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Pricing of the time-change risks

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ABSTRACT

We develop an equilibrium endowment economy with Epstein–Zin recursive utility and a Lévy time-change subordinator, which represents a clock that connects business and calendar time. Our setup provides a tractable equilibrium framework for pricing non-Gaussian jump-like risks induced by the time-change, with closed-form solutions for asset prices. Persistence of the time-change shocks leads to predictability of consumption and dividends and time-variation in asset prices and risk premia in calendar time. In numerical calibrations, we show that the risk compensation for Lévy risks accounts for about one-third of the overall equity premium.

1. Introduction

It has long been known that financial prices display special characteristics, such as stochastic volatility, time-varying risk premium, skewness and excess kurtosis. In this paper, we examine the risk and return properties attributable to these features from a structural perspective. In particular, we explore a time-variation in a one-dimensional measure of current economic conditions, akin an NBER business cycle indicator or Chicago Fed National Activity Index. Similar to Stock (1988), we interpret this state variable as a clock which measures the pace of economic activity. We show that time deformation and Lévy shocks in the time change give rise to non-Gaussian jump-like risks and time-variation in the asset prices and risk premium in calendar time.

We consider a discrete-time, real endowment economy similar to the long-run risks specification of Bansal and Yaron (2004). The preferences of the representative agent are characterized by a recursive utility of Kreps and Porteus (1978) and Weil (1989), in a parametrization of Epstein and Zin (1989). These preferences allow for a separation between risk aversion and intertemporal elasticity of substitution of investors, which goes a long way to explain key features of the asset markets; see Bansal (2007) for a review. We take a rational expectations equilibrium modeling approach and specify the exogenous dynamics of endowment and dividends in the economy. Specifically, we assume that the log consumption and dividend on any asset evolve on the two time scales. In business time, they are i.i.d. Gaussian. The calendar time is connected to a business time scale through a time-change state variable. We model the time-change variable as a Lévy-based subordinator, that is, a non-decreasing and positive process driven by Lévy shocks.

We first consider a random walk specification for the time change. We show that consumption and dividend growth rates and markets returns are i.i.d. in calendar time, but their distribution is non-Gaussian due to the Lévy activity shocks

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in the time deformation. However, as the economy is i.i.d., in equilibrium only immediate consumption risks are priced, and the time-change shocks do not receive a separate risk compensation. Due to a non-Gaussian nature of the economy in calendar time, the equity risk premium reflects the compensations for higher order moments of consumption and dividend dynamics. We decompose the total risk compensation in the economy into its Gaussian part and the non-Gaussian Lévy component. In particular, we express the non-Gaussian component as the sum of the Lévy jump-risk compensations weighted by the expected number of consumption and dividend jumps. The magnitude of the Lévy jump compensation increases exponentially in the left tail, so investors require significant risk compensation for an exposure to large negative consumption jumps.

We then consider a setup when the economic activity variable is a persistent process driven by Lévy shocks. We show that the distribution of consumption and dividend growth rates is conditionally infinitely divisible, and the time-change shocks receive separate risk compensation. The mean and volatility of growth rates as well as the risk premia in calendar time are time-varying and driven by the activity state variable.

The key focus of our paper is the Lévy risk premium. In the calibrations we find that the Lévy risk premium component due to the time-change shocks account for 40% of the total risk compensation on the consumption asset, and about one-third of the risk premium on the dividend asset. The relative importance of the non-Gaussian risks is consistent with other studies; for example, using alternative approaches, Shaliastovich (2010), Broadie et al. (2007) and Pan (2002) also estimate the risk premium due to non-Gaussian jump-risk to be about one-third of the total equity risk premium. Nevertheless, we find that we require relatively high risk aversion (around 40) to match the level of the equity premium. In the model, activity shocks follow tempered stable distribution; the resulting consumption and activity jumps are relatively small and do not receive substantial risk compensation for moderate levels of a risk-aversion coefficient. A fruitful extension of the model is to consider different distributional assumptions on the activity shocks which would assign more weight to the tails of the consumption density. This approach is consistent with structural asset-pricing models developed in the recent literature which entertain large negative moves and non-Gaussian shocks in the economic inputs, such as Eraker and Shaliastovich (2008), Drechsler and Yaron (2011), Bekaert and Engstrom (2010), Bates (2008), Barro (2006), Gabaix (2007), Benzoni et al. (2010), Liu et al. (2005), or beliefs of the agents (Bansal and Shaliastovich, 2010).

Our equilibrium approach based on recursive Epstein–Zin preferences is highly compatible with Lévy-based representation of infinitely divisible probability distributions. Indeed, Lévy-based characteristic function is log affine, so using standard log-linearization of returns we obtain a tractable affine asset-pricing model. This enables us to provide solutions to the asset prices and asset risk premia up to integral operations in general, and closed form in specific cases when time-change shocks follow tempered stable or gamma distributions. These specifications are economically appealing as they do not lead to the break-down of choice theory under fat-tail probability distributions highlighted in Geweke (2001) and Weitzman (2007). In our work, all moments of financial prices exist under a wide range of model parameters.

This paper is related to Martin (2010) and Eraker and Shaliastovich (2008) who analyze the implications of consumption-based asset-pricing models based on Epstein–Zin utility and non-Gaussian, jump-like fundamental shocks. While in these works jumps are directly modeled into the consumption dynamics, in our paper, we start with Gaussian consumption shocks in business time and introduce non-Gaussianity through Lévy time-change shocks. Hansen and Scheinkman (2009) consider a general valuation framework for non-linear continuous-time Markov environments, and use it to characterize the risk–return relationship in the long run. Bidarkota and McCulloch (2003) use stable distribution for consumption errors and derive and analyze the exact solutions for the equilibrium asset prices and risk premia. Bidarkota and Dupoyet (2007) entertain the thick tails in the consumption growth rate process, modeled as a dampened power law, which they show can have considerable impact on the equilibrium returns, while Bidarkota et al. (2009) study power-utility models with incomplete information and α-stable shocks, and explain time-variation in return volatility through non-Gaussian filtering. Unlike these papers, we emphasize the time-change state variable as an economic source for non-Gaussian risks and predictability in the economy. The idea of time deformation is quite popular in the reduced-form finance literature, where the operational time of stock market has been linked to measures of information arrival, such as realized variation in Andersen et al. (2010), order flow in Ané and Geman (2000) and Geman et al. (2000) and cumulative volume in Clark (1973). Lévy-models for economic activity are also entertained in Carr and Wu (2004), Barndorff-Nielsen and Shephard (2006) and Barndorff-Nielsen and Shephard (2001); see also Geman (2008) for a review.

The rest of the paper is organized as follows. In Section 2, we setup preference structure and real economy with a driving time-change variable. In Section 3, we review technical aspects of infinitely divisible distributions. In Section 4, we explore pricing implications for the specifications with i.i.d. and persistent time-change shocks. In Section 5, we use calibrations and provide a numerical analysis of the equity risk premium and compensations for different sources of risks. Conclusion follows.

2. Model setup

2.1. Preferences

We consider a discrete-time real endowment economy. The investor's preferences over the consumption stream C_t in calendar time t can be described by the recursive utility function of Epstein and Zin (1989) and Weil (1989):

$$U_t = \{(1-\delta)C_t^{(1-\gamma)/\theta} + \delta(E_t[U_{t+1}^{1-\gamma}])^{1/\theta}\}^{\theta/(1-\gamma)}, \tag{1}$$

where $\gamma>0$ is a measure of a local risk aversion of the agent, $\psi>0$ is the intertemporal elasticity of substitution and $\delta\in(0,1)$ is the subjective discount factor. For notational convenience, we define

$$\theta=\frac{1-\gamma}{1-\frac{1}{\psi}}. \tag{2}$$

When $\gamma=1/\psi$ (equivalently, $\theta=1$) we obtain standard power-utility specification.

