Cash Flows Discounted Using a Model Free SDF Extracted under a Yield Curve Prior *

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Abstract

We develop a model-free Bayesian extraction procedure for the stochastic discount factor under a yield curve prior. Previous methods in the literature, Bayes or frequentist, use directly or indirectly some particular parametric asset pricing model such as long run risks or habit as the prior. Here, in contrast, we use no such model at all, but rather we adopt a prior that enforces external information about the historically very low levels of U.S. short- and long-term interest rates. For clarity and simplicity our data are annual time series. We use the extracted stochastic discount factor to determine the stripped cash flow risk premiums on a panel of industrial profits and consumption. Interestingly, the results align very closely with recent limited information (bounded rationality) models of the term structure of equity risk premiums, although nowhere did we use any theory of the discount factor other than its implied moment restrictions.

Keywords and Phrases: Stochastic discount factor, Discounting, Cash flows, Yield curve, Moment functions, Bounded Rationality, Bayesian

JEL Classification: C32, C36, E27

1 Introduction

Under very mild conditions there exists a scalar stochastic discount factor (SDF) process that generates moment restrictions on the returns (or cash flows) of traded securities. Knowledge of the SDF process allows one to check if a particular security is priced consistently with other traded assets, and it allows the valuation of uncertain future cash flows on non-traded assets so long one is confident the pricing implications extend appropriately. In the literature there is a plethora of methods either to extract non parametrically the SDF process from historical data or to evaluate particular theories of the SDF process such as long run risk models or habit models.

One difficulty with such such approaches concerns the ex-post implied level of real interest rates. Extant procedures use moment conditions based on asset returns, and the return horizons typically range from monthly to annual. The returns series are like first differenced asset prices and thus nearly white noise processes; information on the level of asset prices, and bond prices in particular, is negligible leaving interest rates ill determined. Left to its own, an extracted SDF can give rise to somewhat implausible levels of real interest rates. As an example, in a recent long run risk application, Christensen (2017) reports the long interest rate as a rather high 7 percent per year; in additional computations using this author's code we found that the entire yield curve from one year on out is essentially flat at just over 7 percent. Poorly determined real yield curves were encountered using the extraction procedure of this paper where, absent prior knowledge, implied yield curves shifted and bent in implausible configurations. As just noted, the moment conditions contain little, if any, level information, and external information from other sources needs to be imposed to discipline the SDF extraction.

An agreed upon fact is that U.S. real interest rates are very low. From Campbell (2003, p. 812) the average short-term U.S. real rate was 0.896 percent over the period 1947–1998, and few would argue for higher real short rates since then. As for longer term real rates, Table 3 of Tesar and Obstfeld (2015) indicates that the 10-year real rate of interest over the period 1930–2014 was often negative, generally fluctuated between 0 and 2.5 percent per year, and only briefly bumped 5 percent during the interwar era and again during dis-inflation period of the early 1980s. Additional information from TIPS real yields is seen in Table 1, which are remarkably low.

(Table 1 about here)

In what follows we implement a Bayesian SDF extraction procedure subject to a prior that enforces these known low values for U.S. real interest rates. The method's mathematical foundation requires a prior to ensure all random variables are actually defined on a proper probability space, and here we elect the yield curve prior in place of a specific model of the SDF. Specifically, the prior centers the one year yield at 0.896 percent with standard deviation 1.00 percent, and it centers the 30-year yield at 2.00 percent with standard deviation 1.00 percent. The prior generally accommodate both the levels and fluctuations in real rates suggested in the above discussion and by Table 1. This prior is maintained throughout the entire sample period, although it would be relatively easy to impose a time-varying prior with possibly higher yields in the earlier parts of the sample if reliable real rates were available to inform development of such a prior.

Post extraction, we use the dynamics of the SDF and related variables in a standard log-Gaussian pricing framework to value various cash flows, with a focus on the risk premiums on stripped cash flows, often termed dividend strips in the literature. The concept is simple: if, from the perspective of period 0, an investment pays off the uncertain stream $\{CF_t\}_{t=1}^{\infty}$ into the indefinite future, then the stripped cash flow is the asset that pays just CF_t in period $t > 0$, and zero in all other periods. Recently, researchers have been investigating the term structure of the equity risk premiums on stripped cash flows to understand better the relationship between risk and reward at short- and long-term horizons. Asset pricing models (van Binsbergen et al., 2012; Maggiori et al., 2015) suggest that the term structure of risk premiums is upward sloping, with more distant cash flows earning higher risk premiums due to a long-run risk (Bansal and Yaron, 2004) mechanism that makes investors fearful of volatility in the distant future. This prediction seems at odds with common sense intuition, but theory alone is not powerful enough to make an unambiguous prediction on the slope of the equity risk premium term structure. Backus et al. (2016) show how a wide range of levels and shapes of the term structures of claims can be achieved by modifying the dynamics of the pricing kernel, the cash flow growth, and their interaction. Empirically, discussions about the true average slope of the equity returns term structure have not yet been settled (Cochrane, 2017; Bansal et al., 2017), and reconciling asset pricing models with possible slopes of term structure of equity returns has recently become a very active area of research. Of particular interest here is Croce et al. (2014), who develop a bounded rationality model with long run risk that appears to explain our findings below.

2 Ex-post Stochastic Discount Factor

The ex-post realized values of $\mathrm{SDF}_{t-1,t}$ are extracted annually 1930–2015 using the methodology developed in Gallant and Hong (2007). The differences are that the data set is longer due to the passage of time, that all of the Fama-French portfolios (Fama and French, 1992, 1993) can be used because a missing data problem has been resolved, and that the prior tilts values toward a specified yield curve instead of toward long run risk dynamics (Bansal and Yaron, 2004). In brief, the ideas are as follows.

