \left( \theta^2_i \theta^3_i, \theta^4_i \right) \eta^1 \left( \theta^2_i, \theta^4_i \right) \theta^2_i$$

$$+ E_{-i} \hat{\rho}^2 \left( \theta^2_i \theta^3_i, \theta^4_i \right) \eta^2 \left( \theta^2_i, \theta^3_i \right) \theta^2_i - t(\theta^2_i, \theta^4_i)$$

$$= U \left( \theta^1_i, \theta^3_i, \theta^2_i, M \right)$$

Hence all participation constraints hold since if $U \left( \theta^1_i, \theta^2_i, \theta^3_i, \theta^4_i; \hat{M} \right) < 0$, then the participation constraint of type $(\theta^2_i, \theta^1_i)$ would be violated for mechanism $M$. Also, all incentive constraint hold since if $U \left( \theta^1_i, \theta^2_i, \theta^3_i, \theta^4_i; \hat{M} \right) < U \left( \theta^1_i, \theta^3_i, \theta^2_i; \hat{M} \right)$, then $U \left( \theta^1_i, \theta^3_i, \theta^2_i, \theta^4_i; \hat{M} \right) < U \left( \theta^2_i, \theta^1_i, \theta^3_i, \theta^4_i; \hat{M} \right)$, so the incentive constraint for type $(\theta^2_i, \theta^1_i)$ would be violated in mechanism $M$. We conclude that $\hat{M}$ is incentive feasible and generates the same social surplus as $M$. Now construct a new mechanism $\hat{M}$ where

$$\hat{\rho}^j(\theta) = \frac{1}{2} \hat{\rho}^j(\theta) + \frac{1}{2} \hat{\rho}^j(\theta) = \frac{1}{2} \hat{\rho}^j(\theta) \quad (A19)$$

$$\hat{\eta}^j(\theta_i) = \frac{1}{2} \hat{\eta}^j(\theta_i) E_{-i} \rho^j(\theta) + \frac{1}{2} \hat{\eta}^j(\theta_i) E_{-i} \hat{\rho}^j(\theta)$$

$$\tilde{t}(\theta_i) = \frac{1}{2} t(\theta_i) + \frac{1}{2} \tilde{t}(\theta_i)$$

$\hat{M}$ is symmetric since

$$\hat{\rho}^1(\theta^1_i, \theta^2_i) = \frac{1}{2} \rho^1(\theta^1_i, \theta^2_i) + \frac{1}{2} \hat{\rho}^1(\theta^1_i, \theta^2_i) = \rho^2(\theta^2_i, \theta^1_i) \quad (A20)$$

$$\hat{\eta}^1(\theta^1_i, \theta^2_i) = \frac{1}{2} \eta^1(\theta^1_i, \theta^2_i) + \frac{1}{2} \eta^1(\theta^2_i, \theta^1_i) + \frac{1}{2} \hat{\eta}^1(\theta^1_i, \theta^2_i) + \frac{1}{2} \hat{\eta}^1(\theta^2_i, \theta^1_i)$$

$$\tilde{t}(\theta^1_i, \theta^2_i) = \frac{1}{2} t(\theta^1_i, \theta^2_i) + \frac{1}{2} \tilde{t}(\theta^1_i, \theta^2_i)$$
We also notice that

\[
\begin{align*}
\mathbb{E} \tilde{\rho}^1(\theta) \tilde{\eta}^1(\theta_i^1, \theta_i^2) \theta_i^1 &= \mathbb{E} \left( \frac{1}{2} \rho^1(\theta) + \frac{1}{2} \tilde{\rho}^1(\theta) \right) \frac{\tilde{\eta}^1(\theta_i^1, \theta_i^2) \mathbb{E}_{-i} \rho^1(\theta) + \frac{1}{2} \tilde{\eta}^1(\theta_i^1, \theta_i^2) \mathbb{E}_{-i} \tilde{\rho}^1(\theta)}{\frac{1}{2} \mathbb{E}_{-i} \rho^1(\theta) + \frac{1}{2} \mathbb{E}_{-i} \tilde{\rho}^1(\theta)}
\end{align*}
\]

