Stochastic Volatility in General Equilibrium

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Abstract

The connections between stock market volatility and returns are studied within the context of a general equilibrium framework. The framework rules out *a priori* any purely statistical relationship between volatility and returns by imposing uncorrelated innovations. The main model generates a two-factor structure for stock market volatility along with time-varying risk premiums on consumption and volatility risk. It also generates endogenously a dynamic leverage effect (volatility asymmetry), the sign of which depends upon the magnitudes of the risk aversion and the intertemporal elasticity of substitution parameters.

Keywords: stochastic volatility, risk aversion, leverage effect, volatility asymmetry

JEL classification: G12, C51, C52.

*Prepared for presentation at the "Conference on Financial Econometrics" in honor of Rob Engle's Nobel Prize and presented at several meetings and conferences. The paper reflects an effort to understand better the underlying economics of stochastic volatility, risk, return, time varying risk premiums, and the volatility risk premium. I have benefitted from many, many, helpful discussions with Ravi Bansal, and thanks also go to Ivan Shaliastovich and Kenneth Singleton for helpful comments. Earlier versions of the manuscript circulated on the web, and far more complicated but much less intuitive quantitative extensions are part of Bollerslev, Tauchen, and Zhou (2009) and Bollerslev, Sizova and Tauchen (2011). Author's Contact: George Tauchen, Department of Economics, Duke University, Box 90097, Durham NC 27708, E-mail george.tauchen@duke.edu.

1 Introduction

Observers of financial markets have long noted that the volatility of financial price movements varies stochastically. The well-established ARCH and GARCH models of Engle (1982) and Bollerslev (1986) and the plethora of descendants provide a very convenient framework for empirical modelling of volatility dynamics. A somewhat different, but essentially equivalent way to model the same phenomenon is to think in terms of the latent stochastic volatility model introduced in embryonic form by Clark (1973), extended and formalized in Taylor (1982, 1986), and studied extensively in the vast literature that follows (See Ghysels et al. (1996) and Shephard (2005)). Financial market empiricists now know that time varying stochastic volatility can account for much of the dynamics of short-term financial price movements.

The empirical volatility literature has proceeded largely in a reduced form statistical manner, with only minimal guidance from economic theory. The role of theory has mainly been to identify important markets and sometimes to provide informal intuitive interpretation of empirical regularities.

This paper studies stock market volatility within a self-contained general equilibrium framework. The economy is a familiar endowment economy with a preference structure assumed to be that of Epstein and Zin (1989) and Weil (1989). The paper is an extension of Bansal et al. (2005), Bansal and Yaron (2004), Campbell and Hentschell (1992), and Campbell (2003). As in those papers, the log-linearization methods of Campbell and Shiller (1988) are used to derive qualitative predictions and gain further insights into the implications of the models under consideration. In this paper, however, the volatility dynamics are more complicated. The paper proceeds through a sequence of models, with each extension motivated by a desire to explore more fully the relationship between stock market returns and volatility. The models, and in particular the most general model considered in Sections 3–4 below, yield some interesting insights that can account for known characteristics of stock market volatility.

For instance, empirical researchers have long known that stock market returns and stock market volatility are negatively correlated. Black (1976) is perhaps the first to call attention to this empirical regularity and attributes it to changing financial leverage associated with equity prices changes, as further studied by Christie (1982). The asymmetric effect has thus been termed the *leverage effect*, and Nelson (1991) highlights its importance by formally building the asymmetry into the E-GARCH model.

Although the term leverage effect, or simply leverage, is the common expression in the econometrics and stochastic volatility literatures, few, if any, economists are comfortable with the original explanation and believe the leverage effect has more to do with risk premiums than balance sheet leverage.

Economic explanations for the leverage effect such as French et al. (1987) employ intuitive traditional CAPM-type reasoning with a presumption that volatility carries a positive risk premium. A related approach Campbell and Hentschell (1992) uses the Merton (1973) model to connect expected returns to volatility along with a GARCH-type model for the evolution of volatility. Bekaert and Wu (2000) provide a comprehensive review of these explanations and a very convenient reduced-form setup for empirical analysis of volatility asymmetry relationships. Wu (2001) develops a self-contained equilibrium model, along with carefully executed empirical work, but the model relies on a pre-specified pricing kernel not directly connected to marginal utility. Also Wu's model permits correlations between cash flow innovations and their volatilities that provide a statistical channel for a leverage effect separate from any economic channels.

The main model developed and analyzed in Section 3–4 indicates that the existence and sign of the leverage effect depend critically on the values of two key economic parameters, the coefficient of risk aversion and the intertemporal elasticity of substitution. In the case of expected utility, these parameters are reciprocals of each other, and the model predicts no leverage effect at all in this case. Thus, the now well-established empirical finding of a negative leverage effect — which is reconfirmed in Figures 1 and Figures 2 below — strongly discredits the expected utility paradigm. Furthermore, economists generally agree that the coefficient of risk aversion exceeds unity; if so, the predicted sign of the leverage effect depends critically on the location of the intertemporal elasticity of substitution relative to unity. If this elasticity parameter exceeds unity, then the leverage effect is negative — exactly as observed in the data. On the other hand, if it is below unity, then the sign of the leverage effect is positive, in direct contrast to empirical findings. The issue of whether the intertemporal elasticity of substitution is below or above unity is contentious; Bansal and Yaron (2005) give details on the debate. Since the negative relationship has been so well documented, e.g., Bekaert and Wu (2000), the findings from reduced-form modeling of asymmetric volatility thereby have sharp consequences for an economic debate regarding the magnitude of a key utility parameter. In addition, the model can also explain the dynamic leverage effect, i.e., the pattern of serial cross correlations between stock market movements and volatility at different leads and lags as documented empirically in Bollerslev et al. (2006).

The model can also account for the empirical finding that stock market volatility appears to follow a two factor structure, with one slowly evolving component and one quickly mean reverting component. Engle and Lee (1999), Gallant, Hsu, and Tauchen (1999), and Alizadeh, Brandt, and Diebold (2002), among many others, adduce evidence on this empirical regularity. The two factor structure emerges naturally from the internal structure of the model.

Finally, the model appears useful for sorting out issues related to time varying risk prices and a volatility risk premium. A common presumption is that increased stock market volatility is associated with increased expected stock market returns. This reasoning is intuitively plausible — riskier investments should demand a higher expected return relative to cash — and a rigorous analysis is Merton (1973). Various expositions of the Merton model appear in the literature, and a convenient summary with easy to interpret log-linear approximations is in Campbell, Lo, and Mackinlay (1997, pp. 291—334). An early effort to model and detect empirically the return-volatility relationship is Engle *et al* (1987), who propose the GARCH-M model for bond returns. The follow-up literature from Nelson (1991) onwards is huge, but, as is well known, the effort to detect an empirical relationship between expected stock returns and volatility has yielded weak and mixed results. Recent efforts, such as Ghysels, Santa-Clara, and Valkanov (2005) and Lundblad (2004) employ more powerful techniques to present evidence for a statistically significant positive risk-return relationship, which has been here-to-for quite difficult to detect. However, as will be seen in Section 4, such efforts are detecting the confounding of time-varying risk premiums on consumption and volatility risk, which clouds the interpretation of the empirical evidence. Scruggs (1998), Guo and Whitelaw (2003), and Guo, Savickas, Wang, and Yang (2005) present evidence that additional factors are needed in the return-volatility equation in order to measure volatility risk reliably, and the main model below suggests the underlying variable for which these factors are likely proxies.

The rest of this paper is organized as follows: Section 2 presents the notatation and two initial models that are useful for understanding the basic structure and ideas. Section 3 sets forth the main model, and Section 4 connects the predictions from that model to the empirical stochastic volatility literature. Section 5 contains the concluding remarks.

2 Setup, Notation, and Two Initial Models

2.1 The MRS and Asset Pricing

Let $\mathcal{M}_{t+1}$ denote the marginal rate of substitution process (MRS), also sometimes termed the stochastic discount factor (SDF), between t and $t+1$, and let R_{t+1} denote the gross return on an asset. The fundamental asset pricing relationship is

$$\mathrm{E}_t\left(\mathcal{M}_{t+1}R_{t+1}\right) = 1. \tag{1}$$

Throughout, we shall work under a conditional lognormality assumption. Let $m_{t+1} = \log(\mathcal{M}_{t+1})$, and let $r_{t+1} = \log(R_{t+1})$ denote the geometric return on the asset. The fundamental asset pricing relationship is then

$$\log\left[\mathrm{E}_t\left(e^{m_{t+1}+r_{t+1}}\right)\right] = 0. \tag{2}$$

We start by working through in this section two models that illustrate the main points about time varying a risk premium on consumption versus a volatility risk premium. We then proceed to the main model in Section 3 below.

