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QUADRATURE-BASED METHODS FOR OBTAINING APPROXIMATE SOLUTIONS TO NONLINEAR ASSET PRICING MODELS[1]

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The paper develops a discrete state space solution method for a class of nonlinear rational expectations models. The method works by using numerical quadrature rules to approximate the integral operators that arise in stochastic intertemporal models. The method is particularly useful for approximating asset pricing models and has potential applications in other problems as well. An empirical application uses the method to study the relationship between the risk premium and the conditional variability of the equity return under an ARCH endowment process.

KEYWORDS: Nonlinear rational expectations model, numerical integration, risk premiums.

1. INTRODUCTION

NONLINEAR DYNAMIC RATIONAL EXPECTATIONS MODELS rarely admit explicit solutions. Techniques like the method of undetermined coefficients or forward-looking expansions, which often work well for linear models, rarely provide explicit solutions for nonlinear models. The lack of explicit solutions complicates the tasks of analyzing the dynamic properties of such models and generating simulated realizations for applied policy work and other purposes.

This paper develops a discrete state-space approximation method for a specific class of nonlinear rational expectations models. The class of models is distinguished by two features: First, the solution functions for the endogenous variables are functions of at most a finite number of lags of an exogenous stationary state vector. Second, the expectational equations of the model take the form of integral equations, or more precisely, Fredholm equations of the second type.

The key component of the method is a technique, based on numerical quadrature, for forming a discrete approximation to a general time series conditional density. More specifically, the technique provides a means for calibrating a Markov chain, with a discrete state space, whose probability distribution closely approximates the distribution of a given time series. The quality of the approximation can be expected to get better as the discrete state space is made sufficiently finer. The term "discrete" is used here in reference to the range space of the random variables and not to the time index; time is always discrete in our analysis.

The discretization technique is primarily useful for taking a discrete approximation to the conditional density of the strictly exogenous variables of a model. The specification of this conditional density could be based on a variety of

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factors depending upon the application. For instance, it could be specified a priori by a researcher interested in exploring the time series properties of a particular model under various assumptions about the dynamics of the driving variables. On the other hand, in the interest of obtaining realistic calibration, the conditional density might be estimated directly from data using either a parametric or nonparametric procedure. In Section 5 below, we present an application where the conditional density is obtained via estimation of an ARCH model.

Asset pricing models without endogenous state variables (Mehra and Prescott (1985), Donaldson and Mehra (1984)) are particularly well suited for approximate solution via discrete methods.[2] For these models, once the state space is made discrete, then solution of the expectational equations only involves matrix inversion. Thus a discrete state space approach essentially maps the "difficult" problem of solving expectational equations into linear "approximating" problems requiring only matrix inversion. This paper goes beyond the just-cited work by identifying and making explicit the mapping from the continuous problem to the discrete linear problem and by providing an efficient method based on numerical quadrature for calibrating the discrete state-space economy.

The general form that describes the class of models considered in this paper can be explained most easily when the dynamics of the M-dimensional driving process $\{y_t\}$ are characterized by a conditional density, $f(y_{t+1}|y_t)$, that depends upon at most one lag. For simplicity we write the conditional density as $f(y|x)$, where y represents the value "one period hence" and x represents the conditioning value. In this case, the equations of the economic models in this class can be written in the form of the integral equation

$$(1.1) \qquad v(x) = \int \psi(y,x) v(y) f(y|x)\, dy + g(x)$$

where $\psi(y,x)$ and $g(x)$ are functions of x and y that depend upon the specific structure of the economic model, and where $v(x)$ is the solution function of the model. The function v may be vector valued, and then the integral in (1.1) is taken elementwise. Equation (1.1) is a general form of the basic equations of the utility-based asset pricing model, though it encompasses other models as well. In particular, (1.1) is of the form analyzed by Lucas and Stokey (1987) and is an important component of the model of Eichenbaum and Singleton (1986).

[2] An endogenous state variable is an endogenous variable of a model whose solution function depends upon the entire infinite past of the exogenous process. For example, in the capital growth model described in Taylor and Uhlig (1990), the decision rule expresses the optimal choice of the current capital stock as a nonlinear function of the lagged capital stock and the exogenous technology shock. After using recursive back substitution through the decision rule to express the current capital stock as a function of the exogenous technology shock only, the dependence extends into the infinite past. With such infinite dependence, discretization of the state space will not reduce the Euler equation to the solution of a finite system of equations in a finite number of unknowns. This reduction, on the other hand, does occur for the asset pricing models that motivate this paper.

The best way to discuss numerical approximation of integral equations is to employ an operator-theoretic notation. Write (1.1) as

$$v = T[v] + g$$

where $T[\cdot]$ is the operator defined by the integral term in (1.1). Under regularity conditions, the operator $[I - T]^{-1}$ exists, where I denotes the identity operator, and the exact solution is

$$v = [I - T]^{-1} g.$$

An approximate solution is obtained using T_N in place of T,

$$v_N = [I - T_N]^{-1} g,$$

where T_N is "close" to T for large N and $[I - T_N]$ is "easy" to invert. In some cases, the function g is of the form $g = T[g_0]$ in which case the approximate solution is taken as $[I - T_N]^{-1} T_N[g_0]$.

The solution method developed in this paper is based on Nystrom's method, also called the "quadrature" method, which is a powerful method for the numerical treatment of integral equations (Atkinson (1976, pp. 88–92), Baker (1978, Chapt. 4), Cryer (1982, pp. 324–332), Wouk (1979, pp. 149–157)). The idea is to use a numerical quadrature rule to approximate the integral operator T, and then inversion of the operator $[I - T_N]$ is equivalent to the straightforward problem of inverting a matrix.

This paper extends that literature by showing how to use the quadrature method for the purpose of calibrating the Markov chain approximation to $f(y|x)$. The use of quadrature to calibrate the Markov chain represents a major extension of Tauchen (1986a, 1986b). Those papers used a simple equispaced grid with the transition probabilities being the areas under a homogeneous error density for a linear VAR. That approach works well in small problems, but it cannot handle large problems efficiently nor can it handle problems with complex dynamics, including, among other things, conditional heteroskedasticity. In this paper, the grid points and transition probabilities are determined by $f(y|x)$ in conjunction with a numerical quadrature rule, and the method can handle large problems and more elaborate dynamics than those of a linear VAR.[3]

A discrete approximation to a general time series law of motion $f(y|x)$ has applications beyond approximating the asset pricing models that motivate this paper. Some of these applications are already underway. As part of a larger study of monetary velocity in cash-in-advance models, Hodrick, Kockerlakota, and Lucas (1989) use the technique of this paper to calibrate a Markov chain model for bivariate money growth and consumption growth. They find that with sixteen states of nature the Markov chain can adequately approximate a

[3] We have coded the Markov chain approximation technique and the asset pricing solution method in both Gauss and Fortran, and the source code is available upon request.

VAR(1) model fitted to annual data. Another application is that of Boudoukh and Whitelaw (1988), who use the quadrature technique and related ideas to study the pricing of mortgage-backed securities and American options. Though their securities have a particularly complicated path-dependent cash flow, in test cases they get very close approximations to exact solutions with state spaces as small as three points. Some other applications are Ghysels and Hall (1990a, 1990b), who extend the approach to solve and test models with nonnested Euler equations, and Kocherlakota (1989), who uses the method as part of a study of the plausibility of the parameter values in commonly used representative agent models. Burnside (1989) uses versions of the method to approximate predicted population moments for method of moments estimation of financial models.

The Markov chain technique should be useful in any solution algorithm that requires a discretization of the law of motion of the exogenous driving processes of the model. These algorithms are designed for models that are not subsumed by (1.1) because, among other reasons, the underlying operators are nonlinear. Hussey (1989), for example, uses the technique to calibrate a Markov chain for the technology shock as part of a solution strategy for a model with non-quadratic adjustment costs. His algorithm combines the ideas of this paper with those of Coleman (1989). The algorithm differs from Tauchen (1990) in that it iterates on the Euler equations instead of the value function. Other solution methods that could potentially use the quadrature discretization technique are those of Baxter, Crucini, and Rouwenhorst (1990) and Christiano (1990). Taylor and Uhlig (1990) contains a complete summary of current work in the general area of solution strategies for nonlinear equilibrium models.

The remainder of the paper is organized as follows. Section 2 introduces more notation and also presents a motivating example. Section 3 presents the details of the approximation method, while Section 4 contains the theoretical results. Section 5 contains applications and related material. Section 6 contains the concluding remarks.