Notably, we specify the preferences of the agent over the calendar time consumption, e.g. consumption over a calendar month or a year. In the paper we specify an evolution of the endowment dynamics in business time, which is different from calendar time and connected to it through a time-change state variable as described in the next section. In principle, one could define the preferences of the agent over the business time consumption; however, most of the paychecks come in on a regular basis (monthly) and many of the economic decisions seem to be made in calendar time, hence, we think it is more natural to specify the preferences in calendar time.

As shown in Epstein and Zin (1989), the logarithm of the intertemporal marginal rate of substitution for these preferences is given by

$$m_{t+1}=\theta\log\delta-\frac{\theta}{\psi}\Delta c_{t+1}+(\theta-1)r_{c,t+1}, \tag{3}$$

where $\Delta c_{t+1}=\log(C_{t+1}/C_t)$ is the log growth rate of aggregate consumption and $r_{c,t+1}=\log R_{c,t+1}$ is the log return on the wealth portfolio, that is, the asset which delivers aggregate consumption as its dividends each time period. The consumption return is not observable in the data. Following the literature, we assume an exogenous process for the consumption growth and use a standard asset-pricing restriction

$$E_t[\exp(m_{t+1}+r_{t+1})]=1, \tag{4}$$

which holds for any continuous return $r_{t+1}=\log(R_{t+1})$, including the one on the wealth portfolio, to solve for an unobserved wealth-to-consumption ratio in the model. This enables us to express the discount factor in terms of the underlying state variables and shocks in the economy. We can then use the solution to the discount factor and the Euler equation (4) to calculate prices of any assets in the economy, such as a risk-free asset and an equity paying a dividend stream D_t. The logarithm of the real risk-free rate $r_{ft}=\log R_{ft}$ can be determined from

$$r_{ft}=-\log E_t e^{m_{t+1}}. \tag{5}$$

To obtain analytical solutions the consumption and dividend asset prices, we apply the Campbell and Shiller (1988) approximation methods to log-linearize the returns:

$$r_{t+1}=\kappa_0+\kappa_1 v_{t+1}-v_t+\Delta d_{t+1}, \tag{6}$$

where v_t is the log price–dividend ratio, $\Delta d_t=\log(D_{t+1}/D_t)$ is the log dividend growth rate, and κ_0 and $0<\kappa_1<1$ are the approximating coefficients.

2.2. Real economy

In this paper we explore a representation of the economy driven by a one-dimensional state variable, which summarizes an intensity of the business activity in the economy. The concept of a univariate state of the economy capturing the slowing down and heating up of the economic activity during the recessions and expansions is quite intuitive and economically appealing, and is exemplified by the NBER business cycle indicator, the index of leading indicators, the consumer confidence index, Chicago Fed National Activity Index and their domestic and international counterparts. Following Stock (1988), we interpret this state variable as a clock which measures the pace of economic activity. The idea behind the stochastic clock is that while macroeconomic data are observed at regular calendar intervals, such as months or years, the real economic activity can take place at its own, potentially different and time-varying pace. This gives rise to the two time scales for the real economic activity, namely, the calendar time where it is observed, and the economic time when it takes place. The connection between the two time scales is achieved by a stochastic clock, a univariate state variable which matches the calendar time to the economic time.

Specifically, we define a stochastic clock variable S_t to be a non-negative and increasing (a.s.) process, driven by a stationary component A_{t+1}:

$$S_{t+1}=S_t+A_{t+1}. \tag{7}$$

The stochastic component A_t captures a change in the pace of economic activity and represents a systematic source of time-change risk, which affects the dynamics of the economy in the observed calendar time.

Denote c_t and d_t the log levels of the consumption and dividend processes. In our time-change specification, the consumption and the dividend evolve on the two time scales, a fictional business time τ and the actual calendar time t, connected by a stochastic clock S_t. In particular, in business time τ, consumption and dividends follow a random walk

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with a drift:

$$\begin{bmatrix} c_\tau^* \\ d_\tau^* \end{bmatrix} = \mu\tau + \Sigma^{1/2} W(\tau), \tag{8}$$

where star superscripts denote the log levels of consumption and dividends in business time and bivariate Brownian motion shock $W(\tau) = [W_c(\tau)\ \ W_d(\tau)]'$ is independent from the activity shocks A_t. Parameter μ denotes the drift of the processes:

$$\mu = [\mu_c\ \ \mu_d]', \tag{9}$$

and the variance-covariance matrix is given by Σ :

$$\Sigma = \begin{bmatrix} \sigma_c^2 & \sigma_{cd} \\ \sigma_{cd} & \sigma_d^2 \end{bmatrix}. \tag{10}$$

We denote $\Sigma^{1/2}$ its lower triangular Cholesky decomposition, and let $\tau_{cd} = \sigma_{cd}/(\sigma_c\sigma_d)$ stand for the correlation between the consumption and the dividend growth in business time.

The calendar time t is connected to business time τ through the stochastic clock $\tau = S_t$. For instance, the observed (log) consumption level in period t=1, 2,… is equal to the consumption level in business time S_t:

$$c_t = c_{S_t}^*. \tag{11}$$

Hence, we can write down the dynamics of consumption and dividends in actual time in the following way:

$$\begin{bmatrix} c_t \\ d_t \end{bmatrix} = \mu S_t + \Sigma^{1/2} W(S_t), \tag{12}$$

so that their rates are given by

$$g_{t+1} \equiv \begin{bmatrix} c_{t+1} - c_t \\ d_{t+1} - d_t \end{bmatrix} = \mu \Delta S_{t+1} + \Sigma^{1/2}(W[S_{t+1}] - W[S_t]) = \mu A_{t+1} + \Sigma^{1/2}(W[A_{t+1} + S_t] - W[S_t]). \tag{13}$$

The amount of variation and predictability of the activity state variable has important implications for the consumption and the dividend dynamics in calendar time. Indeed, when A_t is a constant equal to one, we obtain that the calendar and the business time scales completely coincide, so that the growth rates on both scales are i.i.d. Gaussian:

$$g_{t+1} = \mu + \Sigma^{1/2}(W(t+1) - W(t)) \sim N(\mu, \Sigma). \tag{14}$$

On the other hand, when business activity A_t is time-varying, the pace of the economy in calendar time can run faster or slower than that in business time, so the conditional distributions of the consumption and the dividend streams in calendar and business times are different. For example, when the activity shocks A_{t+1} are i.i.d., the consumption and dividend growth rates g_{t+1} are i.i.d. as well, though, they no longer follow a Gaussian distribution but a mixture of Gaussian, induced by a random component in A_{t+1}. Hence, due to the random activity shocks, the observed distribution of consumption is heavy-tailed, even though the underlying dynamics of the economy in business time is Gaussian. Further, the predictability of the activity component leads to the time-variation of the conditional mean and variance of the consumption and dividend streams, so that in calendar time the consumption and dividends are no longer i.i.d. Indeed, the first two conditional moments of the two streams satisfy

$$\begin{aligned} E_t g_{t+1} &= \mu E_t A_{t+1}, \\ Var_t g_{t+1} &= \Sigma E_t A_{t+1} + \mu\mu' Var_t A_{t+1}. \end{aligned} \tag{15}$$

Then, the time-variation in expected activity $E_t A_{t+1}$ leads to the time-variation of the conditional means and variances of the consumption and dividend growth rates in calendar times. The persistence in the expected growth and variance of consumption is an important feature of the data, as shown in the long-run risks literature (Bansal and Yaron, 2004).