\[= \mathbb{E}_{\theta_i} \left[ \frac{1}{2} \mathbb{E}_{-i} \rho^1(\theta) + \frac{1}{2} \mathbb{E}_{-i} \tilde{\rho}^1(\theta) \right] \frac{\tilde{\eta}^1(\theta_i^1, \theta_i^2) \mathbb{E}_{-i} \rho^1(\theta) + \frac{1}{2} \tilde{\eta}^1(\theta_i^1, \theta_i^2) \mathbb{E}_{-i} \tilde{\rho}^1(\theta)}{\frac{1}{2} \mathbb{E}_{-i} \rho^1(\theta) + \frac{1}{2} \mathbb{E}_{-i} \tilde{\rho}^1(\theta)}
\]

\[= \frac{1}{2} \mathbb{E} \tilde{\eta}^1(\theta_i^1, \theta_i^2) \rho^1(\theta) + \frac{1}{2} \mathbb{E} \tilde{\eta}^1(\theta_i^1, \theta_i^2) \tilde{\rho}^1(\theta)
\]

/\text{(A14)/}\]

\[= \frac{1}{2} \mathbb{E} \eta^1(\theta_i^1, \theta_i^2) \rho^1(\theta) + \frac{1}{2} \mathbb{E} \eta^2(\theta_i^1, \theta_i^2) \rho^2(\theta) \theta_i^2
\]

Symmetrically

\[
\mathbb{E} \tilde{\rho}^2(\theta) \tilde{\eta}^2(\theta_i^1, \theta_i^2) \theta_i^2 = \frac{1}{2} \mathbb{E} \eta^2(\theta_i^1, \theta_i^2) \rho^2(\theta) + \frac{1}{2} \mathbb{E} \eta^1(\theta_i^1, \theta_i^2) \rho^1(\theta) \theta_i^1,
\]

so

\[
\sum_{j=1,2} \mathbb{E} \tilde{\rho}^j(\theta) \tilde{\eta}^j(\theta_i^1, \theta_i^2) \theta_i^j = \sum_{j=1,2} \mathbb{E} \rho^j(\theta) \eta^j(\theta_i^1, \theta_i^2) \theta_i^j.
\]

Since

\[
\mathbb{E} \left( \frac{1}{2} \rho^1(\theta) + \frac{1}{2} \tilde{\rho}^1(\theta) \right) C^1(n) + \mathbb{E} \left( \frac{1}{2} \rho^2(\theta) + \frac{1}{2} \tilde{\rho}^2(\theta) \right) C^2(n) = \mathbb{E} \left( \frac{1}{2} \rho^1(\theta) + \frac{1}{2} \rho^2(\theta) + \frac{1}{2} \tilde{\rho}^1(\theta) + \frac{1}{2} \tilde{\rho}^2(\theta) \right) C^1(n)
\]

\[= \mathbb{E} \rho^1(\theta) C^1(n) + \mathbb{E} \rho^2(\theta) C^2(n)
\]

costs are unaffected, implying that the social surplus generated by mechanism \( \tilde{M} \) is the same as in \( M \). Moreover, and \( \mathbb{E} \tilde{t}(\theta_i) = \frac{1}{2} \mathbb{E} t(\theta_i) + \frac{1}{2} \mathbb{E} \tilde{t}(\theta_i) = \mathbb{E} t(\theta_i) \) (using (A15)), so the resource constraint is unaffected. Finally

\[
U \left( \theta_i, \theta_i'; \tilde{M} \right) = \mathbb{E}_{\theta_i} \sum_{j=1,2} \tilde{\rho}^j(\theta_i', \theta_i) \tilde{\eta}^j(\theta_i') \theta_i' - \tilde{t}(\theta_i')
\]