2. EXAMPLE: ASSET PRICING EQUATIONS

Consider a representative agent/exchange economy asset pricing model in the style of Lucas (1978) and Mehra and Prescott (1985). Let there be I assets and let p_{it} denote the ex-dividend price in period t of the ith asset. The ith asset yields the stochastic dividend stream $\{d_{i,t+1}, d_{i,t+2}, \ldots\}$, $i = 1, 2, \ldots, I$. The first order conditions for the agent's intertemporal utility maximization problem imply that the asset prices follow the expectational equation

$$p_{it} = E_t[(p_{i,t+1} + d_{i,t+1}) mrs_t(c_t, c_{t+1})] \qquad (i = 1, 2, \ldots, I) \tag{2.1}$$

where $E_t[\ ]$ is the conditional expectations operator given information available to the agent through period t, c_t is the agent's consumption in period t, and $mrs_t(c_t, c_{t+1})$ is the agent's marginal rate of substitution between consumption in periods t and $t+1$. The subscript t on mrs reflects possible dependence of the marginal rate of substitution on variables dated period t or earlier due to nonseparabilities in the agent's intertemporal utility function. One should note that, in general, under time nonseparable preferences, $mrs_t(c_t, c_{t+1})$ may depend upon expectations as of time t and $t+1$ of functions of future consumption from time $t+2$ out to some finite horizon. However, this dependence is not made explicit here as it needlessly complicates the notation. In implementing the discrete method of Section 3, one computes these expectations directly from the Markov chain model for the state vector by applying, as need be, the law of iterated expectations.

Following Mehra and Prescott (1985), we rewrite the basic asset pricing equation (2.1) to express it in terms of growth rates, which will be taken to be stationary and Markovian. Specifically, define the consumption growth and dividend growth variables, $q_t = c_t/c_{t-1}$ and $h_{it} = d_{it}/d_{i,t-1}$. In addition assume the agent's intertemporal utility function is homogeneous, so that $mrs_t(c_t, c_{t+1}) = m_t(q_{t+1})$ depends only on the growth variables, i.e., on q_{t+1}, and possibly on $q_t, q_{t-1}, \ldots,$ but not on consumption levels directly. Then letting v_{it} denote the ith asset's price dividend ratio, the asset pricing equation (2.1) can be rewritten as

$$(2.2) \qquad v_{it} = E_t[(1 + v_{i,t+1})h_{i,t+1}m_t(q_{t+1})] \qquad (i = 1, 2, \ldots, I).$$

Consumption growth and dividend growth are assumed to be functions of a finite-memory stationary stochastic process. Specifically, consumption growth is $q_t = \varphi_1(y_t)$, and the vector of dividend growth variables is $h_t = \varphi_2(y_t)$, where φ_1 and φ_2 are functions on $\mathbb{R}^M$, and y_t is an $M \times 1$ strictly stationary process. Letting $x_{t-1} = (y'_{t-1} y'_{t-2} \ldots y'_{t-L})'$, we write the conditional density of y_t given its past as $f(y_t|x_{t-1})$ or simply $f(y|x)$. Generally, the functions φ_1 and φ_2 will be elementary functions.[4] Also, the dimension of y_t might exceed that of $(q_t h'_t)$. This would be the case, for example, in a Monte Carlo study where the modeller allowed for additional variables in y_t that were observed by agents but not by the econometrician.

To write the integral equation corresponding to (2.2), we move the time index back one period and view things from the perspective of period $t-1$ instead of period t, which gives

$$(2.3) \qquad v_{i,t-1} = E_{t-1}[(1 + v_{it})h_{it}m_{t-1}(q_t)].$$

The integral form for the asset pricing equation is

$$(2.4) \qquad v_i(x) = \int [1 + v_i(y, x^-)] h_i(y) m(y, x) f(y|x)\, dy,$$

where the notation is:

x: an $M \cdot L$ vector whose elements correspond to

$$\{y_{t-1}, y_{t-2}, \ldots, y_{t-L}\};$$

[4] With suitable modifications, one can apply the technique to the original setup of Lucas (1978) which assumes stationarity of dividend levels. In this case, y_t would be the vector of dividends and $\varphi_1(y) = \Sigma_t y_t$, $\varphi_2(y) = y$, $y \in \mathbb{R}^M$.

$v_i(x)$: price dividend ratio on asset i as a function of the state of the system;
x^-: an $M \cdot (L-1)$ vector whose elements correspond to

$$\{y_{t-1}, y_{t-2}, \ldots, y_{t-L+1}\};$$

y: an M vector whose elements correspond to y_t;
$h_i(y)$: dividend growth on the ith asset;
$m(y,x)$: marginal rate of substitution between consumption in periods $t-1$ and t;
$f(y|x)$: conditional density of y_t given the system's history.

The notation becomes simpler by letting $\psi_i(y,x) = h_i(y)m(y,x)$, which is the product of dividend growth and the marginal rate of substitution. Then (2.4) becomes

$$(2.5) \qquad v_i(x) = \int [1 + v_i(y, x^-)] \psi_i(y,x) f(y|x)\, dy \qquad (i = 1, 2, \ldots, I).$$

Subsequent sections analyze the properties of the quadrature method for approximating the solutions to equations of the form (2.5).

3. NUMERICAL APPROXIMATION TO RATIONAL EXPECTATIONS INTEGRAL EQUATIONS

The theory of Nystrom's method is closely related to the theory of numerical quadrature, so a brief overview of quadrature is presented first.

3.1. *Numerical Quadrature*

An N-point quadrature rule for integration of functions $g(u)$, $u \in \mathbb{R}^M$, against a density $\omega(u)$ is a set of N abscissa $u_k \in \mathbb{R}^M$ and weights $w_k \in \mathbb{R}^M$ such that

$$(3.1) \qquad \int g(u)\omega(u)\, du \doteq \sum_{k=1}^{N} g(u_k) w_k$$

with convergence for each function g (under regularity conditions) of the approximating sum to the integral as $N \to \infty$. The abscissa u_k and weights w_k depend only on the density ω, and not directly on the function g.

For a classical N-point Gauss rule along the real line, $u \in \mathbb{R}^1$, the abscissa u_k and weights w_k are determined by forcing the rule to be exact for all polynomials of degree less than or equal to $2N-1$. (For details see Davis and Rabinowitz (1975).) The weights for a Gauss rule and most other good rules are nonnegative and the rules integrate the constant function exactly; i.e., the weights sum to unity. Thus a quadrature rule can be viewed as a discrete probability model that approximates the density ω. Indeed, the Gauss rules are discrete approximations to ω determined by the method of moments using moments up through $2N-1$. Gauss rules are close to minimum norm rules and possess several optimum properties (Davis and Rabinowitz (1975)). They are the best that can be done with N points using moments as a criterion, because if two probability distributions have the same moments up through $2N$, and if one of the distributions is a discrete distribution concentrated on N points, then the two distributions must coincide (Norton and Arnold (1985)).

Multivariate quadrature, $u \in \mathbb{R}^M$, is more complex. Stroud (1971, Ch. 3) presents the extension of the one-dimensional Gauss rules, wherein the abscissa and weights are determined by forcing the rule to be exact for monomials of a given degree. Multivariate quadrature becomes much simpler if the density ω can, after an affine transformation of variables, be factored into the product of M one-dimensional densities. (This is analogous to writing $Y = \mu + RZ$, $Y \sim N(\mu, \Omega)$, $Z \sim N(0, I)$, $RR' = \Omega$.) A multivariate product rule can then be formed by combining a set of one-dimensional Gauss rules. A product rule has $N = \prod_{j=1}^{M} J_j$ points, where J_j is the number of points used along the jth axis.

Problems of high dimension typically require use of a nonproduct rule, which generally entails far fewer points than a full product rule. Stroud (1971, Ch. 7 and 8) presents a "toolkit" of nonproduct rules that have been found to work well in a wide class of applied contexts. For instance, a Spherical Lobatto rule for integration against the multivariate normal distribution entails $N = 2^{M+1} - 1$ points and will integrate exactly all polynomials of degree five or less. For a six-dimensional problem, the Spherical Lobatto rule entails $N = 127$ points, while a full Gauss-Hermite product rule requires $N = 3^6 = 729$ points to be exact for polynomials of degree five or less.

3.2. *Approximation for First-Order Dynamics*

For ease of exposition, we present numerical approximation for the special case where the state variable y_t is a first-order vector process with law of motion given by conditional density $f(y|x)$, with $y, x \in \mathbb{R}^M$. The more general case $y \in \mathbb{R}^M$, $x \in \mathbb{R}^{ML}$, entails extra details and is left to Appendix A.