For time $t = 1, \ldots, n = 86$, where $t = 1$ corresponds to 1930 and $t = n$ to 2015, denote real gross returns on the 25 Fama-French portfolios by the vector R_{st}, denote the real gross return on the thirty day T-bill by R_{bt}, denote real per capita consumption growth by $\frac{C_t}{C_{t-1}}$, denote per capita labor income growth by $\frac{L_t}{L_{t-1}}$. Let x_t let a vector of length 29 containing these variables. Let x be an array with these variable as columns; x has dimension 29 by $n = 86$. Let θ_t denote the stochastic discount factor $\mathrm{SDF}_{t-1,t}$ and set $\theta = (\theta_1, \ldots, \theta_{86})$.

Conceptually, a fully articulated dynamic stochastic general equilibrium model (DSGE) with a financial sector would define a probability space on which the random variables $(x, \theta) \in \mathcal{X} \times \Theta$ live. This probability space would determine a conditional distribution of x given θ. For ease of exposition we presume it has density $p^o(x \mid \theta)$ and omit consideration of the structural parameters of the DSGE. Let $p(\theta)$ denote the prior we intend to use for Bayesian inference. Our analysis is with reference to the probability space $(\mathcal{X} \times \Theta, \mathcal{C}^o, P^o)$ where P^o has density $p^o(x \mid \theta)\, p(\theta)$. This construction is necessitated by the fact that θ is endogenous and some care must be taken to define a likelihood $p^o(x \mid \theta)$. Were we willing to undertake a parametric analysis, we would proceed directly to Bayesian inference based on this construction using $p^o(x \mid \theta)$ as the likelihood if it were analytically tractable or using the simulation method of Gallant and Tauchen (2018) if not. Here our preference is to use a model free method in the sense that we only use moment conditions to extract the ex-post realization of θ. Nonetheless, we presume the existence of although not knowledge of the probability space $(\mathcal{X} \times \Theta, \mathcal{C}^o, P^o)$.

Denote the vector of (conditional) moment-equation errors by

$$e_{t,t-1}(\theta_t) = 1 \,-\, \theta_t \begin{pmatrix} R_{s,t} \\ R_{b,t} \end{pmatrix}, \tag{1}$$

where 1 denotes a vector of 1's of length twenty-six. Define the instruments

$$V_t = \begin{pmatrix} R_{st} - 1 \\ R_{bt} - 1 \\ C_t/C_{t-1} - 1 \\ L_t/L_{t-1} - 1 \\ 1 \end{pmatrix},$$

where $R_{st} - 1$ denotes 1 subtracted from each element of R_{st}. Consider the moment conditions

$$m(x_t, x_{t-1}, \theta_t) = V_{t-1} \otimes e_{t,t-1}(\theta), \tag{2}$$

where $\otimes$ denotes Kronecker product, and their sample average

$$\bar{m}(x, \theta) = \frac{1}{n} \sum_{t=2}^{n} m(x_t, x_{t-1}, \theta_t). \tag{3}$$

The length of the vector $m(x_t, x_{t-1}, \theta_t)$ is $K = 754$ so that the number of overidentifying restrictions on $\theta_2, \ldots, \theta_{86}$ is 669. Note that θ_1 is not yet identified because θ_1 does not appear in (3); it is identified by the prior as discussed later in this subsection.

Following Gallant and Hong (2004), we assume that $e_{t,t-1}(\theta)$ has a factor structure. There is one error common to all elements of $\theta_t R_{st}$ and twenty-six idiosyncratic errors, one for each element of $(\theta_t R_{st}, \theta_t R_{bt})$. Denote this matrix by Σ_e (or by $\Sigma_{e,t}$ if one wants to allow for heterogeneity, which makes no difference in what follows). A set of orthogonal eigen vectors U_e for Σ_e are easy to construct (Gallant and Hong, 2004, p. 535), and can be used to diagonalize Σ_e. Similarly U_v and Σ_v for V_t. Let $H_t(\theta) = (U_v \otimes U_e)' m(x_t, x_{t-1}\theta)$ with elements

$h_{i,t}(\theta)$. Diagonalization implies that we can estimate the variance of $H_t(\theta)$ by a diagonal matrix $S_n(\theta)$ with elements

$$s_i(\theta) = \frac{1}{n}\sum_{t=2}^{n}\left(h_{t,i}(\theta) - \frac{1}{n}\sum_{t=2}^{n} h_{t,i}(\theta)\right)^2.$$

Let $S_n^{-1/2}(\theta)$ denote this matrix with the diagonal elements replaced by their square roots.

Extraction of the ex-post realization of the SDF is based on the random variable

$$Z(x,\theta) = \sqrt{n}\, S_n^{-1/2}(\theta)(U_z \otimes U_e)'\bar{m}_n(x,\theta) \tag{4}$$

defined on the aforementioned probability space $(\mathcal{X}\times\Theta, \mathcal{C}^o, P^o)$. $Z(x,\theta)$ is the normalized sum of transformed draws (x_t, θ_t) and will obey a central limit theorem under plausible regularity conditions on $(\mathcal{X}\times\Theta, \mathcal{C}^o, P^o)$. Note specifically that θ_t is random and jointly distributed with x_t so that issues of uniformity in θ do not arise. Thus, it is reasonable to assume that Z follows the standard normal distribution $\Phi(z)$ with density $\phi(z)$.

