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1 Motivation

Since the publication of the Black-Scholes-Merton option pricing formula, the continuous time Geometric Brownian Motion model of stock price became popular in finance. However BM is a continuous model, while stock prices exhibit jumps. The existence of option implied volatility smiles is evidence that the GBM is not a good model of stock prices. But like the BSM formula, BM is popular due to its simplicity and analytical tractability (partly through Ito’s Lemma).

Similar to BM, Levy process is a continuous time model but with jumps, with jump of different intensity and jump sizes superimposed to ensure analytical tractability.

For technical references on Levy process, see Sato [2000], Applebaum [2004] and Kyprianou [2006]. For somewhat technical reference in finance, see Cont and Tankov [2004]. For applications, see the series of papers by Peter Carr and many others.

2 Levy Process

2.1 Definition

Definition 1. (See [Cont and Tankov, 2004, P68]) A cadlag (right continuous with left limit) stochastic process \((X_t)_{t \geq 0}\) is a Levy Process if the following is true:

1. \(P(X(0)=0)=1\)

2. Independent and stationary increments

3. Stochastic continuity

\[
\lim_{t \downarrow 0} P(|X(t)| > a) = 0 \quad \forall a > 0
\]

Definition 2. (See [Cont and Tankov, 2004, P69]) A probability distribution \(F\) is infinitely divisible if for any integer \(n \geq 2\), there exists \(n\) i.i.d random variables \(Y_1, \ldots, Y_n\) such that \(Y_1 + \ldots + Y_n\) has distribution \(F\).

Note that the concept of infinitely divisible distribution (IDD) refers to a distribution, not to a stochastic process.

A convenient tool for analyzing Levy process is the characteristic function (CF) for a random variable \(X\) with distribution \(F\)
\[ \phi_X(u) = E(e^{iuX}) = \int_{-\infty}^{\infty} e^{iux} dF_X(x) = \int_{-\infty}^{\infty} e^{iux} f_X(x) dx \]

There is a one-to-one mapping between CF \( \phi_X \) and CDF \( F_X \). There are maximum likelihood estimates based on CF. See Liu and Nishiyama [2008].

**CF and Fourier Transform:** the characteristic function of a probability density function \( p(x) \) is the complex conjugate of the continuous Fourier transform of \( p(x) \).

**CF and Moment Generating Function:**

\[ M_X(t) = E(e^{tX}) = \phi_X(t/i) \]

How to calculate the moments of \( X \)?

\[ E(X) = E(XM_X(0)) = M'_X(0) \]

In general

\[ E(X^n) = \left. \frac{d^n M_X(t)}{dt^n} \right|_{t=0} \]

If \( X \) and \( Y \) are independent and \( Z = X + Y \) then

\[ \phi_Z(u) = \phi_X(u)\phi_Y(u) \]

Proposition 3. (See [Cont and Tankov, 2004, P70]) Let \((X_t)_{t \geq 0}\) be a Levy process. Then for every \( t \), \( X_t \) has an IDD. Conversely for every IDD \( F \), there is a Levy process \( X_t \) such as the distribution of \( X_1 \) is \( F \).

Examples of Infinitely divisible distribution:

1. **Normal Distribution**
   For \( X \sim N(0, 1) \), the CF is

   \[ \phi_X(u) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{iux} e^{-\frac{1}{2}x^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2+iu x} dx \]

   \[ -\frac{1}{2}x^2 + iux = -\frac{1}{2}(x^2 - 2ix) = -\frac{1}{2}((x - iu)^2 + u^2) = -\frac{1}{2}(x + iu)^2 - \frac{1}{2}u^2 \]
2.1 Definition

Therefore,

\[ \phi_X(u) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+iu)^2} dx \]
\[ = e^{-\frac{1}{2}u^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+iu)^2} dx \]
\[ = e^{-\frac{1}{2}u^2} \]

(2)