2.2 CRR Preferences and Stochastic Volatility

Under constant relative risk aversion (CRR) preferences $\mathcal{M}_{t+1} = \beta(C_{t+1}/C_t)^{-\gamma}$, where C_t is real consumption and β and γ are parameters. Equivalently,

$$m_{t+1} = \delta - \gamma g_{t+1}, \tag{3}$$

where $\delta = \log\beta$ and

$$g_{t+1} = c_{t+1} - c_t, \quad c_t = \log(C_t),$$

so g_{t+1} is the geometric growth rate of consumption. Assume the dynamics of g_{t+1} are

$$\begin{aligned} g_{t+1} &= \mu_g + \sigma_{ct}z_{c,t+1} \\ \sigma^2_{c,t+1} &= a_{\sigma c} + \rho_{\sigma c}\sigma^2_{ct} + \phi_{\sigma c}\sigma_{ct}z_{\sigma,t+1} \end{aligned} \tag{4}$$

where σ_{ct} represents stochastic volatility in consumption that is observed by agents but not by the econometrician, and $z_{c,t+1}$ and $z_{\sigma,t+1}$ are iid $\mathrm{N}(0,1)$ random variables. The above volatility dynamics are not quite the same as those of Bansal and Yaron (2004) and Bansal, Khatchatrian, and Yaron (2003), who have a constant multiplying $z_{\sigma,t+1}$, as do and Brenner, Ou, and Zhou (2004) in their continuous time version of a setup like (4). These other papers use Gaussian volatility dynamics while (4) is a square-root, or CIR, type specification. There are certain consequences to the alternative specifications as discussed farther below. In simulations, of course, care is needed to include an additional reflecting barrier at a small positive number to ensure positivity of simulated $\sigma_{c,t+1}^2$. The above dynamics for volatility are similar to those of Bollerslev and Zhou (2003) for their continuous time assessment of the relationship between the expected stock return volatility relationship.

Let $v_t = \log(P_t/C_t)$ denote log of the price-dividend ratio of the asset that pays the consumption endowment $\{C_{t+j}\}_{j=1}^{\infty}$. Let

$$r_{t+1} = \log\left(\frac{P_{t+1} + C_{t+1}}{P_t}\right) \tag{5}$$

denote the geometric return (hereafter just called the return). The standard Campbell-Shiller (1988) log-linearization is

$$r_{t+1} = k_0 + k_1 v_{t+1} - v_t + g_{t+1} \tag{6}$$

where $k_1 < 1$, $k_1 \approx 1$, is a positive constant. The strategy to solve models of this sort is to conjecture a solution for v_t as a function of the state variables, use the approximation immediately above, impose the fundamental asset pricing equation, and then solve for the coefficients of the conjectured solution.

In this case we conjecture

$$v_t = A_0 + A_\sigma \sigma_{ct}^2 \tag{7}$$

and the solutions for A_0, A_σ are given in Subsection 6.1 of the Appendix. From the solution one can easily derive the familiar relationship for the expected excess return

$$\mathrm{E}_t(r_{t+1}) - r_{ft} = \gamma\sigma_{ct}^2 - \frac{1}{2}\sigma_{rt}^2 \tag{8}$$

where r_{ft} is the riskless rate in geometric form and $\frac{1}{2}\sigma_{rt}^2 = \frac{1}{2}\mathrm{Var}_t(r_{t+1})$ is a geometric adjustment term, also called a *Jensen's Inequality* adjustment (Campbell, Lo, and Mackinlay, 1997, p. 307). The risk premium is thus $\gamma\sigma_{ct}^2$, and, ignoring the geometric adjustment, one can write

$$r_{t+1} - r_{ft} = \alpha + \gamma\sigma_{ct}^2 + \epsilon_{t+1} \tag{9}$$

where α is an intercept and ϵ_{t+1} is a heteroskedastic error term. This expression is the elusive risk-return relationship that has been sought after by Nelson (1991), Lundblad (2004), and Ghysels *el al* (2004), among others.

One has to be very careful on how to interpret the risk premium in (9), however. It is actually a time-varying risk premium on consumption risk, with a variable coefficient that is attributable to the stochastic volatility; this interpretation is emphasized by Bansal and Yaron (2004) and less directly in (Campbell, Lo, and Mackinlay, 1997, p. 307). The fact that stochastic volatility generates time-varying risk premium on other factors (here consumption risk) appears first to have made formal in an econometric sense by Engle *et al* (1987), who use a GARCH-in-mean model to study the risk premium in bond returns.

It proves interesting to examine why (9) does not reflect a volatility risk premium. In this model, any return that depends only on the volatility innovation $z_{\sigma,t+1}$ carries no risk premium, despite the fact that the volatility innovation $z_{\sigma,t+1}$ has an impact on the return; one can easily show that

$$r_{t+1} - \mathrm{E}_t\left(r_{t+1}\right) = \sigma_{ct} z_{c,t+1} + k_1 A_\sigma \phi_{\sigma c} \sigma_{ct} z_{\sigma,t+1} \tag{10}$$

so the volatility innovation $z_{\sigma,t+1}$ affects the return, and possibly substantially. Nonetheless, an arithmetic return that is a pure volatility bet such as

$$R_{\sigma,t+1} = \exp\left(r_{ft} - \frac{1}{2}\varsigma_t^2 + \varsigma_t z_{\sigma,t+1}\right), \tag{11}$$

where ς_t is a constant known at time t, satisfies

$$\mathrm{E}_t\left(R_{\sigma,t+1}\right) = R_{ft}, \tag{12}$$

where $R_{ft} = e^{r_{ft}}$; i.e., $R_{\sigma,t+1}$ carries no risk premium. Also, if $C(\sigma^2_{c,t+1})$ is a cash flow realized at $t+1$ that only depends upon $\sigma^2_{c,t+1}$, then the price (present value) of that cash flow satisfies

$$\mathrm{E}_t\left[\mathcal{M}_{t+1} C(\sigma^2_{c,t+1})\right] = \frac{\mathrm{E}\left[C(\sigma^2_{c,t+1})\right]}{R_{ft}}. \tag{13}$$

There is no reward for bearing volatility risk because that risk is uncorrelated with the MRS process due to the assumption that $z_{c,t+1}$ and $z_{\sigma,t+1}$ are uncorrelated. Of course one could always generate a volatility risk premium by simply correlating $z_{c,t+1}$ and $z_{\sigma,t+1}$, but that seems ad hoc and economically unsatisfactory.

2.3 Epstein-Zin-Weil Preferences and Stochastic Volatility

Bansal and Yaron (2004) and Bansal, Khatchatrian, and Yaron (2003) note that Epstein-Zin-Weil preferences can actually induce an endogenous volatility risk premium. We start with a simplified version of their setups, point out some problems, and then proceed to a more general version in the next section.

Write the log of the marginal rate of substitution as

$$m_{t+1} = b_{m0} + b_{mg} g_{t+1} + b_{mr} r_{t+1} \tag{14}$$

and note that under Epstein-Zin-Weil preferences

$$\begin{aligned} b_{m0} &= \theta \log(\delta) \\ b_{mg} &= -\theta/\psi \\ b_{mr} &= \theta - 1 \end{aligned} \tag{15}$$

where

$$\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}. \tag{16}$$

The parameter γ is the risk aversion parameter; ψ is the coefficient of intertemporal substitution, and δ the subjective discount factor. If $\theta = 1$ then these preferences reduce to the CRR preferences studied above. We retain the same dynamics (4) for consumption growth and volatility. The primary differences between this setup and that of Bansal and Yaron (2004) are that the above entails square-root volatility dynamics instead of the simpler Gaussian dynamics but it excludes the long run risk factor in the consumption growth equation. That factor is excluded only for simplification to concentrate attention on the role of volatility.

The return on the consumption endowment r_{t+1} that appears in the expression for the log of the marginal rate of substitution (14) has to be solved for endogenously. As before, first conjecture a solution for log price-consumption ratio

$$v_t = A_0 + A_\sigma \sigma_{ct}^2. \tag{17}$$

Subsection 6.2 of the Appendix contains the derivation of A_0 and A_σ along with the reduced form expressions for r_{t+1} and m_{t+1}.