\[= \mathbb{E}_{\theta_i} \sum_{j=1,2} \left[ \frac{1}{2} \rho^j(\theta_i', \theta_i) + \frac{1}{2} \tilde{\rho}^j(\theta_i', \theta_i) \right] \frac{\tilde{\eta}^j(\theta_i') \mathbb{E}_{-i} \rho^j(\theta_i', \theta_i) + \tilde{\eta}^j(\theta_i') \mathbb{E}_{-i} \tilde{\rho}^j(\theta_i', \theta_i) \theta_i' - \tilde{t}(\theta_i')}{\mathbb{E}_{-i} \rho^j(\theta_i', \theta_i) + \mathbb{E}_{-i} \tilde{\rho}^j(\theta_i', \theta_i)}
\]

\[= \frac{1}{2} \left( \mathbb{E}_{\theta_i} \sum_{j=1,2} \tilde{\rho}^j(\theta_i') \rho^j(\theta_i', \theta_i) - \tilde{t}(\theta_i') \right) + \frac{1}{2} \left( \mathbb{E}_{\theta_i} \sum_{j=1,2} \tilde{\rho}^j(\theta_i') \tilde{\rho}^j(\theta_i', \theta_i) - \tilde{t}(\theta_i') \right)
\]

\[= \frac{1}{2} U \left( \theta_i, \theta_i'; M \right) + \frac{1}{2} U \left( \theta_i, \theta_i'; \tilde{M} \right)
\]
Truth-telling is incentive compatible in mechanisms $M$ and $\widehat{M}$, so $U(\theta_i, \theta_i; M) \geq U(\theta_i, \theta_i'; M)$ and $U(\theta_i, \theta_i'; \widehat{M}) \geq U(\theta_i, \theta_i'; \widehat{M})$ for all $\theta_i, \theta_i' \in \Theta_i$ and $i \in I$. Incentive compatibility for $\widehat{M}$ follows and the participation constraints are all satisfied by the same token.

**Proof of Lemma 1**

**Proof.** For each $x \in X$, $j = 1, 2, \theta_i \in \Theta_i$ we have that $\nu_j(x) \in [0, 1]$, $\eta^j_\theta \in [0, 1]$. Next, we note that if $\theta < 0$ and all constraints are satisfied, then a deviation where taxes are changed from $t$ to $t' = (t_{hh}, t_{hl}, t_{lh}, 0)$ and where inclusion and provision rules are unchanged will satisfy all constraints in the relaxed program (35). Similarly, if all constraints hold and $t_{lh} < -l - h$ then the deviation to

$$t' = (t_{hh}, t_{hl}, -l - h, \max(0, t_{ll}))$$

will satisfy all constraints (in the relaxed program). A symmetric argument restricts $t_{hl} \geq -h - l$.

Finally, if $t_{hh} < -3h - l$, then a deviation to

$$t' = (-3h - l, \max(t_{hl}, -l - h), \max(t_{lh}, -l - h), \max(0, t_{ll}))$$

will leave all constraints satisfied. We conclude that there is a lower bound $\xi > -\infty$ such that for any mechanism where $\theta < \xi$ for some $\theta_i$, there exists an alternative mechanism that supports the same allocation (and therefore generates the same surplus) where $\theta_i > \xi$. Also, if $\theta_i > \xi = 2h$ for some $\theta_i$ then at least one constraint in (??) must be violated. We therefore conclude that there is no loss in generality to restrict $\theta_i$ to be a number in $[\xi, \eta]$. All constraints and the objective function are linear in the choice variables and therefore continuous so we conclude that the optimization problem has a compact feasible set and a continuous objective. It is easy to check that the feasible set is non-empty, which proves the claim by appeal to the Weierstrass Maximum Theorem.

**Proof of Lemma 7**

**Proof.** Fix an arbitrary $\epsilon > 0$. The most probable value for $y$ is the unique integer $y^*(n)$ satisfying $np - 1 \leq y^*(n) \leq np + 1$, and the corresponding probability is

$$\Pr(y^*(n)) = \frac{n!}{y^*(n)! [n - y^*(n)]!} p^{y^*(n)} (1 - p)^{n - y^*(n)}. \quad (A26)$$