For \( X \sim N(\mu, \sigma^2) \), note that:

\[ -\frac{1}{2\sigma^2}(x - \mu)^2 + iux = -\frac{1}{2\sigma^2}((x - \mu)^2 - 2\sigma^2 iux) \]
\[ = -\frac{1}{2}(x^2 - 2\mu x + \mu^2 - 2\sigma^2 iux) \]
\[ = -\frac{1}{2\sigma^2}(x^2 - 2x(\mu + \sigma^2 iu) + \mu^2) \]
\[ = -\frac{1}{2\sigma^2}((x - (\mu + \sigma^2 iu))^2 + \mu^2 - (\mu + \sigma^2 iu)^2) \]
\[ = -\frac{1}{2\sigma^2}((x - (\mu + \sigma^2 iu))^2 - 2\mu\sigma^2 iu + \sigma^4 u^2) \]
\[ = -\frac{1}{2\sigma^2}(x - (\mu + \sigma^2 iu))^2 + i\mu u - \frac{1}{2}\sigma^2 u^2 \]

Therefore

\[ \phi_X(u) = e^{i\mu u - \frac{1}{2}\sigma^2 u^2} \]

Let

\[ \phi_{\frac{X}{\sigma}}(u) = e^{i\frac{\mu}{\sigma} u - \frac{1}{2}\frac{\sigma^2}{\sigma^2} u^2} \]

then

\[ \phi_X(u) = \left( \phi_{\frac{X}{\sigma}}(u) \right)^N \]

Therefore, \( X \) can be written as

\[ X = X_1 + X_2 + \ldots + X_N \]

and for each \( X_i \)

\[ \phi_{X_i}(u) = \phi_{\frac{X}{\sigma}}(u) \]

That is, the normal distribution \( X \) is infinitely divisible. And the Brownian Motion is a Levy process.

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2.1 Definition

2.1 Exponential Distribution

Exponential pdf:
\[ f(t) = \lambda e^{-\lambda t}, \quad t \geq 0, \quad \lambda > 0 \]

For an exponential rv \( \tau \)
\[ E(\tau) = \frac{1}{\lambda} \]

with CDF \( F(t) = 1 - e^{-\lambda t} \)
If \( \tau \) is the survival time, then the probability of surviving to time \( t \) is \( 1 - e^{-\lambda t} \).
The probability of death prior to time \( t \) is \( e^{-\lambda t} \).

2.2 Poisson Distribution

\( N(t) \): Number of events (say jumps) that occur at or before time \( t \). Events occur randomly. The time interval between events follows the exponential distribution. Events and intervals are independent.

\( \tau_1, \ldots, \tau_k \) i.i.d exponential rv with parameter \( \lambda \). These are the intervals between successive events.

\[ N(t) = \max\{k : \tau_1 + \tau_2 + \ldots + \tau_k \leq t\} \]
Number of events in the interval \([0,t]\) depends on how many exponential random variables can fit into the interval.

A closely related quantity is the time of \( n^{th} \) jump:
\[ S_n = \sum_{k=1}^{n} \tau_k \]
\( S_n \) = has Gamma density:
\[ \Gamma(s; n, \lambda) = \frac{s^{n-1}}{\Gamma(n)} \lambda^{-n} e^{\lambda s} \]

Expected interval between jumps = \( E(\tau) = \frac{1}{\lambda} \)
Average arrival rate = \( \lambda \)

Distribution of a Poisson Process:
\[ P(N(t, \lambda) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \]
The pdf of \( N(1, \lambda) \) is \( \frac{\lambda^k}{k!} e^{-\lambda} \).
CF of a Poisson Distribution:

\[
\phi_{N(1,\lambda)}(u) = E(e^{iuN}) = \sum_{k=0}^{\infty} e^{iku} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{iu})^k}{k!} = e^{-\lambda} e^{\lambda e^{iu}} = e^{\lambda(e^{iu}-1)} \tag{3}
\]

For any integer N, this can be written as

\[
\phi_{N(1,\lambda)}(u) = \left( e^{\frac{\lambda}{N}(e^{iu}-1)} \right)^N
\]