The integral equation is

$$v(x) = \int [1 + v(y)] \psi(y, x) f(y|x)\, dy. \tag{3.2}$$

We assume the solution $v(x)$ exists and consider approximating it. Put

$$\lambda(y, x) = [1 + v(y)] \psi(y, x). \tag{3.3}$$

Define $I[\lambda]$ to be the integral operator given by the right-hand side of (3.2),

$$I[\lambda](x) = \int \lambda(y, x) f(y|x)\, dy. \tag{3.4}$$

Now write (3.4) as

$$I[\lambda](x) = \int \lambda(y, x) \frac{f(y|x)}{\omega(y)} \omega(y)\, dy,$$

where $\omega(y)$ is some strictly positive weighting function.[5] The integral $\int[\cdot]\omega\, dy$ will be approximated by the quadrature rule $\Sigma_k[\cdot]w_k$. There is great latitude in the selection of the weighting function, though clearly one would want to select a weighting function such that the resulting discrete sums are very close to exact integration against $f(y|x)$. For reasons discussed in Subsection 3.3 below, one reasonable choice for the weighting function is $\omega(y)=f(y|0)$, i.e., the conditional density given that the process is at the unconditional mean, which is taken to be zero without loss of generality.

Let $\bar{y}_k$ and w_k, $k=1,2,\dots,N$, denote the abscissa and weights for an N-point quadrature rule for the density $\omega(y)$. This rule may be either a product rule or a nonproduct rule. One should choose an efficient rule like a product Gauss rule for problems of modest dimension or one of the good nonproduct rules as given in Stroud (1971, Ch. 7 and 8) for larger problems.

The approximation based on this rule to $I[\lambda](x)$ in (3.4) is

$$I_N[\lambda](x) = \sum_{k=1}^{N} \lambda(\bar{y}_k, x)\pi_k^N(x), \tag{3.5}$$

where

$$\pi_k^N(x) = \frac{f(\bar{y}_k|x)}{s(x)\omega(\bar{y}_k)} w_k \qquad (k=1,2,\dots,N) \tag{3.6}$$

and

$$s(x) = \sum_{i=1}^{N} \frac{f(\bar{y}_i|x)}{\omega(\bar{y}_i)} w_i.$$

Observe that the weights $\pi_k^N(x)$ in (3.6) are obtained by replacing integration against $\omega(y)$ with summation using the quadrature rule, and then normalizing so that the weights add to unity. Other treatments of Nystrom's method do not utilize the renormalization by $s(x)$, but for our purposes it is essential for the Markov chain interpretation given below.

Given (3.5), the approximation to the solution of (3.2) is obtained by evaluating $I_N[\lambda](x)$ at each of the quadrature abscissa and then solving the implied linear system of equations. For this purpose let

$$\bar{v}_{Nj} = \tilde{v}_N(\bar{y}_j) \qquad (j=1,2,\dots,N),$$

where $\tilde{v}_N$: $\mathbb{R}^M \to \mathbb{R}^1$ denotes the (yet to be determined) approximate solution extended to all $y \in \mathbb{R}^M$, and let $\bar{v}_{Nj} = \tilde{v}_N(\bar{y}_j)$ denote the values of $\tilde{v}_N$ at each of

[5] A reviewer noted that (3.4) is already in the form $\int[\cdot]d\omega$ with ω being the identity function, though the induced measure (Lebesgue) is not a finite measure. One reason for using a weighting function with finite measure is that it allows the user to incorporate prior information about where most of the mass of $\lambda(y,\cdot)f(y|\cdot)$ lies and also to incorporate prior information about the properties of the functional form of $f(y|\cdot)$, including the existence of moments. As noted in Subsection 3.3 below, the user needs to be sure, though, that the tails of $f(y|\cdot)/\omega(y)$ are well behaved relative to $\omega(y)$.

the abscissa. In addition put

$$(3.7) \qquad \psi_{jk} = \psi(\bar{y}_k, \bar{y}_j),$$

$$(3.8) \qquad \pi_{jk}^N = \pi_k^N(\bar{y}_j).$$

Evaluating $I_N(\lambda)$ in (3.5) at x's equal to each of the quadrature abscissa $\bar{y}_j$ and remembering the definition of λ in (3.3) gives

$$(3.9) \qquad \bar{v}_{Nj} = \sum_{k=1}^{N} [1 + \bar{v}_{Nk}]\psi_{jk}\pi_{jk}^N \qquad (j = 1, 2, \ldots, N).$$

The equations (3.9) comprise a system of N linear equations in the $\bar{v}_{Nj}$ which can be solved directly. Their solution provides the values of the approximate solution to the integral equation (3.2) at each of the quadrature points. The Nystrom extension of the solution to the entire domain of x is

$$(3.10) \qquad \tilde{v}_N(x) = \sum_{k=1}^{N} [1 + \bar{v}_{Nk}]\psi(\bar{y}_k, x)\pi_k^N(x), \quad x \in \mathbb{R}^M.$$

The $\{\bar{v}_{Nj}\}_{j=1}^N$ in (3.9) are the solutions to the asset pricing equations if one views the law of motion of the state vector as a discrete Markov chain with range $\{\bar{y}_k\}$ and transition probabilities $\pi_{jk}^N = \Pr(y_t = \bar{y}_k | y_{t-1} = \bar{y}_j)$. Thus, the approximation implicitly provides a means for taking an arbitrary law of motion $f(y|x)$ and calibrating a Markov chain whose law of motion closely approximates $f(y|x)$.

3.3. *The Weighting Function*

As noted above, there is great latitude in the selection of the weighting function, though taking $\omega(y) = f(y|0)$ has proved to work very well in practice. This choice is motivated by Drezner (1978) who is concerned with the much different but related problem of approximating the probability mass of the bivariate normal over rectangles. Another a priori reasonable choice is $\omega(y) = f_s(y)$, where f_s is the unconditional or stationary density of the process. The density $f(y|0)$ places relatively more weight in the central part of the distribution and less weight in the tails than does $f_s(y)$. In numerical evaluation of the accuracy of the approximations, it was found that $\omega(y) = f(y|0)$ gives much better approximations except when the maximum magnitude of the characteristic roots of the autoregression is close to zero, in which case there is little difference between $f(y|0)$ and $f_s(y)$ and the approximations are essentially equivalent.

The choice $f(y|0)$ works well because it balances two conflicting criteria. On the one hand, good approximation requires that the weighting density put a lot of weight near the unconditional mean, which is zero. But at the same time, the ratio of densities $f(y|x)/\omega(y)$ must be well behaved in the y tails; i.e., this ratio should not grow too fast in either tail relative to $\omega(y)$. In particular, suppose y_t is a univariate normal AR(1) process and suppose $\omega(y) = n(y; 0, \theta^2)$, the normal density with mean zero and variance θ^2. So long as θ^2 is at least as large as the variance of $f(y|0)$, then for fixed x, $f(y|x)/\omega(y)$ is at most $\exp[O(y)]$ in either tail, while $\omega(y)=\exp[-O(|y|^2)]$. But if θ^2 is less than the variance of $f(y|0)$, then for fixed x, $f(y|x)/\omega(y)=\exp[O(|y|^2)]$ in one tail. In other words, the choice $\omega(y)=f(y|0)$ puts as much probability mass near the origin as possible while keeping the tails of $f(y|x)/\omega(y)$ heavily damped relative to those of $\omega(y)$.

The selection $\omega(y)=f(y|0)$ does require that the moments of $f(y|0)$ exist and are readily computable if one employs a Gauss quadrature rule. These moments are obviously available if $f(y|x)$ is conditionally Gaussian, as is the case when $\{y_t\}$ is a Gaussian VAR, an ARCH-N process, or a nonlinear autoregression with Gaussian errors. More generally, they are available whenever the error density has easily computed moments. In some instances, though, computing moments beyond the second and calculating the abscissa and weights of a Gauss rule might require extra computational work, as would be the case for the seminonparametric (SNP) models of Gallant and Tauchen (1989). In these instances, one might elect to use either a different quadrature rule or a different weighting function, with the choice of the latter guided by the criteria indicated in the previous paragraph. For SNP models, a reasonable choice would be to leave $\omega(y)=f(y|0)$ and use a Gauss-Hermite rule, even though the error density is non-Gaussian. Such a rule can be expected to work well given that the structure of the SNP model takes the form of a polynomial in y and x times a normal density.