From these expressions one can deduce the expression for the expected excess return

$$\mathrm{E}_t(r_{t+1}) - r_{ft} = -(b_{mr} + b_{mg})\sigma_{ct}^2 - b_{mr} k_1^2 A_\sigma^2 \phi_{\sigma c}^2 \sigma_{ct}^2 - \frac{1}{2}\sigma_{rt}^2 \tag{18}$$

which in the case of Epstein-Zin-Weil preferences reduces to

$$\mathrm{E}_t(r_{t+1}) - r_{ft} = \gamma \sigma_{ct}^2 + (1-\theta) k_1^2 A_\sigma^2 \phi_{\sigma c}^2 \sigma_{ct}^2 - \frac{1}{2}\sigma_{rt}^2 \tag{19}$$

where again $-\frac{1}{2}\sigma_{rt}^2$ is the geometric adjustment term. The expected excess return

$$\gamma \sigma_{ct}^2 + (1-\theta) k_1^2 A_\sigma^2 \phi_{\sigma c}^2 \sigma_{ct}^2 \tag{20}$$

is composed of two terms. The first term represents the familiar time varying risk premium on consumption risk, while the second represents the risk premium on volatility. The volatility risk premium is generated endogenously via the structure of the preferences, and, in fact, is absent in the CRR case where $\theta = 1$. However, both risk premiums are multiples of the same stochastic process, σ^2_{ct}, and would thus be impossible to separately identify empirically. In the expression (20) for the expected excess return, the volatility risk premium gets confounded with the consumption risk premium. The confounding reflects the specification of stochastic volatility in (4) above. By way of contrast, in the models of Bansal and Yaron (2004) and Bansal *et al* (2003) the volatility risk premium gets folded into a constant term. Lettau, Ludvigson and Wachter (2004) also present a model where there is an excess return related to consumption volatility like that of (20). Their model is a stochastic volatility model where volatility is governed by a stochastic Markov regime switching variable. Like (4) above, there is a single variable — the volatility state — that governs both the location and scale of volatility, and the excess expected return is a confounding of the time varying consumption risk premium and the volatility risk premium.

3 Main Model

Consider the following model where consumption growth is

$$g_{t+1} = \mu_g + \sigma_{ct} z_{c,t+1}, \tag{21}$$

as in (4), and the stochastic volatility specification is generalized to

$$\begin{aligned} \sigma^2_{c,t+1} &= a_{\sigma c} + \rho_{\sigma c} \sigma^2_{ct} + q_t^{\frac{1}{2}} z_{\sigma,t+1} \\ q_{t+1} &= a_q + \rho_q q_t + \phi_q q_t^{\frac{1}{2}} z_{q,t+1} \,. \end{aligned} \tag{22}$$

Now we allow for stochastic volatility of the volatility process via the q_t process. This characteristic of volatility is known to be empirically important; see Chernov *et al* (2003) and the references therein.

The log of the marginal rate of substitution remains

$$m_{t+1} = b_{m0} + b_{mg} g_{t+1} + b_{mr} r_{t+1}, \tag{23}$$

where expressions for the coefficients are given in (15) for Epstein-Zin-Weil preferences.

Let v_t denote the log price dividend ratio of an asset paying the consumption endowment and r_{t+1} denote the return. Conjecture a linear expression for v_t

$$v_t = A_0 + A_\sigma \sigma_{ct}^2 + A_q q_t \tag{24}$$

where A_0, A_σ, A_q are constants whose derivation is in Subsection 6.3 of the Appendix. This subsection of the Appendix also contains the reduced form expressions for the marginal rate of substitution and the return. From these expressions one can deduce that the conditional mean excess return is

$$\mathrm{E}_t\left(r_{t+1}\right) - r_{ft} = -\left[(b_{mr} + b_{mg})\sigma_{ct}^2 + b_{mr} k_1^2 (A_\sigma^2 + A_q^2 \phi_q^2) q_t\right] - \frac{1}{2}\sigma_{rt}^2 \tag{25}$$

where $-\frac{1}{2}\sigma_{rt}^2 = -\frac{1}{2}\mathrm{Var}_t(r_{t+1})$ is the geometric adjustment term. For Epstein-Zin-Weil preferences, the conditional mean excess return reduces to

$$\mathrm{E}_t\left(r_{t+1}\right) - r_{ft} = \gamma\sigma_{ct}^2 + (1-\theta)k_1^2(A_\sigma^2 + A_q^2\phi_q^2)q_t - \frac{1}{2}\sigma_{rt}^2 \tag{26}$$

The risk premium

$$\gamma\sigma_{ct}^2 + (1-\theta)k_1^2(A_\sigma^2 + A_q^2\phi_q^2)q_t \tag{27}$$

is composed of two separate terms, where the first term reflects the risk premium on consumption risk and the second the risk premium on volatility risk. The latter is a confounding of a risk premium on shocks to volatility, $z_{\sigma,t+1}$, and shocks to the volatility of volatility, $z_{q,t+1}$, but nonetheless this risk premium (or more precisely the risk price) can be separately identified from that of consumption risk. Indeed, Scruggs (1998) and Guo and Whitelaw (2003) present evidence that additional control factors are needed in the return-volatility equation in order to estimate reliably the relationship between expected return and volatility. In both cases, the additional factors include at least one interest rate variable, which is arguable a proxy for q_t in the equation immediately above, given that the level of interest rates is associated with the turbulence of financial markets. Guo, Savickas, Wang, and Yang (2005) suggests that the value premium needs to be included, which is the likely prediction of a model like this one but with a long run risk factor as in Bansal and Yaron (2004), but has been suppressed for simplicity. In a somewhat different vein, Adrian and Rosenberg (2005) find empirical evidence that a two-factor type structure is helpful for explaining the cross-section of expected asset returns.

Interestingly, the sign of the volatility risk premium depends critically on the sign of $1-\theta$, where θ is defined in (16). Most economists would probably agree that $\gamma > 1$, i.e., the agent is more risk averse than a log investor. If $\gamma > 1$, then the sign of

$$1 - \theta = \frac{\gamma - \frac{1}{\psi}}{1 - \frac{1}{\psi}} \tag{28}$$

depends upon ψ. A sufficient condition for a positive volatility risk premium is $\psi > 1$, which Bansal and Yaron (2004) argue is the most reasonable region for ψ. On the other hand, a number of economists (see Campbell and Koo, 1997, and the references therein) argue that $\psi < 1$. If so, then it would take rather small values of ψ to generate a positive risk premium on volatility since

$$\psi < \frac{1}{\gamma} \quad \Rightarrow \quad 1 - \theta > 0 \tag{29}$$

$$\frac{1}{\gamma} < \psi < 1 \quad \Rightarrow \quad 1 - \theta < 0. \tag{30}$$

4 Stock Price and Volatility Dynamics

Much of the stochastic volatility literature examines the log capital return process

$$\Delta p_t = \log(P_t) - \log(P_{t-1}) \tag{31}$$

instead of the total return process, r_t, which includes the dividend yield. In this section we shall study the dynamics of the volatility of Δp_t implied by the general stochastic volatility model of Section 3; the conclusions are essentially the same for either process, because the capital gain return tends to dominate the total return.

The reduced form expression for the Δp_t process is derived in the Subsection 6.3 of the Appendix and takes the form

$$\begin{aligned} \Delta p_t &= b_{p0} + A_\sigma(\rho_{\sigma c} - 1)\sigma^2_{c,t-1} + A_q(\rho_q - 1)q_{t-1} + \\ &\quad \sigma_{c,t-1}z_{c,t} + A_\sigma q^{\frac{1}{2}}_{t-1}z_{\sigma,t} + A_q\phi_q q^{\frac{1}{2}}_{t-1}z_{q,t} \end{aligned} \tag{32}$$

where b_{p0} is a constant and the other parameters are defined in Subsection 6.3 of the Appendix. Note that from (85) A_σ is of the form

$$A_\sigma = \frac{1}{\theta}h_\sigma, \quad h_\sigma > 0, \tag{33}$$

and from (86) A_q is of the form

$$A_q = \frac{1}{\theta}h_q, \quad h_q > 0, \tag{34}$$

and so the signs of A_σ and A_q are same as those of θ defined in (16) above.

We consider first the dynamic relationship between Δp_t and the consumption volatility process σ^2_{ct}. It follows from the expression (32) and the dynamics (22) that

$$\begin{aligned} \mathrm{Cov}(\Delta p_t, \sigma^2_{\sigma c,t-j}) &= A_\sigma(\rho_\sigma - 1)\mathrm{E}(\sigma^4_{c,t-1})\rho^{j-1}_{\sigma c}, \quad j = 1, 2, \ldots, \infty \\ \mathrm{Cov}(\Delta p_t, \sigma^2_{ct}) &= \rho_q A_\sigma(\rho_{\sigma c} - 1)\mathrm{E}(\sigma^4_{c,t-1}) + A_\sigma \mathrm{E}(q_{t-1}) \\ \mathrm{Cov}(\Delta p_t, \sigma^2_{c\sigma,t+j}) &= A_\sigma(\rho_\sigma - 1)\mathrm{E}(\sigma^4_{c,t-1})\rho^{j}_{\sigma c}, \quad j = 1, 2, \ldots, \infty \end{aligned} \tag{35}$$

The serial cross covariances $\mathrm{Cov}(\Delta p_t, \sigma^2_{c,t+j})$ for $j \neq 0$ are proportional to the auto-covariance function of the σ^2_{ct} process. The sign will be negative if $\theta < 0$, as would be the case if both γ and ψ exceed unity. Thus, in this case a market price decline would signal increased future expected consumption volatility, a result analogous to that of Bansal, Khatchatrian, and Yaron (2003) who study the covariance between the log price dividend ratio, v_t, and subsequent consumption volatility $\sigma_{c,t+j}$, $j > 0$.