It is worthwhile to note that our stochastic clock model specification imposes a restriction on the joint dynamics of the conditional mean and volatility of consumption and dividends in calendar time. Indeed, as the expected activity enters both of the conditional mean and variance of consumption and dividends in (15) with positive loadings, it implies that a rise in expected future activity increases the expectation and the volatility of the two streams in calendar time. In particular, when the conditional variance of the activity shocks $Var_t A_{t+1}$ is constant, the two conditional moments become perfectly positively correlated. To break this one-to-one co-movement of the expected growth and the variance of the two streams, one approach is to introduce a negative correlation between the conditional mean and variance of the activity shocks, so that at times of high expected activity the volatility of activity shocks goes down, which would decrease the conditional volatility of consumption and dividends in calendar times. Another approach is to consider several time-change variables, along the lines of Huang and Wu (2004), which apply separately to the deterministic drift and the innovation portions of the consumption and dividend specifications in business time, though, this might be less straightforward in a general equilibrium context. While breaking a perfect correlation between the conditional mean and variance of consumption would undoubtedly improve the statistical flexibility of the model, for simplicity, in this paper we focus on a specification with one homoscedastic stochastic clock factor, and leave the extensions for a future research.

To complete the model, we need to write down a convenient specification for the activity A_t, which then would allows us to solve for the equilibrium asset prices in the manner outlined in the previous section. We can obtain very tractable models when the activity shocks follow conditional infinitely divisible distributions. The next section presents the key technical ideas used to solve the model.

3. Infinitely divisible distributions

A convenient specification for the time-change shock is given by an infinitely divisible distribution. We provide the key details below; for a comprehensive overview refer to Cont and Tankov (2004), among others.

A univariate infinitely divisible random variable is uniquely specified by its characteristic triplet (b,σ,ν), where σ is the diffusion of the Gaussian part of its distribution, b is drift, and $\nu(dx)$ is a positive measure on $\mathcal{R}$, called Lévy measure, which satisfies $\nu(\{0\})=0$ and $\int_{\mathcal{R}}(x^2\wedge 1)\nu(dx)<\infty$.[1] Intuitively, the infinitely divisible distribution extends the Gaussian one by allowing "jumps". The interpretation of ν in this case is that for any set A in $\mathcal{R}$, $\nu(A)$ specifies the expected number of jumps falling in A per unit of time.

For every infinitely divisible distribution there exists a continuous-time random walk $L(t)$, called Lévy process, such that its increment $\Delta L(t+1)=L(t+1)-L(t)$ possesses this distribution. The reverse is also true: for every Lévy process $L(t)$ its discrete-time increments are infinitely divisible. This allows us to associate with infinitely divisible discrete-time random variables the increments to the continuous-time Lévy processes. In this paper we specialize on infinitely divisible distributions associated with non-decreasing and positive processes $L(t)$ called subordinators. It can be shown that such $L(t)$ has no Brownian motion component, so that $\sigma=0$, and its drift and Lévy measure $\nu(dx)$ are restricted to positive support.

A convenient specification of the subordinator is given by its moment-generating function $\varphi(u)$:

$$Ee^{u\Delta L(t)}=e^{\varphi(u)}. \tag{16}$$

As the variance of the Brownian motion component of the subordinator is zero, ignoring the deterministic drift term, its moment-generating function can be written in the following way:

$$\varphi(u)=\int_0^\infty (e^{ux}-1)\nu(dx). \tag{17}$$

This is well-defined for all $u<0$; for the parametric examples we consider in the paper, the integral can also be extended to positive u below a certain upper bound.

An example of the infinitely divisible distributions includes tempered stable distributions. As shown in Cont and Tankov (2004), the Lévy density for a tempered stable distribution is given by

$$\nu(x)=c\frac{e^{-\pi x}}{x^{\alpha+1}}1_{x>0} \tag{18}$$

for $c>0, 0<\alpha<1$ and $\pi>0$. An intuitive interpretation of c is that of a scale controlling the overall intensity of small and big jumps. The parameter α governs the local behavior of the process: when it is closer to 0 the process moves by big jumps with periods of tranquility between them, while α near 1 implies numerous small oscillations between rare big jumps. The coefficient π represents a tempering parameter dampening the large jumps of the process $L(t)$. It plays a critical role to control the tails of the distribution and ensure the existence of the moments of the distribution. Indeed, the moment-generating function for the tempered stable class can be computed in the closed form as

$$\varphi(u)=c\Gamma(-\alpha)([\pi-u]^\alpha-\pi^\alpha). \tag{19}$$

Notably, it is defined for all $u<\pi$. Hence, the higher the tempering parameter, the less heavy are the tails of the distribution, which guarantees the existence of the moment-generating function for positive and not too large u.

A convenient candidate for the driving shocks to the activity state A_{t+1} are discrete-time increments to the Lévy subordinator, such as a tempered stable or its limiting case of $\alpha=0$, gamma distribution. Such a choice guarantees the positivity of the activity level, and therefore, the positivity and the non-decreasingness of the state S_t. This approach is similar to Barndorff-Nielsen and Shephard (2001), who use distributions with positive support to model the shocks to the volatility processes in the economy.

4. Pricing implications of the time change

To study the effect of the time-change shocks, we first consider a case when the conditional mean of the activity process is constant. We consider an extension of the model which incorporates predictable drift component in Section 4.2.

[1] Multivariate extensions of the infinitely divisible distributions can be incorporated by replacing scalar parameters with their appropriate vector and matrix counterparts.

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4.1. I.I.D. activity

Let us first start with the case when time-change shocks are i.i.d. That is, we write down

$$A_t = m + \xi_t \tag{20}$$

for a constant m and infinitely divisible shocks ξ_t. The time-change shock ξ_{t+1} is a subordinator, that is, we set the drift and the variance of its Brownian component to zero, and restrict the Lévy density v to positive support.

When the consumption and dividends are Gaussian in business time and time-change shocks ξ_{t+1} are infinitely divisible, one can show that the distribution of consumption and dividends in calendar time is infinitely divisible as well. The moment-generating function of the growth rates can be written in the following way:

$$\log E_t e^{u' g_{t+1}} = m\mu' u + \frac{1}{2} u' m \Sigma u + \int_{R^2} (e^{u'x} - 1) v_{cd}(x_c, x_d)\, dx_c\, dx_d. \tag{21}$$

In particular, the drift is $m\mu$, the variance of the diffusion component is $m\Sigma$ and the bivariate Lévy density $v_{cd}(x)$ of consumption and dividend growth rates is given by

$$v_{cd}(x_c, x_d) = \int_{R_+} f([x_c\ x_d]'; \mu s, \Sigma s) v(ds), \tag{22}$$

where $f(x; A, B)$ denotes the multivariate Gaussian pdf with mean A and variance B:

$$f(x; A, B) = \frac{1}{(2\pi)|B|^{1/2}} e^{-1/2(x-A)'B^{-1}(x-A)}. \tag{23}$$

Notably, the consumption and dividend growth rates are i.i.d. in calendar time. However, unlike their dynamics in business time, they are no longer Gaussian. Indeed, the first two terms in (21) represent the drift and the variance of the Gaussian component of the two series. The last term captures the non-Gaussian, Lévy component in their dynamics, and is the main focus of this paper. It is worthwhile to note that this non-Gaussian component is not directly built in into the growth rates in calendar time, but arises due to the time deformation of the observations of Gaussian consumption and dividends through a stochastic clock. Indeed, in the absence of time-change risks, consumption and dividends would be Gaussian in calendar time as well, as we showed in Section 2.2.

We can use the solution to the endowment dynamics in calendar times to solve for the equilibrium asset prices in the economy. As the consumption growth is i.i.d., there is no predictability in the economy, so that the risk-free rate and price-dividend ratios for any asset are constant. In equilibrium, the discount factor is given by

$$m_{t+1} = \theta \log \delta + (\theta - 1) b_{c,0} - \gamma \Delta c_{t+1} \tag{24}$$

for a constant $b_{c,0}$ defined in the A.1. Hence, as in the standard power-utility case, investors are concerned only about shocks in calendar consumption growth, and their price of risks is equal to the risk-aversion coefficient γ. Time change affects consumption process, as its distribution in calendar time is different from that in business time, the time-change shocks ξ_t do not receive a separate risk compensation.