The assumption that $Z(x,\theta)$ has density $\phi(z)$ induces a probability space $(\mathcal{X}\times\Theta, \mathcal{C}, P)$, where $\mathcal{C}$ is the σ-algebra of preimages $\mathcal{C} = \{C = Z^{-1}(B), B \subset \mathcal{X}\times\Theta, B \text{ Borel}\}$ and $P[C = Z^{-1}(B)] = \int_B d\Phi(z)$. Define $\mathcal{C}^*$ to be the smallest σ-algebra that contains all sets in $\mathcal{C}$ plus all sets of the form $R_B = (\mathcal{X}\times B)$, where B is a Borel subset of Θ. Under a semi-pivotal assumption on (4), which is that $\{x : Z(x,\theta) = z\}$ not be empty for any choice of (z,θ) in the parameter space Θ and range space $\mathcal{Z}$, there is an extension of $(\mathcal{X}\times\Theta, \mathcal{C}, P)$ to a space $(\mathcal{X}\times\Theta, \mathcal{C}^*, P^*)$ on which the conditional density of x given θ is

$$f(x\,|\,\theta) = \phi[Z(x,\theta)] \tag{5}$$

(Gallant, 2017a). This density is termed the "method of moments representation" of the likelihood and may be used for Bayesian inference in connection with the prior $p(\theta)$ (Gallant, 2017a, 2017b).

In short, the Bayesian method used in Gallant and Hong (2007) and here uses moment conditions $z = Z(x,\theta)$ given by (4) takes

$$p(x\,|\,\theta) = \phi(z) \tag{6}$$

as the likelihood and proceeds directly to Bayesian inference using a prior $p(\theta)$.

Next we describe the prior.

Let $w_t = \left(\log(\theta_t), \log(\frac{GDP_t}{GDP_{t-1}})\right)'$ where is observed GDP_t growth for $t = 1, 2, \ldots, n = 86$. GDP not involved in the SDF extraction to this point. It is included now as prior information regarding past cycle conditions. Consider the recursion

$$w_t = d_0 + Dw_{t-1} + u_t, \tag{7}$$

where the u_t are independent, bivariate normal with mean zero and variance Σ_d. Markov chain Monte Carlo (MCMC) (Gamerman and Lopes, 2006) is used in the Gallant and Hong (2007) method which means that proposed θ_t are available to compute w_t before the prior and likelihood need to be computed. From the w_t the parameters of (7) can be determined by least squares. With least squares values replacing parameters in (7), a yield curve for maturities one year through thirty years can be computed analytically from (7) conditional on a specified initial condition w_0; see equations (12) and (13) and (15) of Section 3. In particular, the one-year and 30-year yields, $Y^*_{1,t}$ and $Y^*_{30,t}$ can be computed successively for $w_0 = w_t$, $t = 1, ..., n = 86$. Our prior is

$$p(\theta) = \prod_{t=1}^{n} \phi[(Y^*_{1,t} - 0.896)/0.01]\phi[(Y^*_{30,t} - 2.0)/0.01)] \tag{8}$$

Note, in particular, that the prior identifies θ_1.

With likelihood (5) and prior (8) in hand, Bayesian inference can be carried out using MCMC in the usual way; see, e.g., Gamerman and Lopes (2006). After transients died out, we ran a MCMC chain of length 8,000,000. That θ in the chain with the highest value of the posterior was selected as the estimate $\hat{\theta}$ of the ex-post SDFs for the years 1930 through 2015. The estimate is plotted as Figure 1. The shaded areas are NBER recessions.

Figure 1 about here

3 Discounted Cash Flow Estimation

We now illustrate how to use the extracted SDF series to value cash flows on assets outside the span of returns used in the extraction step. For this part we use annual data on corporate profits from various large sectors of the U.S. economy. We assembled annual data for seven sectors but for the shorter period 1959–2015, as data limitations precluded going any farther back. These data are concatenated with the extracted SDF data and various macro aggregates for this shorter time span.

For the valuation step, consider the trivariate series

$$y_t = \begin{pmatrix} \log(SDF_{t-1,t}) \\ \log(GDP_t) - \log(CF_t) \\ \log(CF_t) - \log(CF_{t-1}) \end{pmatrix} = \begin{pmatrix} sdf_{t-1,t} \\ gdp_t - cf_t \\ \Delta cf_{t-1,t} \end{pmatrix} \tag{9}$$

where CF_t denotes a cash flow payoff at time t, such as annual corporate profits in year t, GDP_t denotes gross domestic product, and $SDF_{t-1,t}$ denotes the extracted stochastic discount factor $\hat{\theta}_t$ of Section 2. Note that the second variable in the autoregression, $\log(GDP_t) - \log(CF_t)$, plays no direct effect in subsequent pricing, but it is included because it conveys information of future cash flows. The specification presumes co-integration between GDP and CF, which is discussed more fully in Section 4 below.

The time zero present value of the cash flow CF_t is

$$PV_{0,t}(CF) = \mathcal{E}(S_{0,t}\, CF_t \,|\, \mathcal{F}_0) = \mathcal{E}\left[\exp\left(\sum_{s=1}^{t} \Delta cf_{s-1,s} + \sum_{s=1}^{t} sdf_{s-1,s}\right) \,\Bigg|\, \mathcal{F}_0\right] \tag{10}$$

where $\mathcal{F}_0$ denotes the time 0 information set. The time zero discounted value of the sum of the cash flows through time t is the sum $DCF_{0,t} = \sum_{s=1}^{t} PV_{0,s}(CF)$. For a risk free payoff of one real dollar at time t, the time zero present value is $PV_{0,t}(1) = \mathcal{E}(SDF_{0,t} \,|\, \mathcal{F}_0)$, where $SDF_{0,t} = \exp(\sum_{s=1}^{t} sdf_{s-1,s})$. The corresponding yield is $Y_t^* = -\log[PV_{0,t}(1))]/t$.