Interestingly, the sign of the contemporaneous covariance $\mathrm{Cov}(\sigma_{ct}^2, \Delta p_t)$ is ambiguous because it is the sum of terms of opposite signs.

To tie the theory to the stochastic volatility literature, the most interesting series is the one-step conditional variance process defined as

$$\sigma_{pt}^2 \equiv \mathrm{Var}_t(\Delta p_{t+1}) = \sigma_{ct}^2 + (A_\sigma^2 + A_q^2\phi_q^2)q_t. \tag{36}$$

From (36) it is immediately seen that conditional volatility follows a two-factor structure where it is the superposition of two autoregressive processes. This theoretical representation of volatility corresponds exactly to the two-factor structure developed empirically by Engle and Lee (1999), Gallant, Hsu, and Tauchen (1999), Barndorff-Nielsen and Shephard (2001, and additional papers at www.levyprocess.org), Alizadeh, Brandt, and, Diebold (2002) among others. The typical structure identified empirically contains one factor that is extremely persistent and another that is strongly mean reverting and nearly serially uncorrelated. In (36), σ_{ct}^2 is a likely candidate for the persistent factor while q_t is likely the strongly mean reverting factor. Scruggs (1998) and Guo and Whitelaw (2003) present evidence that additional factor(s) are needed in the return-volatility equation in order to empirically measure volatility risk reliably. These factors are variables that tend to be high when financial volatility is high. Their findings are appear completely consistent with Equation (36), which suggests that these factors should be related to q_t, the volatility of volatility.

The contemporaneous leverage effect pertains to the correlation between the conditional volatility process σ_{pt}^2 and the capital return process Δp_t. An easily computed conditional moment is

$$\mathrm{Cov}_{t-1}(\Delta p_t, \sigma_{pt}^2) = A_\sigma q_{t-1} + A_q(A_\sigma^2 + A_q^2\phi^2)q_{t-1} \tag{37}$$

One sees immediately from these covariances along with with (33), (34), and (16) the role that the utility function parameters γ and ψ play in determining the sign of the leverage effect. The covariances are negative under the parameter values utilized by Bansal and Yaron (2004).

We now consider the dynamic leverage effect. Some direct empirical evidence is seen in Figure 1, which shows the correlations between Δp as proxied by the logarithmic return on S&P 100 Index and leads and lags of the VIX volatility Index, daily, 1990–2004. The VIX index is designed to reflect the implied volatility on S&P 100 Index options with one month to expiration. Evidently, a large price increase is associated with a contemporaneous drop in volatility which then slowly dies away.

Figure 2 shows the same correlations except computed using monthly averages. The pattern remains quite apparent at the monthly frequency. Both figures are consistent with the evidence adduced in Litvinova (2004) and Bollerslev, Litvinova, and Tauchen (2004) using very high frequency data.

In order to compare the predictions from the model to the observed pattern, we need to compute analogous correlations under the model. Convenient analytical approximate results involving moments of σ_{pt}^2 and cross moments with other series appear to be out of reach. Instead, simulation is used to compute unconditional correlations of interest. Given a set of parameter values, the model of Section 3 is simulated for 10000 periods and population moments implied by the model are computed via Monte Carlo. Following the recommendation of Campbell and Koo (1997), the orthogonality conditions of the Euler equation error were checked and found to be negligible.

The correlations of interest are computed for two sets of parameter values, labelled Cases A and B in Table 1, which are based on Campbell and Koo (1997) and Bansal, Gallant, and Tauchen (2004). The only difference between the two cases is that the elasticity of substitution $\psi = 1.50$ in A and $\psi = 0.50$ in B. The risk aversion parameter γ is the same in both cases. The other values of the parameters would be reasonable for a model operating in monthly time. The model could not be expected to fit actual data because it lacks the long run risk component in consumption growth of Bansal and Yaron (2004), which is left out only for simplicity. Figure 3 shows the autocorrelation function of stock market volatility σ_{pt}^2 for the two sets parameter settings. The persistence of volatility is completely consistent with all empirical findings, and comparison across cases indicates that the value of the elasticity of substitution ψ has little effect on volatility persistence.

Figure 4 shows predicted dynamic leverage correlations between Δp_t and leads and lags of σ_{pt}^2. In the upper panel, where $\psi = 1.50$, the contemporaneous leverage effect is a negative and it fades away over time, which is completely consistent with Figures 1 and 2. In the bottom panel, where $\psi = 0.50$, the leverage effect is positive, which is completely counter factual. Interestingly, and somewhat surprisingly, the observed negative dynamic leverage effect is fairly compelling evidence for an intertemporal elasticity of substitution above unity.

5 Conclusion

The characteristics of the relationships between stock market volatility and stock market returns are examined within the context of a general equilibrium framework. The framework only permits connections between volatility and returns that arise through the internal economic structure of the model. All innovations are presumed uncorrelated, thereby ruling out connections that could arise via separate statistical channels. The most general model generates a two-factor structure for volatility along with time-varying risk premiums on consumption and volatility risk. It also generates endogenously a dynamic leverage effect, the sign of which depends upon the magnitudes of the risk aversion (γ) and intertemporal elasticity of substitution (ψ) parameters. In the case of expected utility where $\gamma = 1/\psi$, the leverage effect is absent, which suggest a strong connection between non-expected utility preferences and the leverage effect. The magnitude of ψ relative to unity is an issue of debate in the financial economics literature. Interestingly, if $\gamma > 1$, as is commonly presumed, then the observed negative leverage effect necessarily implies $\psi > 1$, so the well documented finding of negative leverage has bearing on this economic debate.

6 Appendix: Details of the Derivations

6.1 Solution for the Model with CRR Preferences in Subsection 2.2

From the approximation $r_{t+1} = k_0 + k_1 v_{t+1} - v_t + g_{t+1}$ and the presumed dynamics for g_{t+1} and $\sigma^2_{c,t+1}$ it follows that

$$\begin{aligned} r_{t+1} \quad = \quad & k_0 + (k_1 - 1)A_0 + k_1 A_\sigma a_{\sigma c} + \mu_c + A_\sigma (k_1 \rho_{\sigma c} - 1)\sigma^2_{ct} + \\ & k_1 \phi_{\sigma c} A_\sigma \sigma_{ct} z_{\sigma,t+1} + \sigma_{ct} z_{c,t+1} \end{aligned} \tag{38}$$

and

$$m_{t+1} = \delta - \gamma \mu_c - \gamma \sigma_{ct} z_{c,t+1}. \tag{39}$$

Thus

$$\mathrm{E}_t \left(e^{r_{t+1} + m_{t+1}} \right) = 1 \Rightarrow \tag{40}$$

$$\mathrm{E}_t \left(r_{t+1} + m_{t+1} \right) + \frac{1}{2} \mathrm{Var}_t \left(r_{t+1} + m_{t+1} \right) = 0. \tag{41}$$

Computing the conditional first two moments and setting the above to zero gives

$$\begin{aligned} & k_0 + (k_1 - 1)A_0 + k_1 A_\sigma a_{\sigma c} + \delta + (1-\gamma)\mu_c + \\ & \left[A_\sigma (k_1 \rho_{\sigma c} - 1) + \tfrac{1}{2} k_1^2 \phi^2_{\sigma c} A^2_\sigma + \tfrac{1}{2}(1-\gamma)^2 \right] \sigma^2_{ct} = 0. \end{aligned} \tag{42}$$

This can hold for all values of σ^2_{ct} only if

$$A_0 = \frac{k_0 + k_1 A_\sigma a_{\sigma c} + \delta + (1-\gamma)\mu_c}{1 - k_1} \tag{43}$$

where A_σ is a solution to the quadratic

$$(1-\gamma)^2 + 2(k_1 \rho_{\sigma c} - 1)A_\sigma + k_1^2 \phi^2_{\sigma c} A^2_\sigma = 0 \tag{44}$$

There are two roots

$$A_\sigma^{+,-} = \frac{1 - k_1 \rho_{\sigma c} \pm \sqrt{(1 - k_1 \rho_{\sigma c})^2 - (1-\gamma)^2 k_1^2 \phi^2_{\sigma c}}}{k_1^2 \phi^2_{\sigma c}} \tag{45}$$

which are real so long as $\phi^2_{\sigma c}$ is sufficiently small. The root A_σ^+ has the unappealing property that

$$\lim_{\phi_{\sigma c} \to 0} A_\sigma^+ \phi^2_{\sigma c} \neq 0 \tag{46}$$

which would mean the impact of σ_{ct} would grow without bound as stochastic volatility becomes unimportant. Thus we take A_σ^- as the economically meaningful root and set

$$A_\sigma = \frac{1 - k_1 \rho_{\sigma c} - \sqrt{(1 - k_1 \rho_{\sigma c})^2 - (1-\gamma)^2 k_1^2 \phi^2_{\sigma c}}}{k_1^2 \phi^2_{\sigma c}} \tag{47}$$