4. THEORY

We now examine the convergence properties of the Markov chain model and the approximate solution to asset pricing equations. To keep the notation simple, we will do this for the case of first order scalar dynamics, $f(y|x)$, with $y,x\in\mathbb{R}$. The extension of the results to the vector case $y\in\mathbb{R}^M$, $x\in\mathbb{R}^{ML}$, is very straightforward and only involves extra bookkeeping of indexes. We will also assume that the support of $f(y|x)$ is a subset of a rectangle $[a,b]\times[a,b]\subset\mathbb{R}^2$, with $a<b$ and both a and b finite. This restriction is more substantive than assuming y and x are scalars, and in the applications below the support is unbounded. Thus we follow the tradition in this area of research of applying results deduced for the case of bounded support to models with unbounded support. Sloan (1980, p. 56) writes:

> ...the Nystrom method has been restricted to the case of finite intervals. Yet many of the integral equations that occur in practice have infinite integration regions. In many such cases the [method is] often used in practice, but as far as I am aware there exist no error analysis or convergence theory that would justify [its] use. This is a challenge for the theorists...

The main task for extending the results to an unbounded domain is to identify the appropriate spaces and norms such that the very general Anselone-Moore conditions will hold.[6] These conditions are stated and discussed in Atkinson (1976, p. 96) and are also stated in the proof of Theorem 4.3 in Appendix B below.

Let $C_0[a,b]$ denote the space of continuous functions on $[a,b]$ equipped with the usual sup norm, $\|g\| = \sup\{|g(y)|: y \in [a,b]\}$. A bounded linear operator on $C_0[a,b]$ is a mapping T: $C_0[a,b] \to C_0[a,b]$ such that $T[cg] = cT[g]$, $g \in C_0[a,b]$, $c \in \mathbb{R}$, and $\|T\| = \sup\{\|T[g]\|: \|g\| \leqslant 1\} < \infty$.

We will be interested in the properties of N-point quadrature rules for integration against density $\omega(y)$ with support on the interval $[a,b]$. In the remainder of this section the notation will make explicit dependence upon N, since we wish to take limits as N tends to infinity.

DEFINITION 4.1: The triangular array $\{\bar{y}_{Nk}, w_{Nk}\}_{k=1}^{N}$ is a *quadrature rule* if

(a) $\bar{y}_{Nk} \in [a,b]$,

(b) $w_{Nk} \geqslant 0$, and $\sum_{k=1}^{N} w_{Nk} = 1$.

The quadrature rule defines an approximation to integrals against ω,

$$Q_N(g) = \sum_{k=1}^{N} g(\bar{y}_{Nk}) w_{Nk}, \tag{4.1}$$

and the well-known Quadrature Theorem makes precise the convergence properties of Q_N:

THEOREM 4.1 (Quadrature Theorem): *Let ω be a nonnegative weight function such that*: (i) *$\omega(y)$ vanishes for y outside $[a,b]$, and* (ii) *$\int \omega(y)\,dy = 1$. Then*

$$Q_N(g) \to \int g(y)\omega(y)\,dy \quad \textit{for all} \quad g \in C_0[a,b],$$

if and only if

$$Q_N(g_j) \to \int g(y)\omega(y)\,dy, \quad \textit{for each } g_j,$$

where $\{g_j\}_{j=1}^{\infty}$ is a dense subset of $C_0[a,b]$.

This is Theorem 7 of Atkinson (1976, p. 21).

REMARK 4.1: By construction, Gaussian quadrature rules are exact for all polynomials of degree $2N-1$. Thus, taking the $\{g_j\}$ in the theorem to be the set of polynomials with rational coefficients, which is dense, establishes the convergence of $Q_N(g)$ for all $g \in C_0[a,b]$ when Q_N is based on a Gauss rule. Gauss

[6] Anselone and Sloan (1985) is a promising start of a theory for the case of an infinite domain, but their regularity conditions are also too stringent to cover the class of models employed in Section 5 below.

rules thus provide a wide class of rules for which there will be convergence of the approximation sum to the integral. As the theorem indicates, though, other rules are appropriate as well so long as they are convergent on a dense set.

The next two theorems concern the convergence properties of the Markov chain model that approximates the law of motion $f(y|x)$ and the associated approximate solution to the asset pricing equations. Both theorems presuppose the availability of a quadrature rule such that $Q_N(g) \to \int g(y)\omega(y)\,dy$ for all $g \in C_0[a,b]$. The Quadrature Theorem provides a general strategy for constructing such rules, and, as Remark 4.1 indicates, Gauss rules are examples of such rules.

From Section 3, the discrete values of the Markov chain are $\{\bar{y}_{Nk}\}$ and the transition probabilities are of the form $\pi_{jk}^N = \pi_k^N(\bar{y}_{Nj})$, where

$$\pi_k^N(x) = \frac{f(\bar{y}_{Nk}|x)w_{Nk}}{\omega(\bar{y}_{Nk})s_N(x)} \tag{4.2}$$

with $s_N(x) = \sum_{i=1}^N f(\bar{y}_{Ni}|x)w_{Ni}/\omega(\bar{y}_{Ni})$. The function ω is a density on $[a,b]$ and $\{\bar{y}_{Nk}, w_{Nk}\}$ is an N-point quadrature rule for integration against ω. The considerations discussed at the end of Section 3 indicated that $f(y|0)$ is a good choice for the weighting density, though others will work as well and might have to be used if $f(y|0)$ vanishes at some points. For $g \in C_0[a,b]$ define the function e_g on $[a,b]$ by

$$e_g(x) = \int g(y)f(y|x)\,dy,$$

and define e_{gN} by

$$e_{gN}(x) = \sum_{k=1}^{N} g(\bar{y}_{Nk})\pi_k^N(x),$$

which are the expectations of g under $f(\cdot|x)$ and $\{\pi_k^N(x)\}$, respectively.

Under suitable conditions $e_{gN} \to e_g$ uniformly:

THEOREM 4.2: *Let $f(y|x)$ be a transition probability function on $[a,b]\times[a,b]$ such that $f(y|x)$ is jointly continuous in x and y; let $\omega(y)$ be a continuous weight function on $[a,b]$ such that $\omega(y) \geqslant c$, for some $c>0$ and all $y \in [a,b]$, and $\int\omega(y)\,dy = 1$; let $\{\bar{y}_{Nk}, w_{Nk}\}$ be a quadrature rule such that $Q_N(g) \to \int g(y)\omega(y)\,dy$ for all $g \in C_0[a,b]$. Then*

$$\|e_{gN} - e_g\| \to 0 \quad \textit{for each} \quad g \in C_0[a,b].$$

PROOF: Appendix B.

The theorem provides justification for using the Markov chain model to approximate the conditional expectations of interesting nonlinear functions. The condition that ω be strictly bounded away from zero ensures that the ratio in (4.2) is always well defined. The condition could be relaxed, though this would entail added conditions on the zeros of the transition density and the weight function to ensure $f(y|x)/\omega(y)$ is well defined for all y and x.

We now consider the approximate solution of an asset pricing equation. As in Sections 1–3 above, write the asset pricing model as

$$v(x) = \int [1 + v(y)]\psi(y, x) f(y|x)\, dy, \tag{4.3}$$

or as

$$v = T[v] + T[1],$$

where 1 is the identity function and T is the integral operator

$$T[g](x) = \int g(y)\psi(y, x) f(y|x)\, dy, \quad g \in C_0[a, b]. \tag{4.4}$$

We will be assuming that $(I - T)^{-1}$ exists and is a bounded linear operator on $C_0[a, b]$. The asymptotic result concerns the approximation of the exact solution, $v = (I - T)^{-1}T[1]$, by the solution $\tilde{v}_N = (I - T_N)^{-1}T_N[1]$, where T_N is the approximating operator defined by the Markov chain model

$$T_N[g](x) = \sum_{k=1}^{N} g(\bar{y}_k)\psi(\bar{y}_k, x)\pi_k^N(x), \tag{4.5}$$

with $\pi_k^N(x)$ defined as in (4.2) above.

THEOREM 4.3: *Assume*: (i) *the same conditions on* $f(y|x)$ *and* $\omega(y)$ *as in Theorem* 4.2; (ii) $\psi(y, x)$ *is nonnegative and jointly continuous in* x *and* y *on* $[a, b] \times [a, b]$; (iii) *the Markov chain model* $\{\bar{y}_k^N, \pi_{jk}^N\}$ *is constructed as in* (4.2) *using a quadrature rule for* ω *such that* $Q_N(g) \to \int g(y)\omega(y)\, dy$ *for all* $g \in C_0[a, b]$; *and* (iv) $(I - T)^{-1}$ *exists as a bounded linear operator on* $C_0[a, b]$. *Let* T_N *be as in* (4.5). *Then*: (1) *for sufficiently large* N, $(I - T_N)^{-1}$ *exists as a bounded linear operator on* $C_0[a, b]$; *and* (2) *the approximate solution converges to the exact solution uniformly*:

$$\|\tilde{v}_N - v\| = \left\|(I - T_N)^{-1}T_N[1] - (I - T)^{-1}T[1]\right\| \to 0.$$

PROOF: Appendix B.