Let

$$y_t = b_0 + B y_{t-1} + e_t, \tag{11}$$

where the e_t are independent, trivariate normal with mean zero and variance Σ_b. The sum

$$\sum_{s=1}^{t} y_s = \left(\sum_{s=1}^{t} (t+1-s) B^{s-1}\right) b_0 + \left(\sum_{s=1}^{t} B^s\right) y_0 + \sum_{s=1}^{t} \left(\sum_{u=1}^{s} B^{u-1}\right) e_{t+1-s}$$

where $B^0 = I$, $B^1 = B$, $B^2 = B\,B$, etc., has moments

$$\mathcal{E} \sum_{s=1}^{t} y_s = \left(\sum_{s=1}^{t} (t+1-s) B^{s-1}\right) b_0 + \left(\sum_{s=1}^{t} B^s\right) y_0 \tag{12}$$

$$\operatorname{Var} \sum_{s=1}^{t} y_s = \sum_{s=1}^{t} \left(\sum_{u=1}^{s} B^{u-1}\right) \Sigma_b \left(\sum_{u=1}^{s} B^{u-1}\right)' \tag{13}$$

One can use (12) and (13) to evaluate

$$\begin{aligned} PV_{0,t}(CF) &= \exp\left[\mathcal{E}\left(\sum_{s=1}^{t}\Delta cf_{s-1,s}\right)+\mathcal{E}\left(\sum_{s=1}^{t}m_{s-1,s}\right)\right] \\ &\quad + \exp\left[\frac{1}{2}\mathrm{Var}\left(\sum_{s=1}^{t}\Delta cf_{s-1,s}\right)+\frac{1}{2}\mathrm{Var}\left(\sum_{s=1}^{t}m_{s-1,s}\right)\right] \\ &\quad + \exp\left[\mathrm{Cov}\left(\sum_{s=1}^{t}\Delta cf_{t-1,t},\sum_{s=1}^{t}m_{s-1,s}\right)\right] \qquad (14) \\ PV_{0,t}(1) &= \exp\left[\mathcal{E}\left(\sum_{s=1}^{t}m_{s-1,s}\right)+\frac{1}{2}\mathrm{Var}\left(\sum_{s=1}^{t}m_{s-1,s}\right)\right] \qquad (15) \\ EV_{0,t}(CF) &= \exp\left[\mathcal{E}\left(\sum_{s=1}^{t}\Delta cf_{s-1,s}\right)+\frac{1}{2}\mathrm{Var}\left(\sum_{s=1}^{t}\Delta cf_{s-1,s}\right)\right] \qquad (16) \end{aligned}$$

We now describe imposition of a yield curve prior on the estimation of b_0, B, and Σ_b that appear in the VAR (11). Consider a state-space representation of VAR (11)

$$v_t = \begin{pmatrix} sdf_{t-2,t-1} \\ gdp_{t-1} - cf_{t-1} \\ \Delta cf_{t-2,t-1} \\ sdf_{t-1,t} \\ gdp_t - cf_t \\ \Delta cf_{t-1,t} \end{pmatrix} = \begin{pmatrix} 0 \\ b_0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ 0 & B \end{pmatrix} v_{t-1} + \begin{pmatrix} 0 \\ e_t \end{pmatrix} \qquad (17)$$

Estimation of (17) subject to the indicated parameter restrictions gives the same estimates $\hat{b}_0$, $\hat{B}$, $\hat{\Sigma}_b$ as does unconstrained estimation of (11). Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and note that the fourth and fifth elements of Ax_t are $sdf_{t-1,t}$ and $\Delta gdp_{t-1,t} = \log(GDP_t) -$

$\log(GDP_{t-1})$. An implication is that we can insert the parameters b_0^*, B^*, and Σ_b^* of

$$Av_t = b_0^* + B^*Av_{t-1} + e_t^* = A\begin{pmatrix} 0 \\ b_0 \end{pmatrix} + A\begin{pmatrix} 0 & I \\ 0 & B \end{pmatrix} A^{-1}Av_{t-1} + A\begin{pmatrix} 0 \\ e_t \end{pmatrix} \tag{18}$$

into equations (12), (13), and (15) to compute $Y_{1,t}^*$ for year one and yield $Y_{30,t}^*$ for year 30 with y_0 set to v_t successively for $t = 1, ..., n = 86$ and impose the prior

$$p(\theta) = \prod_{t=1}^{n} \phi[(Y_{1,t}^* - 0.896)/0.01]\phi[(Y_{30,t}^* - 2.0)/0.01)] \tag{19}$$

The computational procedure is, within an MCMC loop, for proposed b_0, B, and Σ_b, evaluate the likelihood implied by VAR (11), compute b_0^*, B^*, and Σ_b^* as indicated by expression (18) from the proposed b_0, B, and Σ_b, evaluate prior (19), and use the likelihood and prior so computed to make the accept/reject decision of the MCMC loop.

Serendipitously, the state-space complications can be avoided because it turns out that the yields $Y_{1,t}^*$ and $Y_{30,t}^*$ obtained by applying equations (12), (13), and (15) directly to b_0, B, and Σ_b of VAR (11) are identical to those computed from b_0^*, B^*, and Σ_b^* of VAR (18). Apparently, the reason is that the only difference between the distributions of Av_t and v_t is the location parameter of their fifth element and the location parameter of the fifth element is not involved in equations (12), (13), or (15). For ourselves, we are more comfortable relying on having performed the computations using both (18) and (11) and obtaining identical results than relying on the distributional argument.

4 Empirical Implementation

4.1 Valuation, Expectation, and Risk Premiums

Display (14) shows the valuation operator $PV_{0,t}(CF)$ as the economic value in period 0 of the cash flow CF_t received in t; evidently $PV_{0,t}(\bullet)$ is a linear operation on random variables realized at time t. Likewise (16) shows the conventional statistical expected value operator $EV_{0,t}(CF)$, also linear operator on the same space of time t random variables as $PV_{0,t}(\bullet)$ operates. As usual, two linear operators on a space with minimal structure are connected via a Randon-Nikodym style change of measure/density, which is the usual risk-neutral change of measure, noted, but not used here.