Therefore, the Poisson distribution is an IDD, and the Poisson process is a Levy process. Expected number of jumps:

\[
E(N(t, \lambda)) = \lambda t
\]

**Compensated (Demeaned) Poisson Process**

\[
\tilde{N}(t, \lambda) = N(t, \lambda) - \lambda t
\]

**Compound Poisson Process (CPP) \( Q(t, \lambda) \):**

\[
Y_1, Y_2, \ldots, \text{iid, independent of } N(t, \lambda)
\]

\[
Q(t, \lambda) = \sum_{i=1}^{N(t,\lambda)} Y_i, \quad t > 0
\]

\( N \) is the number of jumps, while the \( Y \)'s are jump sizes.

CF of the CPP:
\[ \phi_Q(u) = E(e^{iuQ}) = E\left(E(e^{iuQ | N(t)})\right) = E\left(\left(\phi_Y(u)\right)^N(t)\right) = \sum_{k=0}^{\infty} \phi_Y(u)^k \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=0}^{\infty} \frac{(\phi_Y(u)\lambda)^k}{k!} e^{-\lambda} = e^{\lambda(\phi_Y(u)-1)} = e^{\lambda f(e^{iu\tilde{y}}-1)f(y)dy} \] (4)

(5)

(6)

(7)

Expected jump impact (combination of number of jumps and sizes):

\[ E(Q(t, \lambda)) = \beta \lambda t \]

where \( \beta = E(Y_i) \) is the mean jump size

\( \tilde{N}(t, \lambda) \) is a martingale, so is \( \tilde{Q}(t, \lambda) = Q(t, \lambda) - \beta \lambda t \)

3 Decomposition of Levy Process

3.1 CF of a Levy process

Proposition 4. (See [Cont and Tankov, 2004, P70]) Let \( (X_t)_{t \geq 0} \) be a Levy process. There exists a continuous function \( \psi \), called the characteristic exponent (CE) of \( X \), such that

\[ \phi_{X_t}(u) = E(e^{iuX_t}) = e^{t\psi(u)} \]

3.2 Levy-Khinchin Representation

Theorem 5. (See [Cont and Tankov, 2004, P83]) Let \( (X_t)_{t \geq 0} \) be a Levy process with CE \( \psi(u) \), then \( \psi(u) \) can be written as:

\[ \psi(u) = iBu - \frac{1}{2} Cu^2 + \int (e^{iux} - 1 - iuxI_{|x|\leq 1}) \nu(dx) \] (8)

where \( b \) is the drift, \( c \) is the variance, \( \nu \) is the Levy measure. And the triplet \( (b, c, \nu) \) is called the Levy triplet of process \( X_t \).
3.3 Levy-Ito Decomposition

Theorem 6. (See [Kyprianou, 2006, P35]) Given a Levy triplet \((b, c, \nu)\), there exist three independent Levy processes \(X^{(1)}, X^{(2)}, X^{(3)}\) where \(X^{(1)}\) is a linear BM with drift \(b\) and variance \(c\), \(X^{(2)}\) is a compound Poisson process, and \(X^{(3)}\) is a martingale with almost surely finite number of jumps in each finite interval with jump magnitude less than 1, such that \(X = X^{(1)} + X^{(2)} + X^{(3)}\) has CE as given in (8).

3.4 Elaboration

Corresponding to the CF for Levy process (8), we define

\[
\psi_1(u) = i bu - \frac{1}{2} cu^2
\]

\[
\psi_2(u) = \int_{|x| \geq 1} (e^{iux} - 1) \nu(dx)
\]

\[
\psi_3(u) = \int_{|x| < 1} (e^{iux} - 1 - iux) \nu(dx)
\]

Then it is clear that \(\psi_1(u)\) corresponds to \(X^{(1)}\) - BM with drift \(b\) and variance \(c\). Now we take a look at \(\psi_2(u)\).