We further show in A.1 that as price-dividend ratios are constant, the returns on the consumption asset $r_{c,t}$ and on a dividend-paying asset $r_{d,t}$ move one-to-one with their respective cash flows:

$$r_{c,t+1} = b_{c,0} + \Delta c_{t+1},$$

$$r_{d,t+1} = b_{d,0} + \Delta d_{t+1}. \tag{25}$$

The return processes inherit the properties of the economic fundamentals. In particular, returns are i.i.d. and follow infinitely divisible distribution.

The risk premia on the consumption and dividend assets reflect the compensation for the systematic consumption risk in the economy. In the Appendix we show that the required compensations for holding the consumption and dividend assets can be expressed in the following way:

$$\log(E_t R_{c,t+1}) - r_{ft} = \gamma m \sigma_c^2 + \int_R (e^{x_c} + e^{-\gamma x_c} - e^{(1-\gamma)x_c} - 1) v_c(x_c)\, dx_c,$$

$$\log(E_t R_{d,t+1}) - r_{ft} = \gamma m \sigma_{cd}^2 + \int_{R^2} (e^{x_d} + e^{-\gamma x_c} - e^{-\gamma x_c + x_d} - 1) v_{cd}(x_c, x_d)\, dx_c\, dx_d. \tag{26}$$

The first component in the two expressions represents a traditional compensation for the Gaussian risks in consumption and dividends, equal to the level of risk-aversion times the covariance between the Gaussian components in returns and the consumption growth in calendar time. On the other hand, the second piece reflects pricing of the Lévy risks in consumption and dividend streams, which arises due to a non-Gaussian time change of the endowment dynamics in calendar time. This Lévy compensation can be intuitively thought as the summation over all possible "jumps" x_c, x_d in consumption and dividends weighted by their risk compensation:

$$e^{x_c} + e^{-\gamma x_c} - e^{(1-\gamma)x_c} - 1$$

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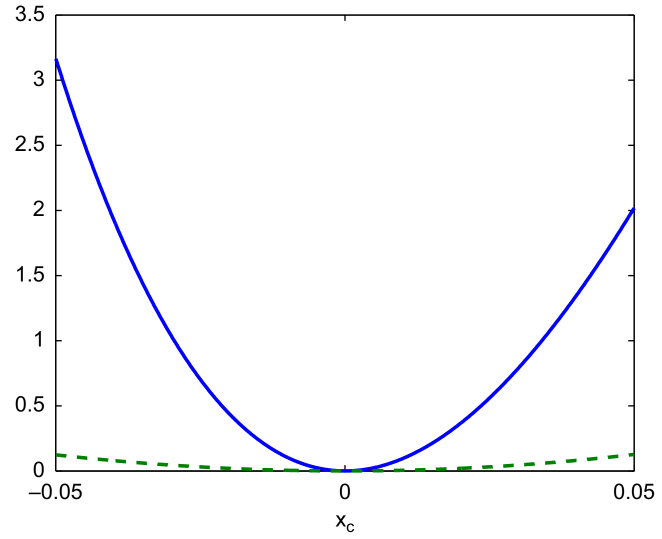

Fig. 1. Lévy risk compensation for consumption asset in the i.i.d. model specification.

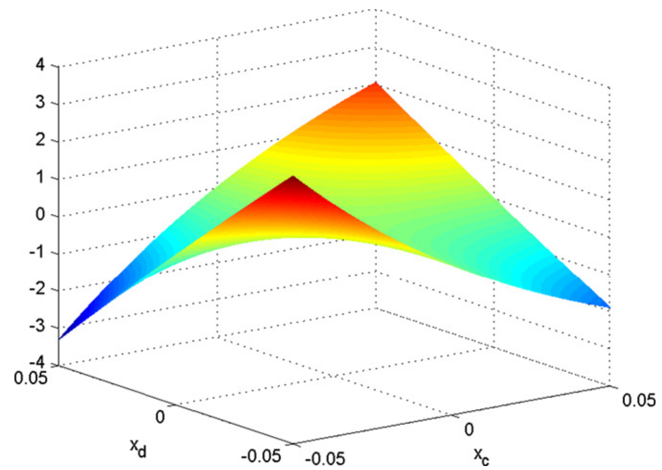

Fig. 2. Lévy risk compensation for dividend asset in the i.i.d. model specification.

for the consumption asset and

$$e^{x_d}+e^{-\gamma x_c}-e^{-\gamma x_c+x_d}-1$$

for the dividend asset. We plot this Lévy risk compensation for consumption asset in Fig. 1 for $\gamma=0.5$ and 10 and for dividend asset in Fig. 2 for $\gamma=10$. As can be seen from the first figure, when the risk aversion is bigger than 1, the agent is very averse to large negative drops in consumption growth, and the risk compensation increases exponentially for large, negative moves x_c.[2] The case of a dividend asset reveals a similar non-linear relation between the risk compensation for holding an asset and systematic non-Gaussian risks in dividend stream. In particular, investors require a substantial Lévy risk compensation for the assets which pay little when consumption growth falls substantially.

The asymmetry and non-linearity in risk premium compensation for positive and negative moves in consumption and dividends, absent in traditional Gaussian models, can be related to the compensations for higher order moments of the consumption and dividend dynamics. We use Taylor expansion to expand the integrand in the risk premia around $x=0$ and, using the properties of Lévy distributions, rewrite the risk premium in terms of the moments (cumulants) of the

[2] Substantial risk compensation for large negative consumption jumps is a central feature of disaster models, see, e.g. Martin (2010), Barro (2006), Rietz (1988).

underlying fundamentals:

$$\log(E_t R_{c,t+1}) - r_{ft} \approx \sum_{j=2}^{\infty} \frac{1+(-\gamma)^j-(1-\gamma)^j}{j!} k_j, \tag{27}$$

and similar for the dividend asset:

$$\log(E_t R_{d,t+1}) - r_{ft} = \gamma k_{11} + \frac{1}{2}(-3\gamma^2 k_{21} + 3\gamma k_{12}) + \frac{1}{12}(2\gamma^3 k_{31} - 3\gamma^2 k_{22} + \gamma k_{13}) + \cdots. \tag{28}$$

In the above expressions, k_j refers to the jth cumulant of consumption growth:

$$k_2 = Var(\Delta c),$$

$$k_3 = E(\Delta c - E(\Delta c))^3,$$

$$k_4 = E(\Delta c - E(\Delta c))^4 - 3V(\Delta c)^2, \tag{29}$$

and k_{ij} is the bivariate cumulant of the consumption and dividend growth rates:

$$k_{11} = E(\Delta c - E(\Delta c))(\Delta d - E(\Delta d)),$$

$$k_{21} = E(\Delta c - E(\Delta c))^2(\Delta d - E(\Delta d)),$$

$$k_{12} = E(\Delta c - E(\Delta c))(\Delta d - E(\Delta d))^2, \tag{30}$$

etc.

Similar to standard Gaussian consumption CAPM model, the risk premium on any asset loads on the covariance of the dividend growth with the consumption growth with a risk-aversion coefficient γ. However, unlike the standard model, skewness, excess kurtosis and higher cumulants and co-cumulants of consumption and dividend growth enter into the risk compensation equation as well. For example, a negative loading of $-3\gamma^2/2$ on k_{21} signifies that, controlling for all other moments and co-moments of consumption and dividend growth, investors dislike assets which tend to pay little in times when the consumption deviates most from its mean, which can also be seen in Fig. 2. These results on compensations for higher order moments are similar to Dittmar (2002) and Harvey and Siddique (2000), who apply the non-linear pricing kernel framework of Bansal and Viswanathan (1993). In our case the pricing kernel is linear, and it is the non-Gaussianity of the time change which leads to the deviation from standard consumption CAPM.