Let $s(r) = \frac{r!}{\sqrt{2\pi e^{-r^2} r^{r+1/2}}}$, where, according to Stirling’s Formula for every $\epsilon > 0$ there exists
we obtain

\[ n! = s(n)\sqrt{2\pi e^{-n}n^{n+\frac{1}{2}}}, \quad (A27) \]

\[ y^*(n)! = s(y^*(n))\sqrt{2\pi e^{-y^*(n)}y^*(n)y^{*(n)+\frac{1}{2}}}, \]

\[ (n - y^*(n))! = s(n - y^*(n))\sqrt{2\pi e^{-(n - y^*(n))}}[n - y^*(n)]^{n - y^*(n)+\frac{1}{2}}, \]

we obtain

\[
\frac{n!}{y^*(n)! [n - y^*(n)]!} = \frac{s(n)}{s(y^*(n)) s(n - y^*(n))} \sqrt{2\pi y^*(n)y^*(n)y^{*(n)+\frac{1}{2}}} \sqrt{2\pi e^{-(n - y^*(n))}}[n - y^*(n)]^{n - y^*(n)+\frac{1}{2}} \\
= \frac{s(n)}{s(y^*(n)) s(n - y^*(n))} \sqrt{2\pi y^*(n)y^{*(n)+\frac{1}{2}}} [n - y^*(n)]^{n - y^*(n)+\frac{1}{2}}. \quad (A28) \]

We note that for any \( p \in (0, 1) \) we have that \( \lim_{n \to \infty} y^*(n) = \infty \) and \( \lim_{n \to \infty} [n - y^*(n)] = \infty \). Hence, there exists \( N < \infty \) such that \( y^*(n) \geq R(\epsilon) \) and \( n - y^*(n) \geq R(\epsilon) \), implying that \( s(n) \leq 1 + \epsilon, s(y^*(n)) \geq 1 - \epsilon, \) and \( s(n - y^*(n)) \geq 1 - \epsilon \). We can thus bound the probability of \( y^*(n) \) by.

\[
\Pr(y^*(n)) = \frac{s(n)}{s(y^*(n)) s(n - y^*(n))} \sqrt{2\pi y^*(n)y^*(n)y^{*(n)+\frac{1}{2}}} \sqrt{2\pi e^{-(n - y^*(n))}}[n - y^*(n)]^{n - y^*(n)+\frac{1}{2}}p^{y^*(n)}(1 - p)^{n - y^*(n)} \\
\leq \frac{(1 + \epsilon)}{(1 - \epsilon)^2} \frac{n^{n+\frac{1}{2}}}{\sqrt{2\pi y^*(n)y^{*(n)+\frac{1}{2}}} \sqrt{2\pi e^{-(n - y^*(n))}}[n - y^*(n)]^{n - y^*(n)+\frac{1}{2}}p^{y^*(n)}(1 - p)^{n - y^*(n)} \\
= \frac{(1 + \epsilon)}{(1 - \epsilon)^2} \frac{p^{y^*(n)}(1 - p)^{n - y^*(n)}}{\sqrt{2\pi y^*(n)y^{*(n)+\frac{1}{2}}} \sqrt{2\pi e^{-(n - y^*(n))}}[n - y^*(n)]^{n - y^*(n)+\frac{1}{2}}. \quad (A29) \]

Since \( \frac{y^*(n)}{n} = \arg\max_{p \in [0, 1]} p^{y^*(n)}(1 - p)^{n - y^*(n)}, \) we know that

\[
\frac{p^{y^*(n)}(1 - p)^{n - y^*(n)}}{\left(\frac{y^*(n)}{n}\right)^{y^*(n)} \left(\frac{n - y^*(n)}{n}\right)^{n - y^*(n)}} \leq 1. \quad (A30) \]