6.2 Solution for Model with EZW Preferences in Subsection 2.3

From the approximation $r_{t+1} = k_0 + k_1 v_{t+1} - v_t + g_{t+1}$ and the presumed dynamics for g_{t+1} and $\sigma^2_{c,t+1}$ it follows that

$$\begin{aligned} r_{t+1} &= k_0 + (k_1 - 1)A_0 + k_1 A_\sigma a_{\sigma c} + \mu_c + A_\sigma(k_1\rho_{\sigma c} - 1)\sigma^2_{ct} + \\ &\quad k_1\phi_{\sigma c} A_\sigma \sigma_{ct} z_{\sigma,t+1} + \sigma_{ct} z_{c,t+1} \end{aligned} \tag{48}$$

with

$$m_{t+1} = b_{m0} + b_{mg} g_{t+1} + b_{mr} r_{t+1}. \tag{49}$$

Then

$$\begin{aligned} m_{t+1} &= b_{m0} + b_{mg}\mu_c + \\ &\quad b_{mr}\left[k_0 + (k_1 - 1)A_0 + k_1 A_\sigma a_{\sigma c} + \mu_c + A_\sigma(k_1\rho_{\sigma c} - 1)\sigma^2_{ct}\right] + \\ &\quad b_{mr} k_1 \phi_{\sigma c} A_\sigma \sigma_{ct} z_{\sigma,t+1} + (b_{mr} + b_{mg})\sigma_{ct} z_{c,t+1} \end{aligned} \tag{50}$$

Thus

$$\begin{aligned} r_{t+1} + m_{t+1} &= (1 + b_{mr})\left[k_0 + (k_1 - 1)A_0 + k_1 A_\sigma a_{\sigma c}\right] + \\ &\quad (1 + b_{mr} + b_{mg})\mu_c + b_{m0} + (1 + b_{mr})\left[A_\sigma(k_1\rho_{\sigma c} - 1)\right]\sigma^2_{ct} + \\ &\quad (1 + b_{mr})k_1\phi_{\sigma c} A_\sigma \sigma_{ct} z_{\sigma,t+1} + (1 + b_{mr} + b_{mg})\sigma_{ct} z_{c,t+1}, \end{aligned} \tag{51}$$

and

$$\begin{aligned} \mathrm{E}_t\left(r_{t+1} + m_{t+1}\right) &= (1 + b_{mr})\left[k_0 + (k_1 - 1)A_0 + A_\sigma k_1 a_{\sigma c}\right] + \\ &\quad (1 + b_{mr} + b_{mg})\mu_c + b_{m0} + (1 + b_{mr})\left[A_\sigma(k_1\rho_{\sigma c} - 1)\right]\sigma^2_{ct}, \end{aligned} \tag{52}$$

$$\mathrm{Var}_t\left(r_{t+1} + m_{t+1}\right) = \left[(1 + b_{mr})^2 k_1^2 \phi^2_{\sigma c} A^2_\sigma + (1 + b_{mr} + b_{mg})^2\right]\sigma^2_{ct}. \tag{53}$$

Imposing

$$\mathrm{E}_t\left(r_{t+1} + m_{t+1}\right) + \frac{1}{2}\mathrm{Var}_t\left(r_{t+1} + m_{t+1}\right) = 0 \tag{54}$$

and equating to zero the constant and coefficient of σ^2_{ct} gives the equations

$$\begin{aligned} 0 &= (1 + b_{mr})\left[k_0 + (k_1 - 1)A_0 + k_1 A_\sigma a_{\sigma c}\right] + \\ &\quad (1 + b_{mr} + b_{mg})\mu_c + b_{m0} \end{aligned} \tag{55}$$

$$\begin{aligned} 0 &= (1 + b_{mr})(k_1\rho_{\sigma c} - 1)A_\sigma + \\ &\quad \tfrac{1}{2}\left[(1 + b_{mr})^2 k_1^2 \phi^2_{\sigma c} A^2_\sigma + (1 + b_{mr} + b_{mg})^2\right]. \end{aligned} \tag{56}$$

Thus

$$A_0 = \frac{k_0 + k_1 A_\sigma a_{\sigma c}}{(1-k_1)} + \frac{(1+b_{mr}+b_{mg})\mu_c + b_{m0}}{(1+b_{mr})(1-k_1)} \tag{57}$$

where A_σ is the solution of the quadratic

$$(1+b_{mr}+b_{mg})^2 + 2(1+b_{mr})(k_1\rho_{\sigma c}-1)A_\sigma + (1+b_{mr})^2 k_1^2 \phi_{\sigma c}^2 A_\sigma^2 = 0 \tag{58}$$

There are two roots

$$A_\sigma^{+,-} = \frac{1-k_1\rho_{\sigma c} \pm \sqrt{(1-k_1\rho_{\sigma c})^2 - (1+b_{mr}+b_{mg})^2 k_1^2 \phi_{\sigma c}^2}}{(1+b_{mr})k_1^2\phi_{\sigma c}^2} \tag{59}$$

The root A_σ^+ has the unappealing property that

$$\lim_{\phi_{\sigma c}^2 \to 0} \phi_{\sigma c}^2 A_\sigma^+ \neq 0 \tag{60}$$

so we take A_σ^- as the economically meaningful root and set

$$A_\sigma = \frac{1-k_1\rho_{\sigma c} - \sqrt{(1-k_1\rho_{\sigma c})^2 - (1+b_{mr}+b_{mg})^2 k_1^2 \phi_{\sigma c}^2}}{(1+b_{mr})k_1^2\phi_{\sigma c}^2} \tag{61}$$

In the case of Epstein-Zin-Weil preferences the coefficients are

$$A_0 = \frac{k_0 + k_1 A_\sigma a_{\sigma c}}{(1-k_1)} + \frac{(1-\gamma)\mu_c + \theta \log(\delta)}{\theta(1-k_1)} \tag{62}$$

$$A_\sigma = \frac{1-k_1\rho_{\sigma c} - \sqrt{(1-k_1\rho_{\sigma c})^2 - (1-\gamma)^2 k_1^2 \phi_{\sigma c}^2}}{\theta k_1^2 \phi_{\sigma c}^2} \tag{63}$$

It is useful to record the reduced form expression for m_{t+1}, r_{t+1}:

$$m_{t+1} = b_{m0}^* + b_{mr}A_\sigma(k_1\rho_{\sigma c}-1)\sigma_{ct}^2 + (b_{mr}+b_{mg})\sigma_{ct}z_{c,t+1} + b_{mr}k_1 A_\sigma \phi_{\sigma c}\sigma_{ct}z_{\sigma,t+1} \tag{64}$$

$$r_{t+1} = b_{r0} + A_\sigma(k_1\rho_{\sigma c}-1)\sigma_{ct}^2 + \sigma_{ct}z_{c,t+1} + k_1 A_\sigma \phi_{\sigma c}\sigma_{ct}z_{\sigma,t+1} \tag{65}$$

where b_{m0}^* and b_{r0} are readily determined constants. In the case of Epstein-Zin-Weil preferences the reduced form expressions are

$$\begin{aligned} m_{t+1} &= b_{m0}^* + (\theta-1)A_\sigma(k_1\rho_{\sigma c}-1)\sigma_{ct}^2 - \\ &\quad \gamma\sigma_{ct}z_{c,t+1} + (\theta-1)k_1 A_\sigma\phi_{\sigma c}\sigma_{ct}z_{\sigma,t+1} \\ r_{t+1} &= b_{r0} + A_\sigma(k_1\rho_{\sigma c}-1)\sigma_{ct}^2 + \sigma_{ct}z_{c,t+1} + k_1 A_\sigma\phi_{\sigma c}\sigma_{ct}z_{\sigma,t+1} \end{aligned} \tag{66}$$

6.3 Solution for the Model with EZW Preferences and General Stochastic Volatility in Section 3

The steps to find the solution start with

$$\begin{aligned} r_{t+1} &= k_0 + (k_1 - 1)A_0 + A_\sigma(k_1\sigma^2_{c,t+1} - \sigma^2_{ct}) + \\ & \quad A_q(k_1 q_{t+1} - q_t) + g_{t+1}. \end{aligned} \tag{67}$$