Notably, we can compute the risk compensations in (26) in a closed form when the activity shocks follow tempered stable distribution:

$$\log E_t R_{c,t+1} - r_{ft} = \gamma\sigma_c^2 m + c\Gamma(-\alpha)\left\{(\pi-\mu_c-\tfrac{1}{2}\sigma_c^2)^\alpha + (\pi+\gamma\mu_c-\tfrac{1}{2}\gamma^2\sigma_c^2)^\alpha - (\pi-(1-\gamma)\mu_c-\tfrac{1}{2}(1-\gamma)^2\sigma_c^2)^\alpha - \pi^\alpha\right\},$$

$$\log E_t R_{d,t+1} - r_{ft} = \gamma\sigma_{cd} m + c\Gamma(-\alpha)\left\{(\pi-\mu_d-\tfrac{1}{2}\sigma_d^2)^\alpha + \left(\pi+\gamma\mu_c-\tfrac{1}{2}\gamma^2\sigma_c^2\right)^\alpha - \left(\pi+\gamma\mu_c-\mu_d-\tfrac{1}{2}(\gamma^2\sigma_c^2+\sigma_d^2-2\gamma\sigma_{cd})\right)^\alpha - \pi^\alpha\right\}. \tag{31}$$

The expressions above are well-defined if tempering parameter π is high enough. For example, for consumption asset the risk premium exists if $\pi > \mu_c + \frac{1}{2}\sigma_c^2$, $\pi > (1-\gamma)\mu_c + \frac{1}{2}(1-\gamma)^2\sigma_c^2$, and $\pi > -\gamma\mu_c + \frac{1}{2}\gamma^2\sigma_c^2$, and similar for the dividend asset.

4.2. Predictable activity

Previously we assumed that the conditional mean of the activity process was constant, $E_t A_{t+1} \equiv m$, so that consumption and returns in calendar times were i.i.d. Now we make the conditional mean to be time-varying, and in particular, we model the activity as a non-negative AR(1) process driven by infinitely divisible innovations ξ_t:

$$A_{t+1} = m + \rho A_t + \xi_{t+1}. \tag{32}$$

The parameter $\rho \in (0,1)$ governs the persistence in A_t, and $m > 0$ determines a non-stochastic drift of the time change. Note that as the activity shocks ξ_t are positive, A_t is guaranteed to take only positive values as well; see Barndorff-Nielsen and Shephard (2003) for further discussion on autoregressive processes with non-Gaussian innovations.

With this modification, the consumption and dividend growth rates are no longer i.i.d. in calendar time: the conditional mean and volatility of the two streams are time-varying and persistent with A_t. The conditional distribution of these two series, however, is still infinitely divisible. The conditionally Gaussian part of their distribution possesses time-varying means and volatilities linear in the activity state A_t, while the pure Lévy shock is characterized by a time-invariant jump measure, which is similar to that in the previous section. The details for the moment-generating functions and related equations are provided in the Appendix.

The time-variation in the activity state variables drives the equilibrium asset prices in the economy. In particular, in Appendix we show that the equilibrium price–consumption ratio is linear in the activity variable A_t:

$$pc_t = H_{c,0} + H_{c,1} A_t. \tag{33}$$

The parameter $H_{c,1}$ measures the sensitivity of the price–consumption ratio to the fluctuations in the activity state, and is given by

$$H_{c,1} = \frac{\rho}{1-\kappa_1\rho}\left(1-\frac{1}{\psi}\right)\left(\mu_c+\frac{1}{2}(1-\gamma)\sigma_c^2\right). \tag{34}$$

The sign of the coefficient depends on the relative magnitudes of the model and preference parameters. For reasonable values of μ_c and σ_c, the expression in the last bracket is positive. Hence, the sign of $H_{c,1}$ depends on the level of intertemporal substitution of the agent, ψ. As in Bansal and Yaron (2004), we require that $\psi > 1$, so that the substitution effect dominates the wealth effect. In this case, $H_{c,1} > 0$, so that the equilibrium prices in the economy rise when the economic activity is high.

Given the solution to the equilibrium price–consumption ratio, we can solve for the equilibrium discount factor, which allows us to price any asset in the economy. We can write down the equilibrium discount factor in the following way:

$$m_{t+1} = m_0 + m_a A_t - \lambda_\xi \xi_{t+1} - \lambda_c \Delta c_{t+1}, \tag{35}$$

where the discount factor loadings m_0 and m_a and market prices of risks λ_ξ and λ_c depend on the model and preference parameters. As in the standard power-utility models, the market price of consumption risk λ_c is equal to the risk-aversion coefficient γ. The novel feature of the model is the pricing of the time-change innovations ξ_{t+1}. Unlike the previous case when the activity state variable is i.i.d., in the presence of persistent time-change shocks investors with recursive utility are concerned with the innovations in the activity variable A_t, and time-change shocks receive a non-zero risk compensation. When agents have preference for early resolution of uncertainty, that is, $\gamma > 1/\psi$, for reasonable parameter values the market price of the time-change risks is positive, $\lambda_\xi > 0$. That is, the agents dislike fluctuations in the activity in the economy, and hence demand risk compensation for the exposure for this source of risk. The intuition for this result is very similar to that in the long-run risks literature, which shows that when investors have preference for the timing of resolution of uncertainty, they dislike fluctuations in the expected growth and require positive risk compensation for these types of risks.

Using the equilibrium discount factor, we can solve for the equilibrium risk-free rate rf_t and price–dividend ratio pd_t. Their solutions are linear in the activity state A_t,

$$pd_t = H_{d,0} + H_{d,1} A_t,$$

$$r_{ft} = F_0 + F_a A_t, \tag{36}$$

and the equilibrium loadings are provided in the Appendix.

We can combine these solutions to the model returns to derive the expressions for the risk premia for holding the consumption and the dividend asset. The risk premia on these assets satisfy

$$\log(E_t R_{c,t+1}) - r_{ft} = \gamma\sigma_c^2(m+\rho A_t) + \int_{R^2} K_c(x_c,x_l)\nu_{cl}(dx_c,dx_l),$$

$$\log(E_t R_{d,t+1}) - r_{ft} = \gamma\sigma_{cd}(m+\rho A_t) + \int_{R^3} K_d(x_c,x_d,x_l)\nu_{cdl}(dx_c,dx_d,dx_l), \tag{37}$$

where the solutions for the Lévy measures of consumption and activity ν_{cl}, consumption, dividend and activity shocks ν_{cdl} as well as the risk compensation kernels K_c and K_d, are provided in the Appendix. The intuitive interpretation of the integrals in the Lévy component is that of a sum of per-jump Lévy risk compensation K_c and K_d, weighted by the expected number of jumps ν_{cl} and ν_{cdl}, respectively. Indeed, the Lévy densities ν measure the expected number of non-Gaussian moves (jumps) in consumption (x_c), dividend (x_d) and activity shocks (x_l). Notably, the integrals in the risk premia expressions can be computed in the closed form when activity shocks follow tempered stable distributions. The technical restriction which guarantees the existence of market prices is that the tempering parameter π is high enough.

As in the previous section, the first part in the risk premium is similar to the standard CCAPM-type risk compensation for the Gaussian components of the return and consumption dynamics. The second term computes the non-Gaussian risk premium component due to the time-change shocks. The risk premium on the assets is time-varying, and is driven by the fluctuations in the activity in the economy. In particular, the required compensations are the highest when the expected activity (expected consumption growth and its volatility) is high.

In the current setup, all the time-variation in the economy is generated by the activity rate A_t which is a sufficient statistics to predict the distribution of the economy in all future periods. We can extend our model of the time change to incorporate several activity factors, stochastic volatility or more complicated moving average specifications, which would enrich the set of the economic states and retain the analytic tractability of the model. We leave these extensions for future research.