For the risk free asset,

$$\text{bond price}_{0,t} = PV_{0,t}(1).$$

The (geometric) risk free yield to maturity $r^f_{0,t}$ at time 0 of the t-maturity zero-coupon bond is defined via the relationship

$$e^{t\, r^f_{0,t}} \times PV_{0,t}(1) = 1,$$

because the investing the amount $PV_{0,t}(1)$ grows at the continuously compounded rate $r_{f,0,t}$ up to the \$1 at time t. Equivalently, the rate is defined by

$$r^f_{0,t} = -\frac{\log(PV_{0,t}(1))}{t}.$$

Just as a coupon bearing bond can be thought of as a portfolio of stripped coupon payments valued as immediately above, finance economists have become interested in "dividend strips," where the dividend asset that pays the owner the infinite stream $\{CF_s\}_{s=1}^{\infty}$ is a portfolio of stripped payments CF_t. The value of each stripped coupon is given by the pricing operation worked out above as $PV_{0,t}(CF)$. Only if agents are neutral to risk would it be the case that $PV_{0,t}(\bullet) = e^{-r_{f,0,t}t} EV_{0,t}(\bullet)$.

By analogy with the pure discount bond, we can define the geometric rate $r_{0,t}$ at which the amount $PV_{0,t}(CF)$ invested at time 0 grows continuously compounded to its statistical expected value at time t by way of

$$\begin{aligned} e^{t\, r_{0,t}} \times PV_{0,t}(CF) = EV_{0,t}(CF) \Rightarrow \\ r_{0,t} = \frac{1}{t} \log\left(\frac{EV_{0,t}(CF)}{PV_{0,t}(CF)}\right) \end{aligned} \tag{20}$$

Note that $r_{0,t}$ is a number known at time 0 that pertains to a cash flow received at time t. The quantity

$$\text{risk premium: } r_{0,t} - r^f_{0,t} \tag{21}$$

is the excess (geometric) return over cash of the investment (stripped cash flow) that pays CF_t. The amount (21) represents the required rate of return above cash necessary to compensate for the economic risk embedded in the investment. To guide interpretation farther below, we recall from elementary asset pricing that in the iid log-Gaussian case,

$$r_{0,t} = r^f_{0,t} + \mathrm{Cov}(\Delta cf_{t-1,t}, sdf_{t-1,t}), \quad \forall t \geq 1, \tag{22}$$

using notation defined in (9). The risk premium in (21) is $-\text{Cov}(\Delta cf_{t-1,t}, sdf_{t-1,t})$, the fundamental notion in finance of reward for bearing covariance risk.

4.2 Cash Flow Data

We now apply the preceding to valuation of eight cash flows: seven industrial profit series, real per capita, and real per capita consumption of nondurables and services, annual, 1959–2015. Treating measured consumption as a cash flow just means that we compute the risk premium on the (endowment) asset that pays out annual consumption. Basic statistics for the cash flows, labeled 1–8 are available in Table 2.

(Table 2 about here)

Some of the industrial cash flows are aggregates but none is a complete aggregate of any of the others. For example, cash flow 1, Total Corporate profits, includes items such as transportation and utilities, which are not among the other categories because of lack of consistent data over the entire sample period. The bottom section of Table 2 also shows the basic statistics for the extracted log-SDF and GDP growth processes.

These cash flows do not correspond to the payoffs of traded securities, but using the above methods we can compute the risk premiums on the stripped cash flows. Among other things, we can then examine issues such as the reasonableness of the risk premiums relative to characteristics of the industries and their term structure at short- and long-term horizons.

The dynamics (9) presumes that log-cash flow and log-GDP processes are co-integrated with $gdp_t - cf_t$ as the stationary error correction process. As a check, the first seven panels (exluding cash flow 8) of Figure 2 show time series plots of the $gdp_t - cf_t$ process for the seven industrial profit series, each of which appears to be reasonably treated as realizations of a stationary process.

Figure 2 about here

We estimated autoregressions of the form

$$gdp_t - cf_t = a + \rho\,(gdp_{t-1} - cf_{t-1}) + e_t, \tag{23}$$

and for the seven industrial profit series the estimates of ρ ranged between 0.71 and 0.93, median 0.86, and generally we rejected quite strongly H_0: $\rho = 1$ in favor of H_1: $\rho < 1$. For the consumption cash flow series the results are different, as consumption of nondurables and services grew steadily relative to GDP during the transition from a production to a service economy over our particular sample period. Thus, for the consumption cash flow we cannot invoke cointegration, and we simply use $gdp_t - gdp_{t-1}$ in place of $gdp_t - cf_t$ as the predictor variable in the vector autoregression (11). The log consumption growth series is displayed in in the lower right corner of Figure 2.

To verify predictability, we estimated forecasting regressions of the form

$$\Delta cf_{t,t+1} = a + b_1 \Delta cf_{t-1,t} + b_2 \left(gdp_t - cf_t \right) + u_t, \tag{24}$$

for each of the seven industrial cash flows, and we found very strong evidence for additional predictability coming from the error correction variable $gdp_t - cf_t$. For consumption (the eighth cash flow) we did the regression as

$$\Delta cf_{t,t+1} = a + b_1 \Delta cf_{t-1,t} + b_2 \left(gdp_t - gdp_{t-1} \right) + u_t, \tag{25}$$

and found only mild evidence for additional predictability coming from the second right-hand variable.