Therefore,

\[
\Pr(y^*(n)) \leq \frac{(1 + \epsilon)}{(1 - \epsilon)^2} \frac{1}{\sqrt{2\pi y^*(n)y^{*(n)+\frac{1}{2}}} \sqrt{2\pi e^{-(n - y^*(n))}}[n - y^*(n)]^{n - y^*(n)+\frac{1}{2}}} \\
\leq \frac{(1 + \epsilon)}{(1 - \epsilon)^2} \frac{1}{\sqrt{2\pi \left(\frac{p - \frac{1}{n}}{1 - p - \frac{1}{n}}\right) \left(1 - p - \frac{1}{n}\right)}} \to 0 \text{ as } n \to \infty. \quad (A31) \]
Hence, there exists \( N' < \infty \) such that
\[
\frac{(1 + \epsilon)}{(1 - \epsilon)^2} \frac{1}{\sqrt{2n\pi} \left( p - \frac{1}{n} \right) \left( 1 - p - \frac{1}{n} \right)} \leq \epsilon. \tag{A32}
\]
Implying that \( \Pr(y^*(n)) \leq \epsilon \) for any \( n \geq \max \{ N, N' \} \). Since \( \epsilon \) was arbitrary the result follows. ■

**Proof of Lemma 8**

**Proof.** We only prove the result for \((\theta_i, \theta_i') = (hh, ll)\). The proof for other type combinations proceed step by step in the same way and are left to the reader. In terms of the multinomial probabiloty distribution in (44) \( \rho_1^i (hh) \) and \( \rho_1^i (ll) \) may be expressed as
\[
\rho_1^i (hh) = \sum_{x \in X_{-1}} a_{-1} (x) \left[ \rho_1^i (x_{hh} + 1, x_{hl}, x_{lh}, x_{ll}) \right] \tag{A33}
\]
\[
\rho_1^i (ll) = \sum_{x \in X_{-1}} a_{-1} (x) \left[ \rho_1^i (x_{hh}, x_{hl}, x_{lh}, x_{ll} + 1) \right]
\]
Let \( \overline{\rho}^i \) maximize the difference between \( \rho_1^i (hh) \) and \( \rho_1^i (ll) \) and let \( \overline{\rho}^i_1 (hh) \) and \( \overline{\rho}^i_1 (ll) \) be the associated perceived probabilities of the project being implemented conditional on the type. That is,
\[
\overline{\rho}^i \in \arg \max_{\rho^i : X \to [0, 1]} \sum_{x \in X_{-1}} a_{-1} (x) \left[ \rho_1^i (x_{hh} + 1, x_{hl}, x_{lh}, x_{ll}) - \rho_1^i (x_{hh}, x_{hl}, x_{lh}, x_{ll} + 1) \right], \tag{A34}
\]
The solution to (A34) is
\[
\overline{\rho}^i (x) = \begin{cases} 
1 & \text{if } a_{-1}(x_{hh} - 1, x_{hl}, x_{lh}, x_{ll}) \geq a_{-1}(x_{hh}, x_{hl}, x_{lh}, x_{ll} - 1) \\
0 & \text{if } a_{-1}(x_{hh} - 1, x_{hl}, x_{lh}, x_{ll}) < a_{-1}(x_{hh}, x_{hl}, x_{lh}, x_{ll} - 1).
\end{cases} \tag{A35}
\]
Using the explicit formula for \( a_{-1} (x) \) in (44), we can express (A35) as
\[
\overline{\rho}^i (x) = \begin{cases} 
1 & \text{if } \frac{x_{hh}}{\alpha^2} \geq \frac{x_{ll}}{(1-\alpha)^2} \\
0 & \text{if } \frac{x_{hh}}{\alpha^2} < \frac{x_{ll}}{(1-\alpha)^2}.
\end{cases} \tag{A36}
\]
Fix an arbitrary \( \epsilon > 0 \) and let \( m = x_{hl} + x_{lh} \leq n - 1 \). Since \( m \) is a binomial random variable with parameters \( p = 2\alpha (1 - \alpha) \) and \( n - 1 \) we know that there exists \( N < \infty \) such that
\[
\Pr \left( \frac{m}{n - 1} \geq 2\alpha (1 - \alpha) + \epsilon \right) \leq \frac{\epsilon}{2} \tag{A37}
\]