5. Model output

5.1. Data and calibration

We use a numerical calibration to explore the implications of our model for the asset prices. While econometric methods are available to estimate the model, such as the ones described in Bidarkota and Dupoyet (2007) and Bidarkota and McCulloch (2003), we do not attempt them in this paper. The reason is that in the current specification, the model is too restrictive to be confronted with the complex dynamic features of the data. For example, as we discussed in Section 2.2, when the conditional variance of the activity shocks is constant, the conditional mean and variance of consumption growth rate are perfectly positively correlated. This is not likely to hold in the data: the empirical evidence suggests that this correlation is negative. Further, our choice of the tempered stable distribution for the activity shocks implies that the skewness of consumption growth in the model is mildly positive, which is also counterfactual. One can enrich the model specification by allowing for a separate stochastic volatility of the activity shocks and entertaining more realistic distributions for the activity shocks. We leave it for future research, and instead calibrate the model to match the key unconditional moments of the data, and analyze the implications for risk premium and asset prices.

We assume an annual decision interval, and use annual macroeconomic and financial data from 1930 to 2007 to calibrate the parameters of the model. In particular, we use annual real consumption series from the BEA tables of real expenditures on non-durable goods and services. The market returns and dividend growth rates, computed for a broad value-weighted portfolio, and the risk-free rate, corresponding to the short-term inflation-adjusted yields, are obtained from CRSP. Summary statistics for the consumption and dividend growth rates and the market return equity premium are presented in Table 1. The mean consumption and dividend growth rate is about 2%. The volatility of consumption is 2%, while that of the dividend growth is much larger and equal to 11%. The consumption and dividend growth rates are positively correlated with a correlation coefficient of 0.6. Finally, the average equity premium in our sample is 5%, and the mean risk-free rate is about 1%.

We calibrated values for the key model parameters reported in Table 2. Specifically, the persistence of the activity shocks is set to $\rho = 0.60$. This is consistent with the persistence of the risk-free rate in the data of 0.59; note that in the model, the risk-free rate is linear in the activity state, so that the persistence in the activity shocks is equal to that of the interest rates. Further, the activity shocks determine the persistence in the conditional drift and volatility of consumption and dividend growth rates; see Eq. (15). The monthly persistence in these conditional moments implied by our calibration is $\rho^{1/12} = 0.96$, which is quite close to the values entertained in the long-run risks literature.

Next, we assume that the activity shocks follow tempered stable distribution characterized by scale and intensity parameters c and α and tempering parameter π; the moment-generating function for this distribution is given in Eq. (19) in Section 3. As the activity shocks are not observed in the data, we calibrate these parameters to $\alpha = 0.1$, $\pi = 11$ and c=3 to target the key moments of the market return data. Notably, the choice of the tempering parameter π guarantees the existence of the asset prices and moments of returns.

The baseline calibration values for the preference parameters are reported in Table 2. Specifically, we let the subjective discount factor δ equal to 0.994. The baseline risk aversion parameter is set at 10, and the intertemporal elasticity of

Table 1
Summary statistics for real consumption and dividend growth rates, market risk premium and risk-free rate. Annual observations from 1930 to 2008. Standard errors are Newey–West corrected using four lags.

	Value	S. E.
$E(\Delta c)$	1.92	(0.29)
$E(\Delta d)$	1.12	(0.96)
$\sigma(\Delta c)$	2.12	(0.59)
$\sigma(\Delta d)$	10.97	(2.91)
$Corr(\Delta c, \Delta d)$	0.60	(0.14)
$E(r_d - r_f)$	5.22	(2.03)
$E(r_f)$	0.64	(0.69)

Table 2
Calibrated and implied model parameter values. Annual frequency.

Calibrated					
ρ	α	π	c	δ	γ
0.60	0.10	11	3	0.994	10
Implied					
m	σ_c	σ_d	τ_{cd}		
0.03	0.02	0.11	0.61		

substitution at 1.5. This configuration implies that the agent has a preference for early resolution of uncertainty, which has important implications for the equilibrium prices, as discussed in the previous section. Notably, high value of γ raises the overall level of the risk premium to help better match the financial data, so we also present the model results for higher levels of risk aversion of 20 and 50.

Given the calibrated activity parameters, we can solve for the implied values of the remaining parameters using the consumption and dividend dynamics. First, we restrict the activity dynamics by assuming that on average, the state of the economy moves one-to-one with the calendar time:

$$E(\Delta S_t) \equiv E(A_t) = 1. \tag{38}$$

We can use this restriction to solve for the drift parameter $m>0$ in the activity specification:

$$m = 1-\rho-E(\xi), \tag{39}$$

where the mean of activity shocks is equal to

$$E(\xi) = -c\Gamma(-\alpha)\alpha\pi^{\alpha-1}. \tag{40}$$

This implies that the unconditional mean and variance of the consumption and dividend shocks in calendar time are given by

$$Eg_{t+1} = \mu,$$

$$Varg_{t+1} = \Sigma + \frac{1+\rho^2}{1-\rho^2} Var(\xi)\mu\mu', \tag{41}$$

where the variance of the activity shocks is equal to

$$Var(\xi) = c\Gamma(-\alpha)\alpha(\alpha-1)\pi^{\alpha-2}. \tag{42}$$

Hence, we can use the observed mean and variance of the two growth rates in calendar time to solve for their mean μ and variance Σ in business time, given the persistence ρ and variance $Var(\xi)$ of the activity shocks. Their values are provided in Table 2.

5.2. Implications for risk premium

The model output for the risk premia and the interest rates is presented in Table 3. When the risk-aversion coefficient is 10, the risk premium on consumption asset is about 0.5%, while that on the dividend asset is about 1.4%. The risk compensations increase to 2.2% and 6.7%, respectively, when the risk-aversion coefficient increases to $\gamma = 50$. Hence, we require a quite high coefficient of the risk aversion to account for the magnitude of the risk premium in the data. The risk-free rate stays at about 1.5% for all the considered values, which broadly matches the data.

The key focus of our paper is on the Lévy part of the risk premium. Notably, the Lévy risk premium component due to the time-change shocks account for 40% of the total risk compensation on the consumption asset, and about one-third of the risk premium on the dividend asset. The relative importance of the non-Gaussian risks is consistent with other studies; for example, using alternative approaches, Shaliastovich (2010). Broadie et al. (2007) and Pan (2002) estimate the risk premium due to non-Gaussian jump-risk to be also about one-third of the total equity premium in the sample. Similarly, Bidarkota and Dupoyet (2007) show that incorporation of the fat tails into the consumption process can raise the equity premium by 80%, relative to standard models.

Table 3
Model-implied risk premium on consumption and dividend asset and its decomposition into Gaussian and Lévy components, and model-implied risk-free rate for a range of risk-aversion coefficients.

	$\gamma = 10$	$\gamma = 20$	$\gamma = 50$
Risk premium			
Consumption asset			
Gaussian	0.27	0.55	1.37
Lévy	0.19	0.36	0.81
Total	0.46	0.91	2.18
Dividend asset			
Gaussian	0.88	1.76	4.41
Lévy	0.51	0.96	2.25
Total	1.39	2.72	6.66
Risk-free rate:	1.53	1.34	1.60

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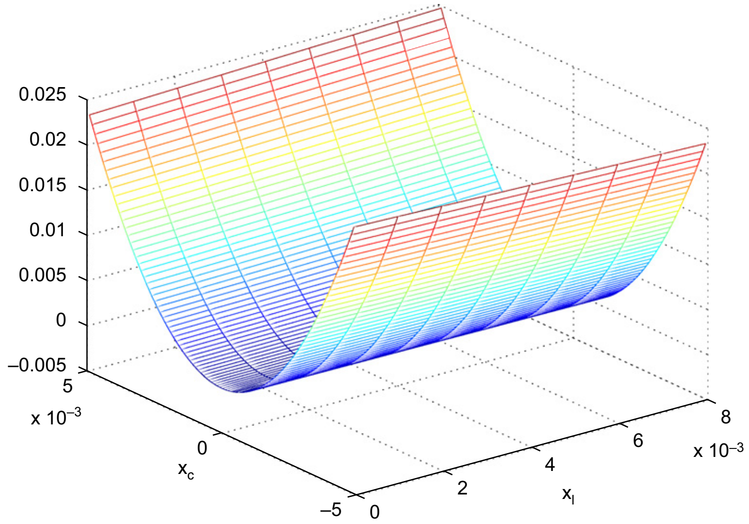

Fig. 3. Lévy risk compensation for consumption asset in a model specification with persistent activity shocks.