4.3 Risk Premiums

Using the methods described in Section 3 we computed for each of the nine cash flows the implied risk premiums (21) at horizons $t = 1, 2, \ldots, 50$ for each available year (1960–2015 after lags). Since the risk premiums show only modest temporal variation, we just report and discuss the full sample averages. For brevity, we consider the average risk premium for horizon 1-year ahead, and the risk premiums averaged over ten year stretches ahead, i.e., 1–10 years, 11–20 years ahead, through 41–50 years ahead. The results are displayed in Table 3. The table also shows $\overline{\text{Cov}(\Delta cf_{t-1,t}, sdf_{t-1,t})}$, i.e., the average covariance of each cash flow growth with the extracted log-SDF, which is the exposure of the cash flow to SDF risk; to ease interpretation, the table shows the average correlations as well.

(Table 3 about here)

As seen from the table, the average 1-year risk premiums are the negatives of the average covariance exposures, which has to be true in this log-Gaussian framework. All cash flows carry positive risk premiums except for Retail Trade, which is plausibly seen to be a hedging cash flow. For easier interpretation, we convert the covariance exposure to beta exposure,

$$\beta = -\overline{\mathrm{Cov}(\Delta cf_{t-1,t}, sdf_{t-1,t})/\mathrm{Var}(sdf_{t-1,t})} \tag{26}$$

and plot the relationship between the risk premiums and the (negative) log-SDF beta exposure in Figure 3.

Figure 3 about here

The essentially exact linear relationship seen in the figure is a mechanical consequence of the computations, but nonetheless it is interesting to note that the price of risk is very close to 0.05, meaning an increase in average return of 5 percent per year per unit exposure of the cash flow to moves in $-sdf_{t-1,t}$.

A far more interesting feature seen in Table 3 is that for the industrial cash flows 1–7, i.e. those other than the hedging cash flow, the risk premiums are seen to decrease with horizon, and they decrease by a factor of one half from 1 to 50 years out; in contrast, for the consumption cash flow, the risk premium increases from about 1 percent per year to 4 percent per year moving from 1 to 50 years out. These results are exactly in line with what one would expect from Croce *et al* (2015, p. 723). In their limited information (bounded rationality) model, there is long run risk embedded in consumption, which under usual parameterizations thereby carries an increasing risk premium at longer horizons, just as seen in Table 2. There is also long run risk embedded in individual cash flows but it is obscured by a high level of cash flow noise that can be correlated with short-term consumption risk. Agents following optimal filtering rules thereby view the assets as much more covariance-riskier in the short run than in the long run, and so we expect to observe higher risk premiums for the short run than the long run.

A final matter is summability of the stripped cash flows, which relates to the issue of whether the asset that pays the entire stream $\{CF_s\}_{s=1}^{\infty}$ is even sensible. Using basic computations like those of Burnside (1998) for log-Gaussian models, in the iid case convergence of $\sum_{t=1}^{\infty} PV_{0,t}(CF)$ is assured if

$$\mathcal{E}(\Delta cf_{t-1,t}) - r^f + \mathrm{Cov}(\Delta cf_{t-1,t}, sdf_{t-1,t}) < 0. \tag{27}$$

In the general case such as that considered by De Groot (2015), the summability conditions are more involved as they involved interactions between conditional means and variances, but the basic intuition of (27) remains: the covariance between the cash flow growth has to be sufficiently negative to overcome any excess of the expected cash flows over the risk free rate. In our case, the cash flows grow at implied rates around 2 percent per year while the long term interest rate prior centers on 2 percent, but the covariances in Table 3 are negative so the sums converge numerically, albeit very slowly. An exception is Retail Trade, the hedging cash flow where the average covariance is positive. Some numerical instability of the partial sums for this cash flow was seen if the range is extended to 100+ years, which is not surprising for an extrapolation so far beyond the range of the data.

5 Conclusion

We develop a model free Bayesian procedure to extract the SDF process using an interest rate prior that enforces the historically very low U.S. short and long term interest rates. Using annual data 1959–2015 we use the extracted SDF to compute the implied stripped cash flow risk premiums on a panel corporate profits for eight major industrial sectors and consumption. The magnitudes of the the risk premiums on the stripped cash flows are plausible, and, with one exception, the risk premiums show a decreasing term structure for 1–50 year horizons. The exception is Retail Trade which is found to be a hedging asset in the short run but not the long run. In contrast, the risk premiums on the stripped consumption cash flow are found to be positive and rather low in the short term but increase with horizon to about 4 percent per year 50 years out. The observed term structures of equity risk premiums generally confirm the limited information (bounded rationality) model of Croce *et al* (2015).

6 Appendix: Data

6.1 SDF Extraction

In the extraction step, all data are annual for the years 1930 through 2015. Raw data are converted from nominal to real using the annual consumer price index obtained from Table 2.3.4 at the Bureau of Economic Analysis web site. Conversions to per capita are by means of the mid-year population data from Table 7.1 at the Bureau of Economic Analysis web site.

The raw data for stock returns are value weighted returns including dividends for NYSE, AMEX, and NASDAQ from the Center for Research in Security Prices data at the Wharton Research Data Services web site (http://wrds.wharton.upenn.edu). Likewise, the raw data for returns on U.S. Treasury 30 day debt are from the Center for Research in Security Prices data at the Wharton Research Data Services web site.

Raw annual returns including dividends on the twenty-five Fama-French (1993) portfolios were obtained from Kenneth French's web site, http://mba.tuck.dartmouth.edu/pages /faculty/ken.french. The portfolios are the intersections of five portfolios formed on market equity and five portfolios formed on the ratio of book equity to market equity. The portfolios are for all NYSE, AMEX, and NASDAQ stocks for which equity data are not missing and book equity data are positive. The portfolios are constructed at the end of each June with breakpoints determined by the NYSE quintiles at the end of June. Complete details are at Kenneth French's web site. The advantage of the Fama-French portfolios here is that they appear to isolate and exhaust the risk factors for holding equities (Fama and French, 1992, 1993).

Raw labor income data is "compensation of employees received" from Table 2.2 at the Bureau of Economic Analysis web site.