The continuity condition on $\psi(y, x)$ will be met in the asset pricing model if the marginal utility of within-period consumption is continuous and does not vanish on the interval $[a, b]$.

REMARK 4.2: As one can see from displays (B.2), (B.3), and (B.4) of Appendix B, the rate at which $\tilde{v}_N \to v$ is essentially the same as the rate at which the quadrature rule converges. Thus, if the Markov chain model is a good approximation to $f(y|x)$, then $\tilde{v}_N$ can be expected to be an equally good approximation of v.

TABLE I

DISCRETE APPROXIMATIONS: AR(1) PROCESSES[a]

Abscissa	AR Coefficient								
(J)	.10	.20	.30	.40	.50	.60	.70	.80	.90
2	.11	.20	.29	.38	.49				
3	.10	.20	.30	.40	.50	.58			
4	.10	.20	.29	.40	.50	.59	.68	.76	.82
5			.30	.40	.50	.59	.70	.78	.85
6						.60	.70	.79	.86
7							.70	.79	.88
8								.79	.88
9								.80	.89

[a] Computed by Monte Carlo using realizations of length 100,000; coefficient std. dev. equals approximately .001.

5. APPRAISAL OF THE APPROXIMATIONS AND APPLICATIONS

5.1. *Markov Chains and Asset Pricing*

One direct way to generate evidence on the quality of the quadrature approximation is to compare the parameters of linear autoregressive models fitted to data simulated from the Markov chain to those of the underlying AR model used to calibrate the Markov chain. Tables I through III display the coefficients of linear regressions for discrete processes associated with different autoregressive models and degrees of fineness of the state space. The regressions were computed by fitting autoregressive models to Monte Carlo realizations of the Markov chains, with each realization of length 100,000. In Tables I and II, the underlying continuous processes are univariate AR(1) and AR(2) models, respectively, while in Table III the underlying processes are bivariate VAR(2) models. In all cases the innovations are independent $N(0,.01)$ variates; the choice of .01 for the variance is immaterial since the regressions are invariant with respect to equiproportionate scale changes in all variables. The

TABLE II

DISCRETE APPROXIMATIONS: AR(2) PROCESSES[a]

Abscissa	AR Coefficients							
(J)	$.60L - .09L^2$		$1.00L - .25L^2$		$1.40L - .49L^2$		$1.80L - .81L^2$	
2	.53	−.06	.74	−.10				
3	.58	−.09	.87	−.20	1.08	−.33	1.23	−.46
4	.60	−.09	.93	−.22	1.16	−.37	1.32	−.51
5			.96	−.23	1.22	−.40	1.39	−.55
6			.98	−.24	1.25	−.41	1.43	−.58
7					1.28	−.43	1.47	−.59
8					1.30	−.44	1.50	−.62
9					1.32	−.45	1.52	−.62

[a] Computed by Monte Carlo using realizations of length 100,000; coefficient std. dev. equals approximately .001.

TABLE III
DISCRETE APPROXIMATIONS: VAR(2) PROCESSES[a]

Abscissa		VAR Coefficients							
		$\begin{bmatrix} .60 & .27 \\ .00 & 1.40 \end{bmatrix} L + \begin{bmatrix} -.09 & .00 \\ .00 & -.49 \end{bmatrix} L^2$				$\begin{bmatrix} .60 & 1.09 \\ .00 & 1.40 \end{bmatrix} L + \begin{bmatrix} -.09 & .00 \\ .00 & -.49 \end{bmatrix} L^2$			
J_1	J_2								
2	2	.51	.20	−.06	.04	.24	.71	−.03	−.01
		.00	.84	.00	−.12	−.01	.84	.01	−.11
3	3	.55	.25	−.09	.00	.36	.68	−.05	.02
		.00	1.08	.01	−.33	.01	1.09	.01	−.35
4	4	.58	.26	−.09	.00	.46	.68	−.08	.00
		.00	1.16	.00	−.34	−.01	1.17	.01	−.37
2	7	.44	.16	−.05	.01	.38	.26	−.04	−.02
		−.01	1.28	.00	−.43	.00	1.28	.00	−.43
2	8	.45	.15	−.05	.00	.40	.24	−.05	−.03
		.00	1.31	.00	−.47	.01	1.30	−.01	−.44
2	9	.44	.15	−.05	.00	.42	.23	−.04	−.03
		.00	1.37	.00	−.46	.01	1.32	−.01	−.45
3	6	.52	.23	−.07	.00	.48	.43	−.08	−.03
		.00	1.25	−.01	−.41	−.02	1.25	.01	−.41
4	5	.57	.25	−.08	.00	.49	.60	−.08	−.03
		.01	1.22	.00	−.40	−.01	1.22	.00	−.40

Computed by Monte Carlo using realizations of length 100,000; coefficient std. dev. equals approximately .001.

leftmost column of each table shows the number of abscissa of the Gauss-Hermite quadrature rule used to calibrate the discrete processes.

The first two tables indicate the importance of the magnitudes of the system's eigenvalues for the accuracy of the approximation. For univariate models (Table I), the approximation is extremely accurate for very coarse state spaces—as coarse as two states of nature—when the eigenvalue is no more than .50 in magnitude. Adequate approximation, though, requires successively finer state spaces for eigenvalues closer to unity. Comparison of AR(2) to AR(1) models (Table II versus Table I) shows that a two-lag system with equal eigenvalues requires a few more points to achieve the same degree of approximation as would be obtained in a one-lag model with an eigenvalue of the same magnitude. Interestingly, Table III reveals that, for bivariate models, not only the magnitudes of the eigenvalues are important for the approximation, but so is the degree of the Granger-Sims feedback. Both of the underlying continuous VARS in Table III have the same eigenvalues. Nevertheless, the quality of the discrete approximations of the coefficients of the first equation differ because the feedback from the second variable to the first is much stronger in the model on the right side of the table.

We now examine how well the method works in terms of approximating the solutions to the asset pricing models. In particular, we assess the rate of convergence of the mean square error from approximating the price dividend ratios on assets in Lucas-style exchange economies for various degrees of fineness of the discrete state-space models. We expect to achieve reasonably rapid convergence so long as the characteristic roots of the system are not too close to unity. As we note in the remarks following Theorem 4.3 above, approximate solutions of asset pricing models converge at the same rate as the quadrature rule itself, which has been seen to be very good so long as the roots are not too close to unity.

The law of motion for the relevant state vector $y_t = (y_{1t} y_{2t})'$ is

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \tag{5.1}$$

where $y_{1t} = \ln(c_t/c_{t-1})$ and $y_{2t} = \ln(d_t/d_{t-1})$ are logarithmic consumption growth and dividend growth, and where ε_{1t} and ε_{2t} are independent normal random variables, each with variance .01. The representative agent's per-period utility function is of the CRR form $u(c) = c^{(1-\gamma)}/(1-\gamma)$ and the agent's subjective discount factor $\beta = .97$. Calculations were made for both $\gamma = 0.30$ and $\gamma = 1.30$, but because the results were so similar we only report those for $\gamma = 0.30$.

Implementation of the discretization is as follows. Given values for the AR parameters in (5.1), the $\pi_k^N(x)$ and the π_{jk}^N are computed in the manner described in Section 3 using a $J \times J$ product Gauss-Hermite rule where $J = 2, 3, \ldots, 8$. The discrete price dividend ratios $\bar{v}_{Nj}$, $j = 1, 2, \ldots, N$, $N = J^2$, are the solutions to the associated system of linear equations.

A measure of accuracy is the mean square approximation error for the Nystrom extension $\tilde{v}_N(x)$, $x \in \mathbb{R}^2$, of $\{\bar{v}_{Nj}\}_{j=1}^N$:

$$\mathrm{MSE}_N = \int_{\{x \in \mathbb{R}^2\}} [\tilde{v}_N(x) - \tilde{v}_\infty(x)]^2 f_s(x)\, dx, \tag{5.2}$$

where f_s is the stationary distribution of $\{y_t\}$. We use the continuous extension $\tilde{v}_N(x)$ to make comparisons across various degrees of fineness of the state space, because the sets of abscissa for J-point Gauss-Hermite quadrature rules are never the same for different J, and therefore there is not direct way to make meaningful comparison of the discrete $\bar{v}_{Nj}$. A measure of relative mean square error is

$$\text{REL-MSE}_N = \frac{\mathrm{MSE}_N}{\mathrm{TVAR}} \tag{5.3}$$

where

$$\mathrm{TVAR} = \int [\tilde{v}_\infty(x) - \mu_v]^2 f_s(x)\, dx \tag{5.4}$$

and where μ_v denotes the unconditional expected value of $v_\infty(x)$ based on $f_s(x)$. The relative mean square error measure is analogous to one minus an R^2 statistic. This measure can exceed unity, which will occur if the approximation of v_∞ is poorer than what would be obtained by using a constant function equal to μ_v.

