Computing gxx and hxx using the Hessian definition of Magnus and Neuberger.

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LaTeXfile: grohe/teaching/ perturbation/MN_Hessian.tex
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This note shows how to use the Magnus and Neuberger matrix notation to find the matrices gxx and hxx. In Schmitt-Grohé and Uribe (*JEDC* 2004) we used some form of tensor notation. Paul Klein (2005) had the original idea of using the Magnus and Neuberger matrix notation for this task.

All notation is taken from our JEDC piece. To find the second-order approximation of the functions $g(x, \sigma)$ and $h(x, \sigma)$, we exploit the fact that $F_{xx}(\bar{x}, 0) = 0$. Instead of working with the three dimensional array F_{xx} we will use a permutation of it known as the Hessian of $F(x)$. According Magnus and Neuberger (1999), and cited in Klein (2005), the Hessian is defined as follows:

$$\mathbf{H}F(x) \equiv \mathbf{D}[\text{vec}[(\mathbf{D}F(x))']]$$

Recall that the function $F(x, 0)$ is defined as follows

$$F(x, 0) \equiv f(g(h(x, 0), 0), g(x, 0), h(x, 0), x)$$

and that $F(x, 0)$ as well as all of its derivatives are equal to zero. For simplicity in what follows we will suppress the second argument of the functions F, h, and g, for it is held fixed at zero.

Further recall the expression for $\mathbf{D}F(x)$.

$$\mathbf{D}F(x) = f_{y'}g_x h_x + f_y g_x + f_{x'}h_x + f_x$$

To apply the Magnus and Neudecker definition of the Hessian we need to start by taking the transpose

$$[\mathbf{D}F(x)]' = h'_x g'_x f'_{y'} + g'_x f'_y + h'_x f'_{x'} + f'_x$$

Now we need to take the vec of $[\mathbf{D}F(x)]'$. We will apply extensively the rule $vec(ABC) = (C' \otimes A) * vec(B)$. And then we need to apply the $\mathbf{D}$ operator.

Let's perform these two operations for each term. Start with the first term which is:

$$part1 = [h_x]'[g_x]'[f_{y'}]'$$

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Note that

$$[h_x]'[g_x]'[f_{y'}]' = I_{n_x}[h_x]'([g_x]'[f_{y'}]') = ([h_x]'[g_x]')[f_{y'}]'I_n$$

Now we need to take the vec.

$$([f_{y'}] \otimes [h_x]')\text{vec } [g_x]' = ((([f_{y'}][g_x]) \otimes I_{n_x})\text{vec } [h_x]' = (I_n \otimes ([h_x]'[g_x]'))\text{vec } [f_{y'}]'$$

Next apply the **D** operator.

$$([f_{y'}] \otimes [h_x]')\mathbf{D}\text{vec } [g_x]' + ((([f_{y'}][g_x]) \otimes I_{n_x})\mathbf{D}\text{vec } [h_x]' + (I_n \otimes ([h_x]'[g_x]'))\mathbf{D}\text{vec } [f_{y'}]'$$

Note that $\mathbf{D}\text{vec } [g_x]'$ is a function of $h(x)$ so we need to take account of this.

$$\mathbf{D}\text{vec } [g_x]' = g_{xx}h_x$$

Note that g_{xx} is the Hessian of $g(x)$ again using the Magnus and Neuberger definition. that is:

$$g_{xx} = \begin{bmatrix} \mathbf{H}g_1(x) \\ \mathbf{H}g_2(x) \\ \vdots \\ \mathbf{H}g_{n_y}(x) \end{bmatrix}$$

Similarly, $f_{y'}$ is a function of 4 arguments and some of the arguments are again functions of x, so that we get

$$\mathbf{D}\text{vec } [f_{y'}]' = f_{y'y'}g_xh_x + f_{y'y}g_x + f_{y'x'}h_x + f_{y'x}$$

Again f_{xx}, is the Hessian of $f(.,.,.,x)$ again using the Magnus and Neuberger definition. that is:

$$f_{xx} = \begin{bmatrix} \mathbf{H}f_1(.,.,.,x) \\ \mathbf{H}f_2(.,.,.,x) \\ \vdots \\ \mathbf{H}f_n(.,.,.,x) \end{bmatrix}$$

Collecting terms we have that

$$\begin{aligned} \mathbf{D}(\text{vec } part1) \quad = \quad & (I_n \otimes h_x'g_x')[f_{y'y'}g_xh_x + f_{y'x'}h_x + f_{y'y}g_x + f_{y'x}] + \\ & (f_{y'} \otimes [h_x]')g_{xx}h_x + (f_{y'}g_x \otimes I_{n_x})h_{xx} \end{aligned}$$

Now let's do the second term:

$$part2 = g_x'f_y' = g_x'f_y'I_n = I_{n_x}g_x'f_y'$$

$$\text{vec } (part2) = (I_n \otimes g_x')\text{vec } f_y' = (f_y \otimes I_{n_x})\text{vec } g_x'$$

$$\begin{aligned} \mathbf{D}(\text{vec } part2) \quad = \quad & (I_n \otimes g_x')[f_{yy'}g_xh_x + f_{yx'}h_x + f_{yy}g_x + f_{yx}] + \\ & +(f_y \otimes I_{n_x})g_{xx} \end{aligned}$$