Real Treasury Inflation Protected Securities (TIPS yields displayed in Table 1 are annual averages of daily values from `https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=realyieldAll`. According to Treasury, "These rates are commonly referred to as "Real Constant Maturity Treasury" rates, or R-CMTs. Real yields on Treasury Inflation Protected Securities (TIPS) at "constant maturity" are interpolated by the U.S. Treasury from Treasury's daily real yield curve. These real market yields are calculated from composites of secondary market quotations obtained by the Federal Reserve Bank of New York. This method provides a real yield for a 10 year maturity, for example, even if no outstanding security has exactly 10 years remaining to maturity."

6.2 Valuation

Raw data for the cash flow valuation step are annual 1959–2015, for 56 observations net after the provision for the initial lag.

The industrial cash flow data are annual corporate profits with inventory valuation adjustment and without capital consumption allowance for major sectors. Data were spliced together for consistency from Table B-6 of the 2017 *Economic Report of the President* and Table B-91 of the 2004 *Economic Report of the President*. GDP data are from NIPA Table 1.1.5; consumption of nondurable goods services data are from NIPA Table 2.3.5; nominal data are converted to real using the implicit GDP deflator (Price Index for Gross Domestic Product) from NIPA Table 1.1.4. All data series converted to per capita using total U.S. resident population plus armed forces overseas (annual average of monthly estimates) obtained from FRED `https://fred.stlouisfed.org/series/B230RC0A052NBEA`.

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Tables and Figures

Table 1. TIPS (real) Yields

Year	5-year	7-year	10-year	20-year
2004	1.02	1.39	1.76	2.13
2005	1.50	1.63	1.81	1.97
2006	2.28	2.30	2.31	2.31
2007	2.15	2.25	2.29	2.36
2008	1.30	1.63	1.77	2.18
2009	1.06	1.32	1.66	2.21
2010	0.26	0.68	1.15	1.73
2011	-0.41	0.10	0.55	1.20
2012	-1.20	-0.88	-0.48	0.21
2013	-0.76	-0.30	0.07	0.75
2014	-0.09	0.32	0.44	0.86
2015	0.15	0.36	0.45	0.78

Displayed are annual average of daily fixed term Treasury Inflation Protected Securities (TIPS) yields. See Section 6 for details.

Table 2. Summary Statistics

	Industry	$gdp_t - cf_t$		$\Delta cf_{t-1,t}$	
		Mean	Std Dev	Mean	Std Dev
1	Total Corporate	2.572	0.227	0.019	0.131
2	Federal Reserve Banks	5.834	0.343	0.045	0.131
3	Other Financial	4.366	0.468	0.028	0.360
4	Total NonFinancial	2.828	0.272	0.016	0.146
5	Manufacturing	3.626	0.473	0.004	0.239
6	Wholesale Trade	5.194	0.260	0.027	0.137
7	Retail Trade	5.141	0.332	0.028	0.252
8	Consumption (NDS)	0.597	0.063	0.022	0.013
		$sdf_{t-1,t}$		$gdp_t - gdp_{t-1}$	
	log-MRS and GDP growth	-1.2231	1.8595	0.0196	0.0202

Displayed are sample means and standard deviations. See display (9) of the text for notation and the Appendix sources.

Table 3. Averaged Risk Premiums on Stripped Cash Flows

	Industry	Exposure		Average Risk Premium, Horizons in Years					
		Cov	Corr	1	1–10	11–20	21–30	31–40	41–50
1	Total Corporate	-0.083	-0.37	0.083	0.104	0.079	0.059	0.049	0.042
2	Fed Reserve Banks	-0.042	-0.21	0.042	0.107	0.099	0.073	0.056	0.045
3	Other Financial	-0.138	-0.23	0.138	0.174	0.115	0.077	0.057	0.045
4	Total NonFinancial	-0.087	-0.36	0.087	0.102	0.086	0.071	0.060	0.054
5	Manufacturing	-0.161	-0.41	0.161	0.158	0.129	0.108	0.094	0.085
6	Wholesale Trade	-0.082	-0.34	0.082	0.104	0.088	0.070	0.056	0.046
7	Retail Trade	0.044	0.11	-0.044	-0.070	-0.053	-0.035	-0.023	-0.014
8	Consumption	-0.010	-0.35	0.010	0.022	0.033	0.037	0.039	0.040