TABLE IV

RELATIVE MEAN SQUARE ERRORS FROM APPROXIMATING PRICE / DIVIDEND RATIOS

Abscissa	Coefficient on Dividend Growth (a_{22})[a]						
(J)	.50	.30	.10	−.10	−.30	−.50	−.70
2	12.836	1.682	.104	.038	.269	2.118	10.738
3	1.780	1.342E − 02	9.859E − 06	2.349E − 06	1.438E − 03	.121	2.291
4	.118	7.350E − 05	7.073E − 10	1.290E − 10	9.633E − 06	7.060E − 03	.531
5	5.411E − 03	3.493E − 07	4.438E − 14	7.344E − 15	7.287E − 08	4.563E − 04	1.015E − 01
6	2.016E − 04	1.571E − 09	2.673E − 18	4.788E − 19	4.783E − 10	2.676E − 05	1.618E − 02
7	5.247E − 06	6.029E − 12	3.828E − 22	1.067E − 22	2.185E − 12	9.569E − 07	1.804E − 03

[a] $a_{11} = -.10$, $a_{12} = a_{21} = 0.0$; $\sigma_{11} = .01$, $\sigma_{12} = \sigma_{21} = 0.0$, $\sigma_{22} = .01$.

Table IV displays the relative MSE's for various values of J and a_{22} with $a_{11} = -0.10$, $a_{12} = a_{21} = 0$. The limiting v_∞ cannot be computed exactly, and for the purposes of the calculation the $\tilde{v}_\infty$ is approximated by the Nystrom extension for $J = 8$. This is quite reasonable as convergence always sets in well before then. The integrals in (5.2) and (5.4) were approximated using an 8×8 Gauss-Hermite rule for the stationary density f_s.

The rate of convergence shown in Table IV is quite rapid. So long as the autoregressive coefficient a_{22} on dividend growth is 0.30 or smaller, then convergence is achieved at $J = 4$; when the magnitude of the coefficient is .50, convergence is achieved at $J = 6$. Note the asymmetry in the rates of convergence depending upon the sign of a_{22}, with the convergence being less rapid when a_{22} is positive. This is due to the nonlinearity of the asset pricing equations. In fact, when $a_{22} = 0.50$ the model is very close to a situation where the dividend growth is so strongly positively autocorrelated that the infinite sum of the dividend capitalization formula is not convergent. If a_{22} is increased to 0.55 or larger, then the discrete price/dividend ratios are negative, which is indicative of a lack of convergence of the capitalization sum. Still, the convergence is reasonably rapid at $a_{22} = 0.50$.[7]

5.2. *Risk-Return Relations with ARCH Endowment Processes*

The nature of the relationships between risk premiums on financial assets and the conditional variances and covariances of returns has been the subject of intensive empirical investigation. Bollerslev, Engle, and Wooldridge (1988), French, Schwert, and Stambaugh (1987), along with many others use ARCH-in-mean specifications for returns to relate risk premiums to conditional second moments. Much of this effort is directed towards measurement and interpretation of simple monotonic relationships between risk premiums and conditional second moments, often between the risk premium on a single asset and its own

[7] As an additional accuracy check, we computed the equilibrium price dividend ratios for the parameter settings shown in Table IV for the case of logarithmic utility, $\gamma = 1.0$. In every instance, the program computed the correct price dividend ratio, which is $\beta/(1-\beta)$, to within machine accuracy.

conditional variance. The existence of such relationships, though, has been contested both empirically (Pagan and Hong (1990)) and theoretically (Backus and Gregory (1988)). This debate is perhaps not surprising, given that equilibrium asset pricing models relate the conditional means of asset returns to generalized notions of a marginal rate of substitution (Hansen and Jagannathan (1989)), and not directly to their conditional second moment structure.

In this subsection we explore further the nature of the relationship between the risk premium on an equity asset and its conditional variance. Our strategy is to use the quadrature method to solve a small scale equilibrium asset pricing model. We then examine how the risk premium covaries with the conditional variability of the equity return in the model economy under various assumptions about risk aversion.

The analysis differs in several respects from that of Abel (1988) and Backus and Gregory (1988), primarily in the way we calibrate and solve the asset pricing model. Abel (1988) shows that the asset pricing model admits an exact solution under special assumptions about the law of motion of the endowment process, but does not actually undertake estimation and calibration. Backus and Gregory (1988) employ a tightly parameterized small-state Markov model for the consumption endowment but do not directly link it to a fully specified law of motion for the endowment process. Our approach is to specify and estimate an actual time-series model with conditional heteroskedasticity for the consumption endowment. We use the quadrature method to find an approximating Markov chain for the estimated endowment process, and we solve the asset pricing equations that underlie the discrete approximation.

We consider asset pricing relationships for a one-good, single-agent endowment economy where the law of motion of the endowment process $\{c_t\}$ is

$$\ln(c_{t+1}/c_t) = b + a\ln(c_t/c_{t-1}) + u_{t+1}, \quad u_{t+1} \sim N(0, h_{t+1}),$$
$$h_{t+1} = \alpha_0 + \alpha_1 u_t^2.$$

This is the basic ARCH model of Engle (1982) with one lag each in the mean and variance equations. We calibrated this model using annual data, 1889–1983, taking as $\{c_t\}$ annual observations on the consumption of nondurables and services.[8] Point estimates and standard errors are

$$\hat{b} = \underset{(0.004)}{0.023}, \qquad \hat{a} = \underset{(0.099)}{-0.298}, \qquad \bar{R}^2 = 0.08,$$

$$\hat{\alpha}_0 = \underset{(0.00027)}{0.00086}, \qquad \hat{\alpha}_1 = \underset{(0.099)}{0.287}, \qquad \bar{R}^2 = 0.08.$$

The estimates were obtained by using simple regression to fit the AR(1) model for $\ln(c_t/c_{t-1})$ and by using simple regression to fit an AR(1) model to the squared residuals. These estimates are consistent but not necessarily fully efficient method of moments estimates. Additional analysis using longer lag

[8] The data series is from Muoio (1988), who constructed the series from data in Kuznets (1961) and from the National Income and Product Accounts.

lengths uncovered no evidence for lags extending beyond the first. In particular, when an extra lag is introduced in the mean equation, the $\bar{R}^2$ only increases to 0.09; when the same is done in the variance equation, the $\bar{R}^2$ remains essentially unchanged.

The asset pricing structure is as follows. Let $v_t = p_t/c_t$ denote the price dividend ratio at time t on the equity asset that pays the stochastic dividend stream $\{c_{t+j}\}_{j=1}^{\infty}$. We employ the CRR utility function, $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$, which is commonly used in this debate. The asset pricing equation is

$$v_t = \beta E_t\left[(1 + v_{t+1})\left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma}\right],$$

where $E_t[\cdot]$ denotes conditional expectations given time t information calculated from the ARCH model for the endowment. The gross one-period return on the equity asset is $r_{e,t+1} = [(1+v_{t+1})/v_t](c_{t+1}/c_t)$. The gross one-period return on a conditionally risk free pure discount one-period bond is $r_{f,t+1} = [\beta E_t(c_{t+1}/c_t)]^{-1}$, which is observed by the agent in period t. Throughout, we define the risk premium on the equity to be the random variable $E_t[r_{e,t+1}] - r_{f,t+1}$ and the equity premium to be the constant $E[r_{e,t+1}] - E[r_{f,t+1}]$.

Some preliminary analytical evidence on the characteristics of this model can be obtained by considering the special case of logarithmic utility ($\gamma = 1$), in which case the model admits an exact solution. Using methods similar to those of Abel (1988), one can show that whenever the endowment growth c_{t+1}/c_t is lognormally distributed conditional on time t information, then, if $\gamma = 1$, the following relation holds:

$$\frac{E_t[r_{e,t+1}]}{r_{f,t+1}} = 1 + \xi_{t+1}^2$$

where ξ_{t+1} is the coefficient of variation (standard error divided by the mean) of the conditional distribution of $r_{e,t+1}$. In this case there will be a positive risk premium, which will be time varying if $\ln(c_{t+1}/c_t)$ displays conditional heteroskedasticity as was found to be the case in the empirical work reported above.