\[
\psi_2(u) = \lambda \int_{|x| \geq 1} (e^{iux} - 1) \frac{\nu(dx)}{\lambda} = \lambda \int_{-\infty}^{\infty} (e^{iux} - 1) I_{|x| \geq 1} \frac{\nu(dx)}{\lambda} = \lambda \int_{-\infty}^{\infty} (e^{iux} - 1) f(x) dx
\]

where \(\lambda = \nu(R - (-1, 1))\) and \(f(x) dx = |x|_{\geq 1} \frac{\nu(dx)}{\lambda}\). This is in the same form as the CE of a compound Poisson process. Therefore, \(X^{(2)}\) is found.

\(\psi_3(u)\) and \(X^{(3)}\) is a little tricky. It deals with jumps of sizes smaller than one.

4 Variance Gamma Process

4.1 Subordination - Time Change

If \(S_t\) is a nonnegative Levy process, then \(S_t\) is called a subordinator (can be used as "time").

Let the triplet of \(S_t\) be \((b, 0, \nu)\), it is CE is

\[
\psi_{S_t}(u) = i bu + \int (e^{iux} - 1) \nu(dx)
\]

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or the Laplace transform (exponent):

\[ l(u) = \psi_{S_t}(u/i) = bu + \int(e^{ux} - 1)\nu(dx) \]

Let the Levy process \( X_t \) have CE \( \psi_X(u) \), then the subordinated (time changed) process \( Y_t = X(S_t) \) is a Levy process with CE:

\[ \psi_Y(u) = l(\psi_X(u)) \]

See (See [Cont and Tankov, 2004, P108])

For the BM, recall from the normal CF that the CE is

\[ \psi_X(u) = -\frac{1}{2}u^2 \]

### 4.2 Subordinator - Gamma Process

For the Gamma subordinator \( S_t \), its probabilistic properties are:

(see http://en.wikipedia.org/wiki/Gamma_distribution):

**PDF:**

\[ \Gamma(x, t, \alpha, \beta) = \frac{x^{\alpha t-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha t)\beta^{\alpha t}} \]

At \( t = 1 \), the PDF is:

\[ \Gamma(x, 1, \alpha, \beta) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}} \]

Its mean and variance are:

\[ E(S_1) = \alpha\beta \]
\[ Var(S_1) = \alpha\beta^2 \]

Normalizing the Gamma process by \( E(S_1) = 1 \), we have:

\[ E(S_1) = \alpha\beta = 1 \]
\[ Var(S_1) = \beta \]

**CF:**

\[ \Phi(u) = (1 - i\beta u)^{-\alpha} \]

Laplace transform is:

\[ L(u) = (1 - \beta u)^{-\alpha} \]
4.3 Subordinated Process - Variance Gamma Process

Hence its Laplace exponent is

\[ l(u) = \ln(L(u)) = -\alpha \ln(1 - \beta u) \]

It is CE is:

\[ \psi_{\Gamma(\alpha, \beta)}(u) = -\alpha \ln(1 - i\beta u) \]

4.3 Subordinated Process - Variance Gamma Process

The Gamma-subordinated Brownian Motion \( Y(t) = B(S(t)) \) has the following CE:

\[ \psi_Y(u) = l(\psi_{BM}(u)) = -\alpha \ln(1 + \frac{1}{2}\beta u^2) \]

which can be rewritten as:

\[ \psi_Y(u) = \left( -\alpha \ln(1 - i\sqrt{\frac{\beta}{2}} u) \right) + \left( -\alpha \ln(1 + i\sqrt{\frac{\beta}{2}} u) \right) = \psi_{\Gamma(\alpha, \sqrt{\frac{\beta}{2}})}(u) + \psi_{\Gamma(\alpha, \sqrt{\frac{\beta}{2}})}(-u) \]

Note that by definition:

\[ \Phi_X(u) = E(e^{iuX} = e^{i\psi(u)}) \]
\[ \Phi_X(-u) = E(e^{-iuX} = E(e^{iu-X} = e^{-i\psi(u)}) \]