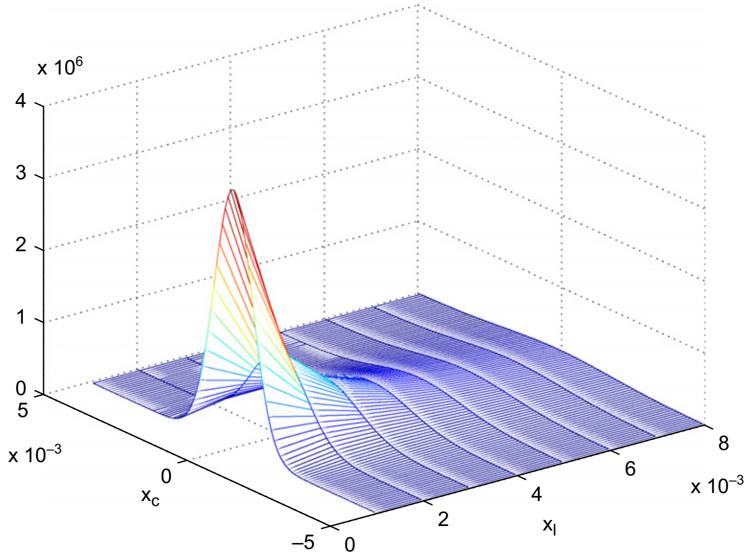

Fig. 4. Lévy joint density for consumption and activity shocks.

To get further insight on the Lévy risk compensation in the case with persistent activity shocks, we plot the risk compensation kernel for the consumption asset K_c in Fig. 3. As in the i.i.d. case, the agent demands risk premium for being exposed to non-Gaussian consumption jumps. In addition to that, when agents have preference for early resolution of uncertainty, the activity shocks receive a separate risk premium, so the risk compensation kernel also depends on the activity jumps. The total Lévy risk premium is equal to the sum of the compensations for consumption and activity jumps weighted by the expected number of jumps. The expected number of jumps, given by the joint Lévy density of consumption and activity shocks, is plotted in Fig. 4. Notably, in our specification, most of the distribution mass is concentrated on small consumption and activity shocks. Large jump shocks which demand a large jump premium do not receive much weight in the total sum. Hence, we require a large risk-aversion coefficient which steepens the kernel function in Fig. 3 to match the magnitude of the risk premium in the data.

An alternative way to match the equity premium and keep risk aversion at a low level is to entertain different distributional assumptions on the activity shocks and their impact on the consumption and dividend streams, which would assign more weight to the tails of the density in Fig. 4. Many of the structural asset-pricing models developed in the recent literature entertain large negative moves in the economic inputs such as Eraker and Shaliastovich (2008), Drechsler and Yaron (2011), Gabaix (2007), Bates (2008), Barro (2006), Benzoni et al. (2010), Liu et al. (2005), or beliefs of the agents (Bansal and Shaliastovich, 2010), which impact the left tail of the distributions. We leave the extension of our time-deformation model along these lines to further research.

6. Conclusion

We develop a discrete-time endowment economy featuring Epstein–Zin utility function and non-Gaussian risks driven endogenously by economic separation of time scales along the lines of Stock (1988). While consumption and dividends are i.i.d. Gaussian in business time, in calendar time their dynamics is non-Gaussian because of the Lévy time-change clock that connects business time to calendar time. This provides a convenient and tractable extension of standard equilibrium models for pricing non-Gaussian risks. We show that using log-linearization methods we can obtain solution for financial prices up to integral operations in general, or in closed form for the tempered stable distributions.

The deviations from Gaussianity imply that the agents require compensations for higher order moments and co-moments of consumption and dividend growth rates of the assets. Further, when activity shocks are persistent, this gives rise to the variation in the expected consumption growth and its conditional volatility, similar to the long-run risks model. These fluctuations lead to the time-variation in the risk premium and the volatilities of the returns, driven by the activity shocks.

In the calibration, we show that the Lévy risk premium accounts for about one-third of the overall premium in the economy. The model can match the risk-free rate, however, it still required relatively high risk aversion to match the level of the risk premium. One way to resolve that is to extend the model to assign more weight to the large non-Gaussian jumps in consumption and activity shocks.

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Appendix A. Model solution

A.1. I.I.D. case

Let us first solve the model in when the time change is i.i.d.

Conjecture that the price–consumption ratio is constant. In this case, we can express the log return on the consumption asset in the following way:

$$r_{c,t+1} = b_{c,0} + \Delta c_{t+1}, \tag{A.1}$$

where $b_{c,0}$ is related to the level of the price–consumption ratio.

Using the distributional properties of consumption growth in (21) and the Euler condition (4), we can solve for the level of the return process:

$$b_{c,0} = -\log\delta - \frac{m}{\theta}((1-\gamma)\mu_c + \frac{1}{2}(1-\gamma)^2\sigma_c^2) - \frac{1}{\theta}\int_R (e^{(1-\gamma)x_c} - 1)\nu_c(x_c)\,dx_c, \tag{A.2}$$

where we defined the univariate Lévy density of consumption growth

$$\nu_c(x_c) = \int_{R_+} f(x_c; \mu_c s, \sigma_c^2 s)\nu(ds).$$

The equilibrium risk-free rate satisfies

$$r_{ft} = -\left[\theta\log\delta + (\theta-1)b_{c,0} + m\left(-\gamma\mu_c + \frac{1}{2}\gamma^2\sigma_c^2\right)\right] + \int_R (e^{-\gamma x_c} - 1)\nu_c(x_c)\,dx_c. \tag{A.3}$$

Finally, the logarithm of the expected return on the wealth portfolio $R_{c,t+1} = \exp(r_{c,t+1})$ is equal to

$$\log E_t R_{c,t+1} = b_{c,0} + m\left(\mu_c + \frac{1}{2}\sigma_c^2\right) + \int_{R^2} (e^{x_c} - 1)\nu_c(x_c)\,dx_c. \tag{A.4}$$

Therefore, the risk premium on consumption asset is given by

$$\log(E_t R_{c,t+1}) - r_{ft} = \gamma m \sigma_c^2 + \int_R (e^x + e^{-\gamma x} - e^{(1-\gamma)x} - 1)\nu_c(x_c)\,dx_c. \tag{A.5}$$

The expression for the risk premium on a dividend-paying asset is obtained in a similar way.