Thus

$$\begin{aligned} m_{t+1} + r_{t+1} &= b_{m0} + (1 + b_{mr})\left[k_0 + (k_1 - 1)A_0\right] + (1 + b_{mg} + b_{mr})g_{t+1} + \\ & \quad (1 + b_{mr})[A_\sigma(k_1\sigma^2_{c,t+1} - \sigma^2_{ct}) + A_q(k_1 q_{t+1} - q_t)], \end{aligned} \tag{68}$$

and so

$$\begin{aligned} \mathrm{E}_t(m_{t+1} + r_{t+1}) &= b_{m0} + (1 + b_{mr})\left[k_0 + (k_1 - 1)A_0\right] + (1 + b_{mg} + b_{mr})\mu_g + \\ & \quad (1 + b_{mr})k_1[A_\sigma a_{\sigma c} + A_q a_q] + \\ & \quad (1 + b_{mr})[A_\sigma(k_1\rho_{\sigma c} - 1)\sigma^2_{ct} + A_q(k_1\rho_q - 1)q_t], \end{aligned} \tag{69}$$

and

$$\begin{aligned} \mathrm{Var}_t(m_{t+1} + r_{t+1}) &= \mathrm{Var}_t[(1 + b_{mg} + b_{mr})g_{t+1}] + \\ & \quad \mathrm{Var}_t[(1 + b_{mr})(A_\sigma k_1\sigma^2_{c,t+1} + A_q k_1 q_{t+1})]. \end{aligned} \tag{70}$$

This can be expressed as

$$\begin{aligned} \mathrm{Var}_t(m_{t+1} + r_{t+1}) &= (1 + b_{mg} + b_{mr})^2\sigma^2_{ct} + \\ & \quad (1 + b_{mr})^2(A^2_\sigma k_1^2 q_t + A^2_q k_1^2\phi^2_q q_t) \end{aligned} \tag{71}$$

The asset pricing equation is

$$0 = \mathrm{E}_t(m_{t+1} + r_{t+1}) + \frac{1}{2}\mathrm{Var}_t(m_{t+1} + r_{t+1}) \tag{72}$$

Setting to zero the constant term yields

$$A_0 = \frac{b_{m0} + (1 + b_{mr})[k_0 + k_1(A_\sigma a_{\sigma c} + A_q a_q)] + (1 + b_{mg} + b_{mr})\mu_c}{(1 + b_{mr})(1 - k_1)}. \tag{73}$$

The term for σ^2_{ct} is

$$(1 + b_{mr})(k_1\rho_{\sigma c} - 1)A_\sigma\sigma^2_{ct} + \frac{1}{2}(1 + b_{mg} + b_{mr})^2\sigma^2_{ct} \tag{74}$$

and setting it to zero gives

$$A_\sigma = \frac{\frac{1}{2}(1 + b_{mg} + b_{mr})^2}{(1 + b_{mr})(1 - k_1\rho_{\sigma c})}. \tag{75}$$

The term for q_t is

$$(1+b_{mr})(k_1\rho_q-1)A_q q_t+\frac{1}{2}(1+b_{mr})^2(A_\sigma^2 k_1^2 q_t+A_q^2 k_1^2\phi_q^2 q_t). \tag{76}$$

and setting it to zero gives a quadratic in A_q:

$$(1+b_{mr})A_\sigma^2 k_1^2+2(k_1\rho_q-1)A_q+(1+b_{mr})k_1^2\phi_q^2 A_q^2=0. \tag{77}$$

There are two real solutions

$$A_q^+,\, A_q^- = \frac{1-k_1\rho_q \pm \sqrt{(1-k_1\rho_q)^2-(1+b_{mr})^2 k_1^4\phi_q^2 A_\sigma^2}}{(1+b_{mr})k_1^2\phi_q^2} \tag{78}$$

so long as

$$\phi_q^2 \le \frac{(1-k_1\rho_q)^2}{(1+b_{mr})^2 k_1^4 A_\sigma^2}. \tag{79}$$

Note that

$$\lim_{\phi_q\to 0}\phi_q^2 A_q^+ \neq 0, \tag{80}$$

which is economically unappealing, so we take the other root and set

$$A_q = \frac{1-k_1\rho_q - \sqrt{(1-k_1\rho_q)^2-(1+b_{mr})^2 k_1^4\phi_q^2 A_\sigma^2}}{(1+b_{mr})k_1^2\phi_q^2}. \tag{81}$$

The solution for the log price dividend ratio is thus

$$v_t = A_0 + A_\sigma\sigma_{ct}^2 + A_q q_t \tag{82}$$

where expressions for A_0, A_σ, A_q are given immediately above. Under Epstein-Zin-Weil preferences the coefficients are

$$A_0 = \frac{\theta[\log(\delta)+k_0+k_1(A_\sigma a_{\sigma c}+A_q a_q)]+(1-\gamma)\mu_c}{\theta(1-k_1)} \tag{83}$$

or

$$\boxed{A_0 = \frac{\log(\delta)+k_0+k_1(A_\sigma a_{\sigma c}+A_q a_q)}{1-k_1}+\frac{(1-\gamma)\mu_c}{\theta(1-k_1)}} \tag{84}$$

and

$$\boxed{A_\sigma = \frac{\frac{1}{2}(1-\gamma)^2}{\theta(1-k_1\rho_{\sigma c})}} \tag{85}$$

$$\boxed{A_q = \frac{1-k_1\rho_q-\sqrt{(1-k_1\rho_q)^2-\theta^2 k_1^4\phi_q^2 A_\sigma^2}}{\theta k_1^2\phi_q^2}} \tag{86}$$

The reduced form expressions for the MRS, the return, and the price change, are derived as follows. Start with

$$m_{t+1} = b_{m0} + b_{mg} g_{t+1} + b_{mr} r_{t+1} \tag{87}$$

$$m_{t+1} = b_{m0} + b_{mg}\mu_c + b_{mg}\sigma_{ct} z_{c,t+1} + b_{mr}(k_0 + k_1 v_{t+1} - v_t + g_{t+1}). \tag{88}$$

Hence

$$\begin{aligned} m_{t+1} &= b^*_{m0} + b_{mr} A_\sigma (k_1\rho_{\sigma c} - 1)\sigma^2_{ct} + b_{mr} A_q (k_1 \rho_q - 1) q_t + \\ & (b_{mg} + b_{mr})\sigma_{ct} z_{c,t+1} + b_{mr} k_1 A_\sigma q_t^{\frac{1}{2}} z_{\sigma,t+1} + b_{mr} k_1 A_q \phi_q q_t^{\frac{1}{2}} z_{q,t+1} \end{aligned} \tag{89}$$

where

$$\begin{aligned} b^*_{m0} &= b_{m0} + (b_{mg} + b_{mr})\mu_c + b_{mr}[k_0 + k_1(A_\sigma a_{\sigma c} + A_q a_q)\,] + \\ & b_{mr}(k_1 - 1) A_0 \end{aligned} \tag{90}$$

Under Epstein-Zin-Weil preferences the solution for the MRS is

$$\begin{aligned} m_{t+1} &= b^*_{m0} + (\theta - 1) A_\sigma (k_1\rho_{\sigma c} - 1)\sigma^2_{ct} + (\theta - 1) A_q (k_1\rho_q - 1) q_t + \\ & -\gamma\sigma_{ct} z_{c,t+1} + (\theta - 1) k_1 A_\sigma q_t^{\frac{1}{2}} z_{\sigma,t+1} + (\theta - 1) k_1 A_q \phi_q q_t^{\frac{1}{2}} z_{q,t+1} \end{aligned} \tag{91}$$

with

$$\begin{aligned} b^*_{m0} &= \theta \log(\delta) - \gamma\mu_g + (\theta - 1)[k_0 + k_1(A_\sigma a_{\sigma c} + A_q a_q)\,] + \\ & (\theta - 1)(k_1 - 1) A_0 \end{aligned} \tag{92}$$

To obtain the reduced form expressions for the return start with

$$r_{t+1} = k_0 + k_1 v_{t+1} - v_t + g_{t+1} \tag{93}$$

$$r_{t+1} = k_0 + k_1(A_0 + A_\sigma \sigma^2_{c,t+1} + A_q q_{t+1}) - (A_0 + A_\sigma \sigma^2_{ct} + A_q q_t) + g_{t+1} \tag{94}$$

which gives

$$\begin{aligned} r_{t+1} &= b_{r0} + A_\sigma (k_1\rho_{\sigma c} - 1)\sigma^2_{ct} + A_q (k_1\rho_q - 1) q_t + \\ & \sigma_{ct} z_{c,t+1} + k_1 A_\sigma q_t^{\frac{1}{2}} z_{\sigma,t+1} + k_1 A_q \phi_q q_t^{\frac{1}{2}} z_{q,t+1} \end{aligned} \tag{95}$$

where

$$b_{r0} = \mu_g + k_0 + (k_1 - 1) A_0 + k_1 (A_\sigma a_{\sigma c} + A_q a_q) \tag{96}$$