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for every $n \geq N$. Moreover, conditional on $m$, $x_{hh}$ is binomially distributed with parameters $p' = \frac{\alpha}{2(1-\alpha)}$ and $n - 1 - m$. From (A36) it follows that there is a single value $\overline{E}_{hh}(m)$ such that $\overline{p}'(\overline{E}_{hh}(m) + 1, x_{hl}, x_{lh}, x_{ll}) = 1$ and $\rho^2(\overline{E}_{hh}(m), x_{hl}, x_{lh}, x_{ll} + 1) = 0$, while for all other realizations the of $x_{hh}$ the type of agent $i$ is irrelevant for the provision decision. Applying Lemma ?? we conclude that there exists $N' < \infty$ such that

$$\Pr(x_{hh} = \overline{E}_{hh}(m) | m) \leq \frac{\epsilon}{2} \tag{A38}$$

for all $m$ such that $n - 1 - m \geq N'$. Consider $n \geq \max\left\{N, \frac{N'}{1 - 2\alpha(1 - \alpha) - \epsilon} + 1\right\} < \infty$. Then, $N' \leq (n - 1)[1 - 2\alpha(1 - \alpha) - \epsilon]$, so

$$\Pr[n - 1 - m \leq N'] = \Pr[m \geq (n - 1) - N'] \leq \Pr[m \geq (n - 1) - (n - 1)[1 - 2\alpha(1 - \alpha) - \epsilon]] = \Pr \left[ \frac{m}{n - 1} \geq 2\alpha(1 - \alpha) + \epsilon \right] \leq \frac{\epsilon}{2} \tag{A39}$$

Hence, for $n \geq \max\left\{N, \frac{N'}{1 - 2\alpha(1 - \alpha) - \epsilon} + 1\right\}$ we have that $n - 1 - m \leq N'$ with probability of at least $1 - \frac{\epsilon}{2}$ so

$$\sum_{x \in X_{-1}} \sum_{m=0}^{n-1} \Pr(m) \Pr(x_{hh} = \overline{E}_{hh}(m) | m) \leq \sum_{m=0}^{n-1-N'} \Pr(m) \leq \frac{\epsilon}{2} \Pr[n - 1 - m \geq N'] + \Pr[n - 1 - m \leq N'] \leq \epsilon$$

since $\Pr[n - 1 - m \leq N'] \leq \frac{\epsilon}{2}$ by (A39). Next, let $\rho^1$ solve

$$\rho^1 \in \arg \min_{p: X \rightarrow [0,1]} \sum_{x \in X_{-1}} a_{-1}(x) \left[ \rho^1(x_{hh} + 1, x_{hl}, x_{lh}, x_{ll}) - \rho^1(x_{hh}, x_{hl}, x_{lh}, x_{ll} + 1) \right] \tag{A41}$$

and let $\rho^1_{hh}(hh)$ and $\rho^1_{ll}(ll)$ be the associated perceived probabilities for types $hh$ and $ll$. A
solution to (A41) is
\[
\rho^1(x) = \begin{cases} 
1 & \text{if } x_{hh} \frac{1}{\alpha^2} < x_l \frac{1}{(1-\alpha)^2}, \\
0 & \text{if } x_{hh} \frac{1}{\alpha^2} \geq x_l \frac{1}{(1-\alpha)^2}
\end{cases}
\]  
(A42)

which is just reversing the provision rule \(\overline{\rho}^1\). Hence in this case \(\overline{\rho}^1(\overline{x}_{hh}(m)+1, x_{hl}, x_{lh}, x_{lt}) = 0\) and \(\rho^1(\overline{x}_{hh}(m), x_{hl}, x_{lh}, x_{lt}+1) = 1\) whereas for all other values for \(x_{hh}\) the announcement of agent \(i\) is irrelevant. It thus immediately follows from our previous calculations that

\[
\rho^1_i(hh) - \rho^1_i(ll) = -\sum_{m=0}^{n-1} \Pr(m) \Pr(x_{hh} = \overline{x}_{hh}(m)|m) \geq -\epsilon. 
\]  
(A43)

Combining (A40) and (A43) it follows that for any conceivable provision rule,

\[
-\epsilon \leq \rho^1_i(hh) - \rho^1_i(ll) \leq \rho^1_i(hh) - \rho^1_i(ll) \leq \overline{\rho}^1_i(hh) - \overline{\rho}^1_i(ll) \leq \epsilon.
\]  
(A44)

\[\blacksquare\]

References