Therefore,

\[ \psi_{\Gamma(\alpha, \sqrt{\frac{\beta}{2}})}(-u) = \psi_{\Gamma(\alpha, \sqrt{\frac{\beta}{2}})}(u) \]

This is the sum of the CE of two independent Gamma random variables. \( Y_t \) can be written as:

\[ Y_t = \Gamma_1(t, \alpha, \beta_1) - \Gamma_2(t, \alpha, \beta_1) \]

where \( \beta_1 = \sqrt{\frac{\beta}{2}} \)

This subordinated process is called Variance Gamma, because it can be interpreted as the "variance" or "difference" of two gamma processes. The first process is interpreted as the market forces pushing up stock prices, while the second process pushing down the stock prices.

See Madan et al. [1998] for a more general Variance Gamma process with three parameters.

In general the Variance Gamma has three parameters: the drift \( \mu \) and variance \( \sigma^2 \) of the BM, and the variance of the Gamma distribution \( \beta \). The CE is in the
following form:

$$\psi_Y(u) = l(\psi_{BM}(u)) = -\frac{1}{\beta} \ln \left(1 - \beta \left(i\mu u - \frac{1}{2}\sigma^2 u^2\right)\right)$$

Assume the following decomposition of the quadratic form:

$$1 - \beta \left(i\mu u - \frac{1}{2}\sigma^2 u^2\right) = (1 - i\eta_p u)(1 + i\eta_n u)$$

Then, as before, the Variance Gamma process is the difference of two Gamma processes:

$$Y_t = G_p(t, \mu_p, \nu_p) - G_n(t, \mu_n, \nu_n)$$

where the parameters are defined as:

$$\mu_p = \eta_p/\nu, \mu_n = \eta_n/\nu, \nu_p = \mu_p^2 \nu, \nu_n = \mu_n^2 \nu$$

See Carr et al. [2002]

5 CGMY Process

For the Levy triplet \((b, c, \nu)\), the parameter \(\nu\) is the only one that determines the jump behavior of the Levy process. By modifying the \(\nu\), we can create Levy processes with different jump behavior.

There are at least three ways to build Levy processes:
1. Subordinating a BM
2. Specifying a Levy measure
3. Specifying the density for IID increments (Gaussian vs Gamma distribution)
See [Cont and Tankov, 2004, P.106].

5.1 Tail Behavior

The Levy density of the VG process is, see [Carr et al., 2002, Eq.(6)],

$$\nu_{VG}(x) = \begin{cases} 
\frac{\mu_n^2 e^{-\frac{\mu_n}{\nu_n} |x|}}{\nu_n |x|} & \text{for } x < 0 \\
\frac{\mu_p^2 e^{-\frac{\mu_p}{\nu_p} |x|}}{\nu_p |x|} & \text{for } x > 0 
\end{cases}$$
And the CGMY process has the following Levy density, see [Carr et al., 2002, Eq.(7)],

\[ \nu_{CGMY}(x) = \begin{cases} 
C e^{-G|x|/|x|^\gamma} & \text{for } x < 0 \\
C e^{-M|x|/|x|^\gamma} & \text{for } x > 0 
\end{cases} \]

C: the overall level of activity
G: the rate of exponential decay on the left side
M: the rate of exponential decay on the right side, when \( G \neq M \), there is skewness
Y: measure of fine structure (monotonicity of the density)

The four parameters CGMY are named after the model inventors (Carr, Geman, Madan and Yor). I found a good introductory presentation on the CGMY model here [www.mathematik.uni-kl.de/~seifried/teach_07ss/05_hodnekvam.ppt](http://www.mathematik.uni-kl.de/~seifried/teach_07ss/05_hodnekvam.ppt)

Levy process has been used in almost all areas of quantitative finance, including option pricing, term structure, and corporate defaults, etc. There are many technical and not-so-technical notes available, with just a few clicks.

## 6 Option Pricing with Levy Process
References