For the price change start with

$$\Delta p_{t+1} = p_{t+1} - p_t = v_{t+1} - v_t + g_{t+1} \tag{97}$$

which leads to

$$\begin{aligned}\Delta p_{t+1} &= b_{p0} + A_\sigma(\rho_{\sigma c} - 1)\sigma_{ct}^2 + A_q(\rho_q - 1)q_t + \\ & \sigma_{ct} z_{c,t+1} + A_\sigma q_t^{\frac{1}{2}} z_{\sigma,t+1} + A_q \phi_q q_t^{\frac{1}{2}} z_{q,t+1}\end{aligned} \tag{98}$$

where

$$b_{p0} = \mu_c + A_\sigma a_{\sigma c} + A_q a_q \tag{99}$$

The riskless rate, r_{ft}, is the solution to

$$-r_{ft} = \mathrm{E}_t(m_{t+1}) + \frac{1}{2}\mathrm{Var}_t(m_{t+1}), \tag{100}$$

which works out to

$$\begin{aligned}-r_{ft} &= b_{m0}^* - (1-\theta)\left[A_\sigma(k_1\rho_{\sigma c} - 1)\sigma_{ct}^2 + A_q(k_1\rho_q - 1)q_t\right] + \\ &= \tfrac{1}{2}\gamma^2\sigma_{ct}^2 + \tfrac{1}{2}(\theta-1)^2 k_1^2(A_\sigma^2 + \phi_q^2 A_q^2)q_t.\end{aligned} \tag{101}$$

7 Tables and Figures

Table 1: Parameter Settings Cases A and B

Parameter	Case A	Case B
$a_{\sigma c}$	0.10e-06	0.10e-06
$\rho_{\sigma c}$	0.98	0.98
a_q	0.20e-06	0.20e-06
ρ_q	0.20	0.20
ϕ_q	0.10e-06	0.10e-06
δ	0.9949	0.9949
γ	8.00	8.00
ψ	1.50	0.50
μ_c	0.163e-02	0.163e-02

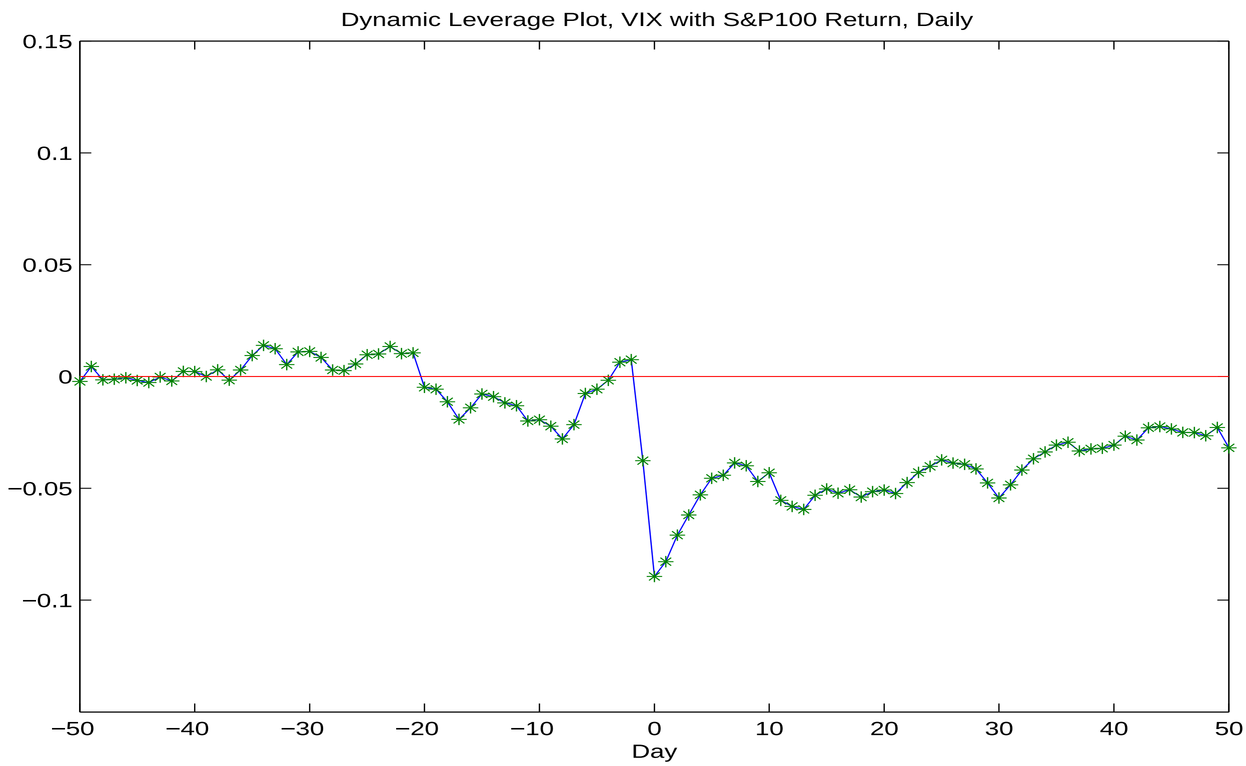

Figure 1: The figure shows shows the correlation $\mathrm{Corr}(\Delta p_t, \mathrm{vix}_{t+k})$, at the daily frequency for $k = 0, 1, \ldots 50$ and for $k = -1, -2, \ldots, -50$. The daily log-price change is for the S&P 100 Index and the VIX index is the daily closing value. The sample period is 1990-01-02–2004-05-21

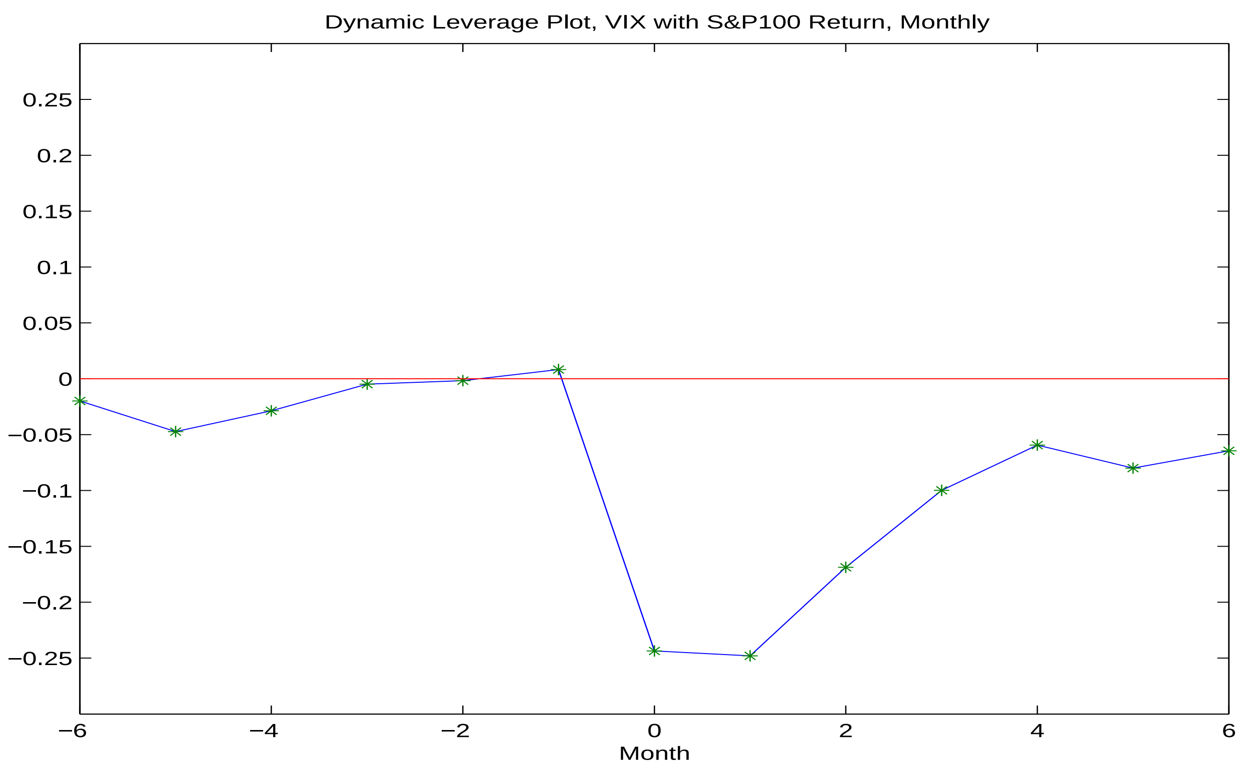

Figure 2: The figure shows shows the correlation Corr($\overline{\Delta p}_j$), $\overline{vix}_{j+l}$), at the monthly frequency for $l = 0, 1, \ldots 5$ and for $l = -1, -2, \ldots, -5$. The monthly average log-price change $\overline{\Delta p}$ is for the S&P 100 Index and the monthly VIX value $\overline{vix}$ is the average of daily closing values. The sample period is 1990-01–2004-05