In the general case, $\gamma \neq 1$, no known analytical solution is available and we use the quadrature method outlined in Sections 3 and 4. We calibrate a discrete economy with endowment dynamics defined by a Markov chain that approximates the law of motion, $f(y_t | y_{t-1}, y_{t-2})$, implied by the fitted ARCH model for endowment growth. The weighting function is $\omega(y) = f(y|0)$ and the quadrature rule is an 8-point Gauss-Hermite rule. Because of the ARCH error structure, the lag length is two, and with an 8-point rule the number of discrete states is 64.

Table V shows, for various values of γ, the population unconditional means of the equity return and the conditionally risk-free return in the discrete economy. For these calculations the value $\beta = 0.97$ is used, though additional calculations showed the conclusions are quite insensitive to the choice of β.

TABLE V

COVARIANCE STRUCTURE BETWEEN THE EQUITY PREMIUM AND THE CONDITIONAL VARIANCE OF THE EQUITY RETURN

γ	$E[r_{e,t+1}]$	$E[r_{f,t+1}]$	Intercept	Slope	ρ^a
			Fitted ARCH Model		
0.1	1.0329	1.0328	−0.0001	0.0077	0.9943
1.0	1.0502	1.0489	−0.0015	0.0781	0.9936
2.0	1.0692	1.0661	−0.0038	0.1560	0.9920
3.0	1.0879	1.0824	−0.0067	0.2313	0.9895
3.5	1.0971	1.0902	−0.0084	0.2674	0.9879
4.0	1.1063	1.0978	−0.0101	0.3022	0.9862
4.5	1.1153	1.1051	−0.0119	0.3355	0.9843
5.0	1.1243	1.1122	−0.0138	0.3673	0.9824
			Sign Reversed AR(1) Coefficient		
0.1	1.0329	1.0328	−0.0002	0.0079	0.9938
1.0	1.0502	1.0489	−0.0015	0.0781	0.9936
2.0	1.0677	1.0661	−0.0018	0.1504	0.9930
3.0	1.0833	1.0824	−0.0008	0.1879	0.9914
3.5	1.0905	1.0902	−0.0003	0.1251	0.9663
4.0	1.0971	1.0978	0.0000	−0.0908	−0.8370
4.5	1.1033	1.1051	0.0016	−0.2633	−0.9774
5.0	1.1089	1.1122	0.0034	−0.3335	−0.9740

[a] $\rho = \mathrm{Corr}(E_t[r_{e,t+1} - r_{f,t+1}], \sigma_t(r_{e,t+1}))$.

Also shown in the table are the slope and intercept from the population linear regression of the risk premium, $E_t[r_{e,t+1}] - r_{f,t+1}$, on the conditional standard error, $\sigma_t(r_{e,t+1})$, of the equity return, along with the correlation coefficient between these two random variables. The conditional standard error is used instead of the conditional variance only to achieve better scaling of the calculations. Note that both the risk premium and conditional standard deviation are exact functions of the state of the system at time t, and thus in general there is an exact relationship between these two variables. The linear regression should therefore be interpreted as an $\mathscr{L}_2$ approximation to this relationship and is reported as a convenient summary measure. Inspection of the correlation coefficients shows that the linear approximation is quite close.

Sensitivity checks with 4-point and 12-point rules revealed that the selected 8-point rule generally yields close to four digit accuracy in Table V, which is the conventional standard for numerical work. This includes the measurement of the equity premium, as the approximation errors for the mean returns are strongly positively correlated, and hence cancel in the calculation of the equity premium. Exceptions are the calculated intercepts and slopes in the bottom panel in the region $\gamma \in [3.0, 4.0]$, where the sign of the slope coefficient changes abruptly as γ increases. Here the accuracy drops to roughly two digits, which is still more than adequate.[9]

[9] The total computational time required to compute a typical row in the table—including calculation of the Markov chain, solution of the asset pricing equations, and computation of the relevant population statistics—is 18.9 seconds using Gauss 1.49b on a Compaq 386-25.

The top panel in the table shows the population statistics computed for a discrete economy with the Markov chain model calibrated directly from the fitted ARCH model for the endowment process. These endowment dynamics are arguably realistic, but, not surprisingly in view of Mehra and Prescott (1985), the model generates too small of an equity premium. At the same time, though, the model does predict a tight and monotonic increasing relationship between the risk premium and the conditional variability of the equity return over the wide range of values considered for the risk aversion parameter γ.

The bottom panel shows the same statistics computed from a model economy calibrated with the same ARCH specification for the endowment process except that the sign is reversed on the autoregressive parameter in the mean equation and the constant is adjusted to retain the same unconditional mean of $\ln(c_{t+1}/c_t)$. This model economy thus differs from the previous one only in that $\ln(c_{t+1}/c_t)$ is positively autocorrelated instead of negatively autocorrelated. As one should expect, for γ near unity the predictions from this second model economy are very similar to those from the first economy. Interestingly, though, this second economy can generate a negative relationship between the risk premium and conditional variability, which confirms Backus and Gregory (1988). But it does so only at relatively high values of the risk aversion parameter and simultaneously predicts a negative equity premium.

Some interesting conclusions emerge from our analysis. When the underlying endowment dynamics are calibrated from actual data with a model that allows for conditional heteroskedasticity, then the equilibrium asset pricing model predicts a positive relationship between the risk premium and conditional volatility. Furthermore, the slope of the relationship becomes larger as risk aversion increases. As in Backus and Gregory (1988), though, we can find alternative configurations of the parameters where the model will predict a negative relationship. But in the alternative configurations the autocorrelation structure of $\ln(c_{t+1}/c_t)$ is counterfactual, the risk aversion parameter is relatively high, and a negative equity premium is generated. This finding adds some theoretical support for empirically based efforts directed at measuring the relationship between risk premia and conditional second moments, though some caution is needed in interpreting the finding. The model's conditional moments are computed given current and lagged consumption, which in the context of the model is an information set that subsumes all relevant current information. In empirical work, on the other hand, the moments are typically computed using an information set comprised of variables observed by the econometrician, e.g., current and lagged returns, which is presumably a smaller information set than that actually used by economic agents.

6. CONCLUSION

This paper has developed a discrete method for finding approximate solutions to a class of nonlinear rational expectations models. The class of models is wide enough to include many asset pricing models and some interesting monetary models. Excluded models are those with endogenous state variables where the dependence of the exogenous variables implicitly extends into the infinite past. These models require more elaborate solution methods, though, as noted in the Introduction, the Markov chain approximation method developed here is a potentially useful component of such a method.

In our main application we employ the discretization technique for the purpose of analyzing the covariation between the risk premium on a financial asset and the variability of its return. In particular, we estimate a time series model with conditional heteroskedasticity for annual per capita consumption. This fitted time series model is taken as input to the discretization technique to make calibration of the Markov chain realistic. We then use the solution method to solve the asset pricing equations and thereby calculate risk premiums and related statistics. With this calibration of the Markov chain, we find a positive relationship between the risk premium of the equity return and its conditional variability given current and lagged consumption, which is the single driving variable of the model. In addition, the relationship becomes stronger at higher values of the risk aversion parameter. The findings help buttress the motivation for empirical efforts aimed at relating risk premiums to conditional second moments, which has been a topic of contention. Subsequent research should investigate the sensitivity of the findings to the use of a single-shock endowment economy. It should also investigate the covariance structure of conditional moments computed from past returns only, as these are the conditional moments commonly utilized in empirical work. Quadrature-based methods could be used for this work.

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APPENDIX A

Approximation for Finite-Memory Conditional Densities

We now consider the most general case where the law of motion for the state vector is a finite memory Markov process with conditional density $f(y_t|y_{t-1}, y_{t-2}, \ldots, y_{t-L})$ defined on $\mathbb{R}^M$. The most general case is treated as an extension of the first-order multivariate case. This is done despite the fact that by suitably defining a new state vector, a system with lag length L can always be re-expressed as a first order system, and the first order system was handled in Section 3.2. The reason for not defining a new state vector is that this device produces no gain here. The device produces a new system with a singular conditional density concentrated on a lower dimensional space. Development of a quadrature rule that properly accounts for the singularity is very awkward and notationally cumbersome. The best strategy is not to take the step of reducing the system's lag length to unity in the conventional manner, but rather just to leave the system in general form with L lags.