A.2. AR(1) case

The joint conditional moment-generating function of consumption and dividend streams g_{t+1} and activity shocks ξ_{t+1} can be written in the following form:

$$\begin{aligned}\log E_t e^{u'_{cd} g_{t+1}+u_l \xi_{t+1}} &= [m+\rho A_t]\left(\mu' u_{cd}+\frac{1}{2}u'_{cd}\Sigma u_{cd}\right)+\int_{R^+}(e^{[\mu' u_{cd}+1/2u'_{cd}\Sigma u_{cd}+u_l]z}-1)v(z)\,dz \\ &= [m+\rho A_t]\left(\mu' u_{cd}+\frac{1}{2}u'_{cd}\Sigma u_{cd}\right)+\int_{R^3}(e^{u'x}-1)v_{cdl}(x)\,dx,\end{aligned} \tag{A.6}$$

where the joint Lévy density $v_{cdl}(x)$ is given by

$$v(x_c,x_d,x_l)=f\left(\begin{bmatrix} x_c \\ x_d\end{bmatrix};\mu x_l,\Sigma x_l\right)v(x_l). \tag{A.7}$$

Integrating out the dividend component, we can obtain the conditional moment-generating function for consumption and activity shocks:

$$\log E_t e^{u_c \Delta c_{t+1}+u_l\xi_{t+1}}=[m+\rho A_t]\left(\mu_c u_c+\frac{1}{2}u_c^2\sigma_c^2\right)+\int_{R^2}(e^{u'x}-1)v_{cl}(x_c,x_l)\,dx \tag{A.8}$$

for a joint Lévy density of consumption growth and activity shocks,

$$v_{cl}(x_c,x_l)=f_n(x_c;\mu_c x_l,\sigma_c^2 x_l)v(x_l). \tag{A.9}$$

To solve for the equilibrium asset prices, we log-linearize the return on consumption asset, which can be conveniently expressed in the following form:

$$r_{c,t+1}=-\log\kappa_1+\kappa_1(pc_{t+1}-E(pc_t))-(pc_t-E(pc_t))+\Delta c_{t+1}, \tag{A.10}$$

where κ_1 is an endogenous log-linearization coefficient, related to the unconditional level of the price–consumption ratio.

Conjecture that the price–consumption ratio is affine in the activity state A_t:

$$pc_t=H_{c,0}+H_{c,1}A_t. \tag{A.11}$$

Then, we can express the consumption return in terms of the underlying state variables and shocks in the economy:

$$r_{c,t+1}=-\log\kappa_1+H_{c,1}(\kappa_1(m-E(A))+E(A))+H_{c,1}(\kappa_1\rho-1)A_t+\kappa_1 H_{c,1}\xi_{t+1}+\Delta c_{t+1}, \tag{A.12}$$

where the unconditional mean of the activity state $E(A)$ is equal to

$$E(A)=\frac{m+E(\xi_t)}{1-\rho}.$$

We can use the Euler equation (4) to solve for the equilibrium coefficients in the price–consumption ratio. The loading $H_{c,1}$ satisfies

$$H_{c,1}=\frac{\rho}{1-\kappa_1\rho}\left(1-\frac{1}{\psi}\right)\left(\mu_c+\frac{1}{2}(1-\gamma)\sigma_c^2\right), \tag{A.13}$$

while the log-linearization coefficient, which is related to the unconditional level of the price–consumption ratio, is given by the recursive equation:

$$\log\kappa_1=\log\delta+H_{c,1}(\kappa_1(m-E(A))+E(A))+m\left(1-\frac{1}{\psi}\right)\left(\mu_c+\frac{1}{2}(1-\gamma)\sigma_c^2\right)+\frac{1}{\theta}\int_{R^2}(e^{(1-\gamma)x_c+\theta\kappa_1 H_{c,1}x_l}-1)v_{cl}(x_c,x_l)\,dx. \tag{A.14}$$

We can now express the discount factor in (3) in terms of the underlying state variables and innovations in the economy:

$$m_{t+1}=m_0+m_a A_t-\lambda_\xi\xi_{t+1}-\lambda_c\Delta c_{t+1}, \tag{A.15}$$

where the discount factor loadings m_0 and m_a and market prices of risks λ_ξ and λ_c depend on the model and preference parameters:

$$\begin{aligned} m_0 &= \theta\log\delta+(\theta-1)(-\log\kappa_1+H_{c,1}(\kappa_1(m-E(A))+E(A))), \\ m_a &= (\theta-1)(\kappa_1\rho-1)H_{c,1}, \\ \lambda_\xi &= (1-\theta)\kappa_1 H_{c,1}, \\ \lambda_c &= \gamma. \end{aligned} \tag{A.16}$$

Using the Euler equation, we obtain that the risk-free rate is given by

$$r_{ft}=F_0+F_a A_t \tag{A.17}$$

for

$$F_0 = -m_0 + m\left(\lambda_c\mu_c - \frac{1}{2}\lambda_c^2\sigma_c^2\right) - \int_{R^2}(e^{-\lambda_c x_c - \lambda_\xi x_l} - 1)\nu_{cl}(x_c, x_l)\,dx,$$

$$F_a = \rho\left(\frac{1}{\psi}\mu_c - \frac{1}{2}(\gamma + \frac{1}{\psi}(\gamma - 1))\sigma_c^2\right). \tag{A.18}$$

Combining the equations for the consumption asset (A.12) and risk-free rate (A.17), we obtain that the risk premium on consumption asset satisfies

$$\log(E_t R_{c,t+1}) - r_{ft} = \gamma\sigma_c^2(m + \rho A_t) + \int_{R^2} K_c(x_c, x_l)\nu_{cl}(dx_c, dx_l) \tag{A.19}$$

for

$$K_c(x_c, x_l) = e^{x_c + \kappa_1 H_{c,1} x_l} + e^{-\lambda_c x_c - \lambda_\xi x_l} - e^{(1-\lambda_c)x_c + \theta\kappa_1 H_{c,1} x_l} - 1. \tag{A.20}$$

We use similar methods to compute the price–dividend ratio and the risk premium on a dividend asset. Conjecture that the price–dividend ratio is linear in the activity variable A_t:

$$pd_t = H_{d,0} + H_{d,1}A_t. \tag{A.21}$$

Using the Euler equation for the log-linearized dividend return, we can solve for the equilibrium loadings in the price–dividend ratio:

$$H_{d,1} = \frac{\rho}{1 - \kappa_{d,1}\rho}\left(\mu_d - \frac{1}{\psi}\mu_c + \frac{1}{2}\left[\left(\gamma + \frac{1}{\psi}(\gamma - 1)\right)\sigma_c^2 - 2\gamma\sigma_{cd} + \sigma_d^2\right]\right), \tag{A.22}$$

where the log-linearization coefficient for the dividend return $\kappa_{d,1}$ satisfies the recursive equation

$$\log\kappa_{d,1} = m_0 + H_{d,1}(\kappa_{d,1}(m - E(A)) + E(A)) + m\left(\mu'\begin{bmatrix}-\lambda_c \\ 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-\lambda_c \\ 1\end{bmatrix}'\Sigma\begin{bmatrix}-\lambda_c \\ 1\end{bmatrix}\right)$$
$$+ \int_{R^3}(e^{-\lambda_c x_c + x_d + (\kappa_{d,1}H_{d,1} - \lambda_\xi)x_l} - 1)\nu_{cdl}(x)\,dx. \tag{A.23}$$

The risk premium on a dividend asset is given by

$$\log(E_t R_{d,t+1}) - r_{ft} = \gamma\sigma_{cd}(m + \rho A_t) + \int_{R^3} K_d(x_c, x_d, x_l)\nu_{cdl}(dx_c, dx_d, dx_l), \tag{A.24}$$

for

$$K_d(x_c, x_d, x_l) = e^{x_d + \kappa_{d,1}H_{d,1}x_l} + e^{-\lambda_c x_c - \lambda_\xi x_l} - e^{-\lambda_c x_c + x_d + (\kappa_{d,1}H_{d,1} - \lambda_\xi)x_l} - 1. \tag{A.25}$$

To obtain closed-form solutions for the asset prices in case when the activity shocks follow tempered stable distribution, we use the result that

$$\int_{R^3}(e^{u_c x_c + u_d x_d + u_l x_l} - 1)\nu_{cdl}(x)\,dx = \varphi\left(\mu'\begin{bmatrix}u_c \\ u_d\end{bmatrix} + \frac{1}{2}\begin{bmatrix}u_c \\ u_d\end{bmatrix}'\Sigma\begin{bmatrix}u_c \\ u_d\end{bmatrix} + u_l\right), \tag{A.26}$$

where φ_u is the moment-generating function of the tempered stable distribution defined in (19).

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