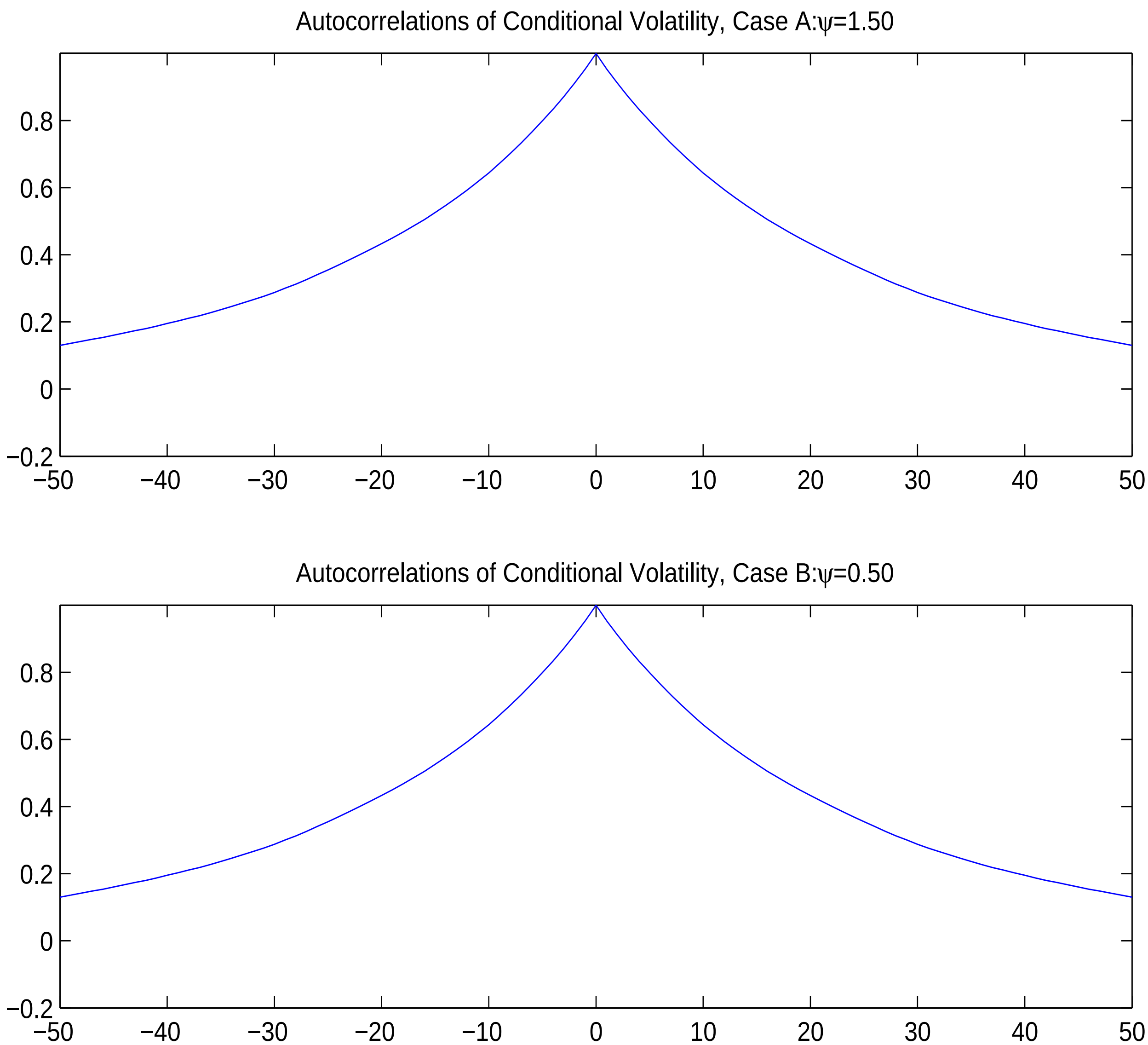

Figure 3: Each panel shows the autocorrelation function of the conditional variance process: $\mathrm{Corr}(\sigma^2_{pt}, \sigma^2_{p,t+j})$, $j = -50, \ldots, 50$. The top panel is computed under the parameter settings A of Table 1, where the inter-temporal elasticity of substitution is $\psi = 1.50$; the bottom panel is computed under settings B of Table 1, where the inter-temporal elasticity of substitution is $\psi = 0.50$. Comparison of the two panels indicates that the the autocorrelation function of the conditional variance process is very insensitive to the value of ψ.

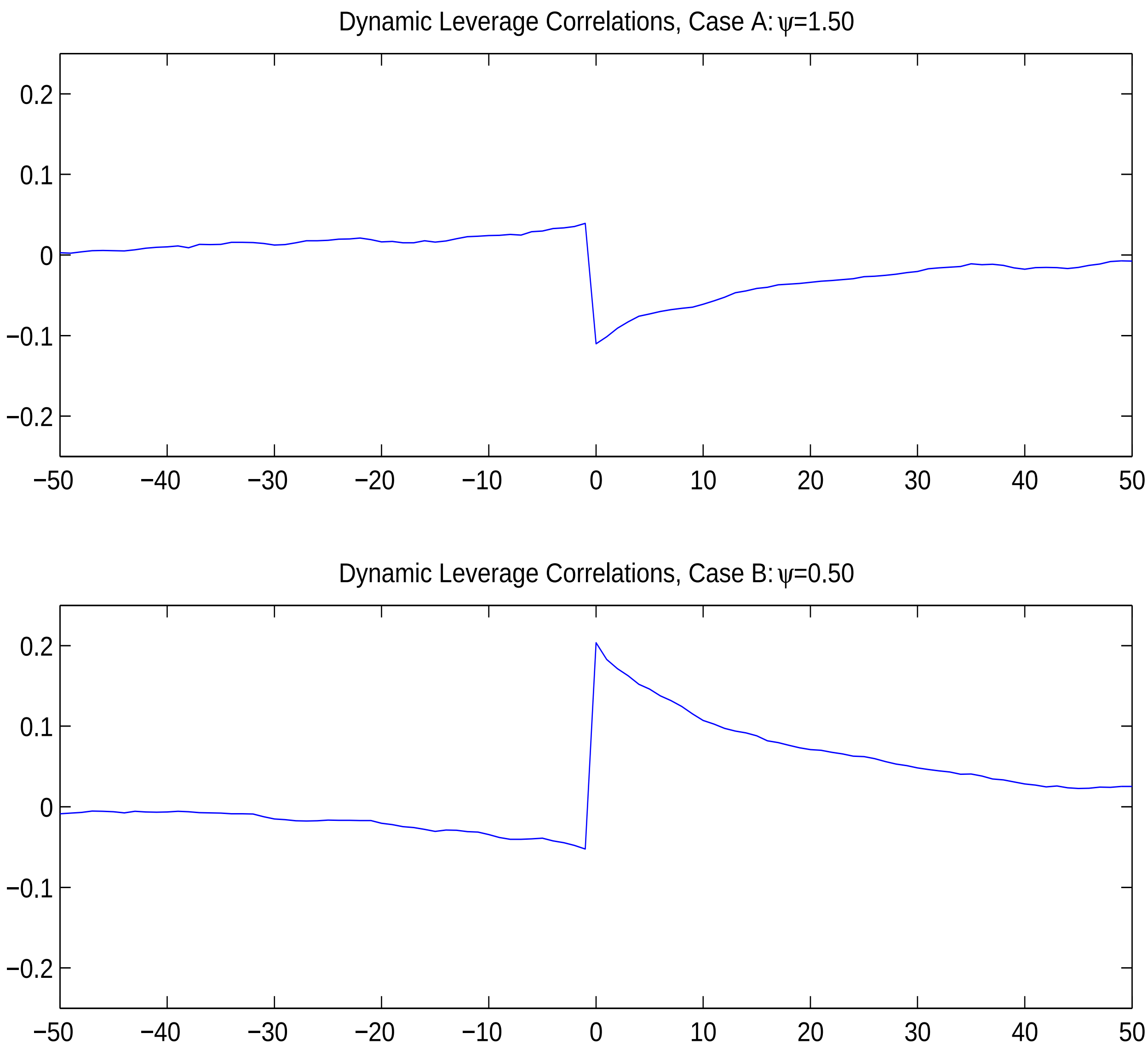

Figure 4: The right side of each panel shows the unconditional correlation of the price change with subsequent volatility, $\mathrm{Corr}(\Delta p_t, \sigma^2_{p,t+j})$, $j = 1, 2, \ldots 50$, and the left side shows the correlation of the price change with lagged volatility, $\mathrm{Corr}(\Delta p_t, \sigma^2_{p,t-j})$, $j = 1, 2, \ldots 50$. The top panel pertains to the parameter settings A of Table 1, where the inter-temporal elasticity of substitution is $\psi = 1.50$; the bottom panel pertains to the parameter settings B of Table 1, where the inter-temporal elasticity of substitution is $\psi = 0.50$. Comparison of the two panels indicates that the sign of the leverage effect depends upon the value of ψ relative to unity.

8 References

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