From (2.4) of Section 2 the most general integral equation is

$$v(x) = \int [1 + v(y, x^-)]\psi(y, x) f(y|x)\, dy \tag{A.1}$$

where x is an $M \cdot L$ vector that corresponds to $(y_{t-1}, y_{t-2}, \ldots, y_{t-L})$, x^- is an $M \cdot (L-1)$ vector that corresponds to $(y_{t-1}, y_{t-2}, \ldots, y_{t-L+1})$, and the rest of the notation is given after (2.4) in Section 2.

Write the integral in (A.1) as an integral against a weighting density $\omega(y)$:

$$v(x) = \int [1 + v(y, x^-)] \psi(y, x) \frac{f(y|x)}{\omega(y)} \omega(y)\, dy. \tag{A.2}$$

Let $\bar{y}_k$ and w_k denote the abscissa and weights for an N-point quadrature rule on $\mathbb{R}^M$ for the density $\omega(y)$. Define

$$\alpha_k(x) = \frac{f(\bar{y}_k|x)}{\omega(\bar{y}_k)} w_k \qquad (k = 1, 2, \ldots, N), \tag{A.3}$$

and

$$\pi_k^N(x) = \frac{\alpha_k(x)}{s(x)} \qquad (k = 1, 2, \ldots, N), \tag{A.4}$$

where $s(x) = \sum_{k=1}^N \alpha_k(x)$.

The discretization of the integral equation (A.2) is as follows. Let

$$y_s^* = (\bar{y}'_{j_1}, \bar{y}'_{j_2}, \ldots, \bar{y}'_{j_L})',$$

where the range of the indices are

$$1 \leqslant j_i \leqslant N, \quad i = 1, 2, \ldots, L,$$

$$1 \leqslant s \leqslant N^*, \quad N^* = N^L.$$

Here $\bar{y}_s^*$, which is of dimension $M \cdot L$, is an arrangement of L possible choices of the $\bar{y}_k$. There are a total of $N^* = N^L$ such arrangements for the y_s^* and it is understood that there is an invertible labeling scheme,

$$s = s(j_1, j_2, \ldots, j_L), \tag{A.5}$$

that maps the subscripts $(j_1, j_2, \ldots, j_L)$ into $s \in \{1, 2, \ldots, N^*\}$.

Now put

$$\psi_{sk} = \psi(\bar{y}_k, y_s^*), \tag{A.6}$$

$$\pi_{sk}^N = \pi_k^N(y_s^*), \tag{A.7}$$

for $k = 1, 2, \ldots, N$ and $s = 1, 2, \ldots, N^*$. Then the $\bar{v}_{Ns}$ in the discretization are given by the solution to the N^* linear equations

$$\bar{v}_{Ns} = \sum_{k=1}^{N} \left[1 + \bar{v}_{N, \sigma(s,k)}\right] \psi_{sk} \pi_{sk}^N \qquad (s = 1, 2, \ldots, N^*), \tag{A.8}$$

where $\sigma(s, k)$ selects the state label from (A.5) that corresponds to the arrangement of subscripts $(k, j_1, j_2, \ldots, j_{L-1})$. In the asset pricing example, $\bar{v}_{Ns}$ is the price dividend ratio when the (discrete) state is $(\bar{y}_{j_1}, \bar{y}_{j_2}, \ldots, \bar{y}_{j_L})$, and $\bar{v}_{N, \sigma(s,k)}$ is the price dividend ratio when the next state is $(\bar{y}_k, \bar{y}_{j_1}, \ldots, \bar{y}_{j_{L-1}})$. The reason for introducing the selection function $\sigma(s, k)$ becomes apparent when one observes that in VAR models where $(y_{t-1}, \ldots, y_{t-L})$ is viewed as the state of the system as of period $t-1$, then the only possible states of the system in period t are of the form $(y_t, y_{t-1}, \ldots, y_{t-L+1})$.

APPENDIX B

PROOF OF THEOREM 4.2: Define the function ρ_N on $[a,b]$ by

$$\rho_N(x) = \sum_{k=1}^{N} \frac{g(\bar{y}_{Nk})f(\bar{y}_{Nk}|x)}{\omega(\bar{y}_{Nk})} w_{Nk}.$$

For each fixed x, the assumptions imply $g(y)f(y|x)/\omega(y)$ is bounded and continuous in y, and $\rho_N(x) = Q_N(g(\cdot)f(\cdot|x)/\omega(\cdot))$. By the assumption on the quadrature rule, $\rho_N(x) \to e_g(x)$ pointwise in x. The assumptions also imply that the ρ_N are uniformly bounded. Furthermore, for $x_1, x_2 \in [a,b]$,

$$|\rho_N(x_2) - \rho_N(x_1)| \leqslant c^{-1}\|g\| \sum_{\iota=1}^{N} w_{N\iota}|f(\bar{y}_{N\iota}|x_2) - f(\bar{y}_{N\iota}|x_1)|$$

and equicontinuity of the ρ_N thereby follows from the (uniform) continuity of $f(y|x)$, together with $\sum_{\iota=1}^{N} w_{N\iota} = 1$. Hence, $\rho_N \to e_g$ uniformly. By a similar argument with g replaced by the identity function, $s_N \to 1$ uniformly. Hence, $e_{gN} = \rho_N/s_N \to e_g$ uniformly. *Q.E.D.*

PROOF OF THEOREM 4.3: We need to establish the Anselone-Moore conditions stated as A1, A2, and A3 of Atkinson (1976, p. 96). Condition A1 is automatically satisfied because $C_0[a,b]$ is a Banach space. Condition A2 requires that $T_N[g] \to T[g]$ for each $g \in C_0[a,b]$, which follows from Theorem 4.2 above. Condition A3 requires that $\{T_N\}$ be collectively compact; specifically, the set $\{T_N(g)\colon \|g\| \leqslant 1\}$ must be precompact in $C_0[a,b]$. To show this, note that

$$T_N[g](x) = \sum_{k=1}^{N} g(\bar{y}_k) \frac{\psi(\bar{y}_k, x)f(\bar{y}_k|x)w_{Nk}}{\omega(\bar{y}_k)s_N(x)}$$

is uniformly bounded because ψ and f are bounded from above, ω is bounded from below, and, from the argument used to prove Theorem 4.2, $s_N \to 1$ uniformly. Furthermore, the continuity of $f(y|x)$ implies that $s_N(x)$ is equicontinuous in x and this, together with the continuity of $\psi(y,x)f(y|x)$, implies that $T_N[g](x)$ is equicontinuous in x. Hence, by the Arzela-Ascoli Theorem (Wouk (1979, p. 83)) the set $\{T_N(g)\colon \|g\| \leqslant 1\}$ is precompact and so $\{T_N\}$ is collectively compact.

Given that the sequence of operators $\{T_N\}$ satisfies the Anselone-Moore conditions, then conclusion (1) of the theorem holds by Atkinson (1976, Theorem 4, pp. 97–98). Thus for sufficiently large N, $(I - T_N)^{-1}$ exists and

$$\begin{aligned} \text{(B.1)} \qquad \|\tilde{v}_N - v\| &= \left\|(I - T_N)^{-1}T_N[1] - (I - T)^{-1}T[1]\right\| \\ &\leqslant \left\|(I - T_N)^{-1}T[1] - (I - T)^{-1}T[1]\right\| + \left\|(I - T_N)^{-1}(T_N[1] - T[1])\right\|, \end{aligned}$$

by the triangle inequality. Now put

$$\text{(B.2)} \qquad \varepsilon_N = \left\|(I - T)^{-1}\right\| \|(T_N - T)T_N\|,$$

$$\text{(B.3)} \qquad \eta_N = \left\|(I - T)^{-1}\right\| \|T_N - T\|.$$

By Atkinson (1976, Lemma 4, p. 96), $\varepsilon_N \to 0$ and $\eta_N \to 0$. Furthermore, by applying displays (3.26)–(3.28) of Atkinson's Theorem 4 (pp. 96–97) to each of the two terms on the right side of the inequality (B.1), we get

$$\text{(B.4)} \qquad \|\tilde{v}_N - v\| \leqslant \frac{\varepsilon_N\|v\| + \eta_N\|T\|}{1 - \varepsilon_N} + \frac{\eta_N\left\|(I - T)^{-1}\right\|^{-1} + \eta_N\|T_N\|}{1 - \varepsilon_N}.$$

The boundedness of the sequence of numbers $\{\|T_N\|\}$ was established in the proof of the Anselone-Moore conditions, and hence $\|\tilde{v}_N - v\| \to 0$. *Q.E.D.*

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