The third term:

$$part3 = h_x' f_{x'}' = h_x' f_{x'}' I_n = I_{n_x} h_x' f_{x'}'$$

$$\text{vec } (part3) = (I_n \otimes h_x')\text{vec } (f_{x'}') = (f_{x'} \otimes I_{n_x})\text{vec } h_x'$$

$$\mathbf{D}(\text{vec } part3) = (I_n \otimes h_x')[f_{x'y'} g_x h_x + f_{x'y} g_x + f_{x'x'} h_x + f_{x'x}] + (f_{x'} \otimes I_{n_x}) h_{xx}$$

The fourth term:

$$part4 = f_x'$$

$$\text{vec } (part4) = \text{vec } (f_x')$$

$$\mathbf{D}(\text{vec } part4) = [f_{xy'} g_x h_x + f_{xy} g_x + f_{xx'} h_x + f_{xx}]$$

Collecting terms we end up with

$$\begin{aligned}\mathbf{H}F(x) = {} & (I_n \otimes h_x' g_x')[f_{y'y'} g_x h_x + f_{y'x'} h_x + f_{y'y} g_x + f_{y'x}] + (f_{y'} \otimes [h_x]')g_{xx} h_x + (f_{y'} g_x \otimes I_{n_x}) h_{xx} + \\ & (I_n \otimes g_x')[f_{yy'} g_x h_x + f_{yx'} h_x + f_{yy} g_x + f_{yx}] + (f_y \otimes I_{n_x}) g_{xx} + \\ & (I_n \otimes h_x')[f_{x'y'} g_x h_x + f_{x'y} g_x + f_{x'x'} h_x + f_{x'x}] + (f_{x'} \otimes I_{n_x}) h_{xx} + \\ & [f_{xy'} g_x h_x + f_{xy} g_x + f_{xx'} h_x + f_{xx}]\end{aligned}$$

Now let's rearrange to get something of the form:

$$\begin{aligned}\mathbf{H}F(x) = {} & (I_n \otimes h_x' g_x')[f_{y'y'} g_x h_x + f_{y'x'} h_x + f_{y'y} g_x + f_{y'x}] + \\ & (I_n \otimes g_x')[f_{yy'} g_x h_x + f_{yx'} h_x + f_{yy} g_x + f_{yx}] + \\ & (I_n \otimes h_x')[f_{x'y'} g_x h_x + f_{x'y} g_x + f_{x'x'} h_x + f_{x'x}] + \\ & [f_{xy'} g_x h_x + f_{xy} g_x + f_{xx'} h_x + f_{xx}] + \\ & (f_{y'} \otimes [h_x]') g_{xx} h_x + \\ & (f_y \otimes I_{n_x}) g_{xx} + \\ & [(f_{y'} g_x \otimes I_{n_x}) + (f_{x'} \otimes I_{n_x})] h_{xx}\end{aligned}$$

Letting

$$\begin{aligned}A = {} & (I_n \otimes h_x' g_x')[f_{y'y'} g_x h_x + f_{y'x'} h_x + f_{y'y} g_x + f_{y'x}] + \\ & (I_n \otimes g_x')[f_{yy'} g_x h_x + f_{yx'} h_x + f_{yy} g_x + f_{yx}] + \\ & (I_n \otimes h_x')[f_{x'y'} g_x h_x + f_{x'y} g_x + f_{x'x'} h_x + f_{x'x}] + \\ & [f_{xy'} g_x h_x + f_{xy} g_x + f_{xx'} h_x + f_{xx}] \\ B = {} & [(f_{y'} \otimes [h_x]') \\ C = {} & (f_y \otimes I_{n_x})] \\ D = {} & [(f_{y'} g_x \otimes I_{n_x}) + (f_{x'} \otimes I_{n_x})]\end{aligned}$$

this in turn we can write as

$$A + B g_{xx} h_x + C g_{xx} + D h_{xx} = 0$$

Applying vec one last time, we have

$$\begin{aligned} \text{vec } 0 &= \text{vec } [A + Bg_{xx}h_x + Cg_{xx} + Dh_{xx}] \\ &= \text{vec } A + (h_x' \otimes B)\text{vec } g_{xx} + (I_{n_x} \otimes C)\text{vec } g_{xx} + (I_{n_x} \otimes D)\text{vec } h_{xx} \end{aligned}$$

Finally we have:

$$-\text{vec } A = [(h_x' \otimes B) + (I_{n_x} \otimes C) \quad (I_{n_x} \otimes D)] \begin{bmatrix} \text{vec } g_{xx} \\ \text{vec } h_{xx} \end{bmatrix}$$

or

$$\begin{bmatrix} \text{vec } g_{xx} \\ \text{vec } h_{xx} \end{bmatrix} = -Q^{-1}\text{vec } A$$

where

$$Q = [(h_x' \otimes B) + (I_{n_x} \otimes C) \quad (I_{n_x} \otimes D)]$$