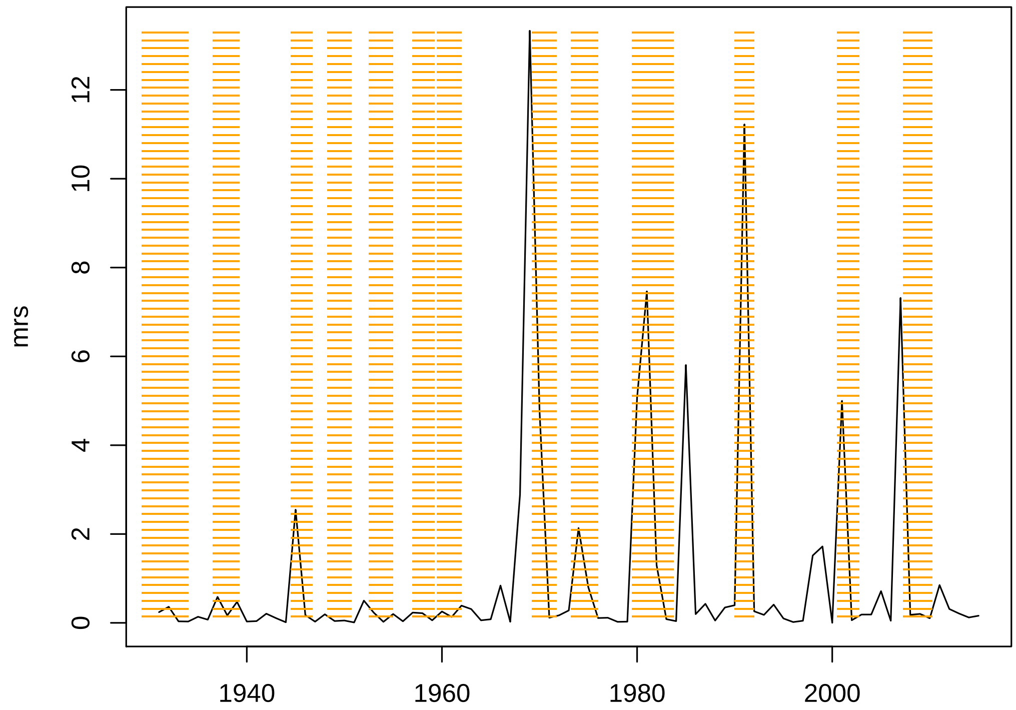

Figure 1. Posterior Valuation Plotted is the posterior mode of $\theta_1 = \mathrm{SDF}_{1930,1931}$ through $\theta_{86} = \mathrm{SDF}_{2014,2015}$.

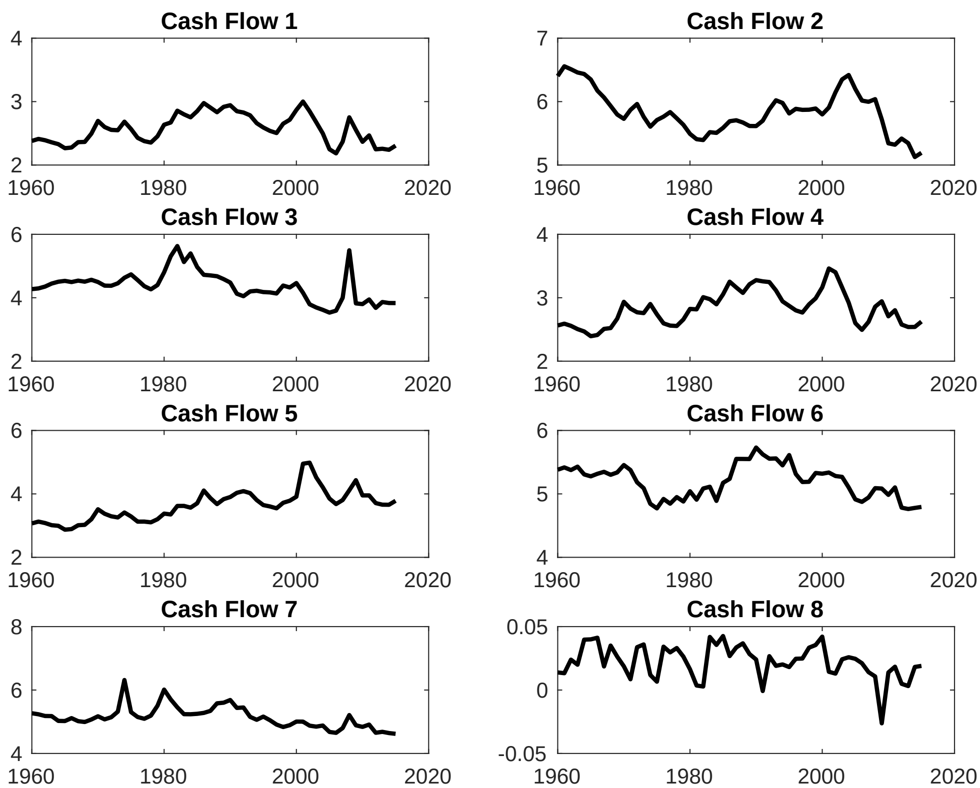

Figure 2. Plotted are $gdp_t - cf_t$ for the seven industrial cash flows 1–7 and $\Delta cf_{t-1,t}$ for cash flow 8 (consumption), where the variables are defined via display (9) of the text.

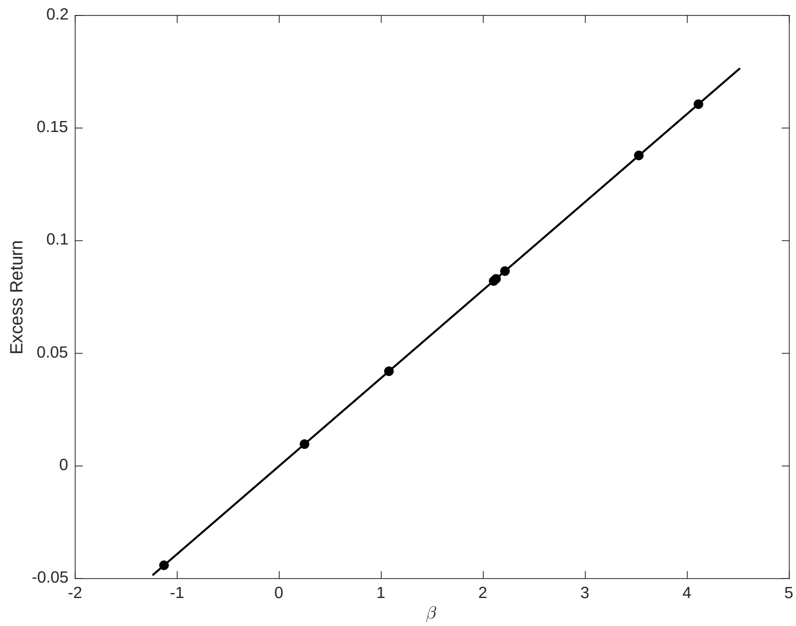

Figure 3. Plotted are the average 1-year risk premiums versus the cash flow beta exposures to *sdf* risk. Points for cash flows 6 and 8 nearly over plot creating the illusion of seven points, while there are actually eight.