Econometrics of Testing for Jumps in Financial Economics Using Bipower Variation

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ABSTRACT
In this article we provide an asymptotic distribution theory for some nonparametric tests of the hypothesis that asset prices have continuous sample paths. We study the behaviour of the tests using simulated data and see that certain versions of the tests have good finite sample behavior. We also apply the tests to exchange rate data and show that the null of a continuous sample path is frequently rejected. Most of the jumps the statistics identify are associated with governmental macroeconomic announcements.

KEYWORDS: bipower variation, jump process, quadratic variation, realized variance, semimartingales, stochastic volatility

In this article we will show how to use a time series of prices recorded at short time intervals to estimate the contribution of jumps to the variation of asset prices and form robust tests of the hypothesis that it is statistically satisfactory to regard the data as if it has a continuous sample path. Being able to distinguish between jumps and continuous sample path price movements is important as it has implications for risk management and asset allocation. A stream of recent articles in financial econometrics has addressed this issue using low-frequency return data based on (i) the parametric models of Andersen, Benzoni, and Lund (2002), Chernov et al. (2003), and Eraker, Johannes, and Polson (2003); (ii) the Markovian,
nonparametric analysis of Aït-Sahalia (2002), Bandi and Nguyen (2003), and Johannes (2004); (iii) options data [e.g., Bates (1996) and the review by Garcia, Ghysels, and Renault (2005)]. Our approach will be nonparametric and exploit high-frequency data. Monte Carlo results suggest that it performs well when based on empirically relevant sample sizes. Furthermore, empirical work points us to the conclusion that jumps are common.

Traditionally, in the theory of financial economics, the variation of asset prices is measured by looking at sums of squared returns calculated over small time periods. The mathematics of that is based on the quadratic variation process [e.g., Back (1991)]. Asset pricing theory links the dynamics of increments of quadratic variation to the increments of the risk premium. The recent econometric work on this topic, estimating quadratic variation using discrete returns under the general heading of realized quadratic variation, realized volatility, and realized variances, was discussed in independent and concurrent work by Andersen and Bollerslev (1998), Comte and Renault (1998), and Barndorff-Nielsen and Shephard (2001). It was later developed in the context of the methodology of forecasting by Andersen et al. (2001), while central limit theorems for realized variances were developed by Jacod (1994), Barndorff-Nielsen and Shephard (2002), and Mykland and Zhang (2005). Multivariate generalizations to realized covariance are discussed by, for example, Barndorff-Nielsen and Shephard (2004a) and Andersen et al. (2003a). See Andersen, Bollerslev, and Diebold (2005) and Barndorff-Nielsen and Shephard (2005) for surveys of this area and references to related work.

Recently Barndorff-Nielsen and Shephard (2004b) introduced a partial generalization of quadratic variation called bipower variation (BPV). They showed that in some cases relevant to financial economics, BPV can be used, in theory, to split up the individual components of quadratic variation into that due to the continuous part of prices and that due to jumps. In turn, the bipower variation process can be consistently estimated using an equally spaced discretization of financial data. This estimator is called the realized bipower variation process.

In this article we study the difference or ratio of realized BPV and realized quadratic variation. We show we can use these statistics to construct nonparametric tests for the presence of jumps. We derive the asymptotic distributional theory for these tests under quite weak conditions. This is the main contribution of this article. We will also illustrate the jump tests using both simulations and exchange rate data. We relate some of the jumps to macroeconomic announcements by government agencies.

A by-product of our research is an appendix that records a proof of the consistency of realized BPV under substantially weaker conditions than those used by Barndorff-Nielsen and Shephard (2004b). The appendix also gives the first derivation of the joint limiting distribution for realized BPV and the corresponding realized quadratic variation process under the assumption that there are no jumps in the price process. The latter result demonstrates the expected conclusion that realized BPV is slightly less efficient than the realized quadratic variation as an estimator of quadratic variation in the case where prices have a continuous sample path.

Since we wrote the first draft of this article, we have teamed up with the probabilists Svend Erik Graversen, Jean Jacod, and Mark Podolskij to relax these conditions further. In particular, we report in Barndorff-Nielsen et al. (2005) some
results on generalized BPV that largely removes the regularity assumptions used in this article. To do this we use very advanced probabilistic ideas that make the results rather inaccessible. Hence we think it is still valuable to publish the original results, which can be proved using relatively elementary methods. We will augment these results with comments that show how far Barndorff-Nielsen et al. (2005) were able to go in removing the assumptions we use in this article.

In the next section we will set out our notation and recall the definitions of quadratic variation and BPV. In Section 2 we give the main theorem of the article, which is the asymptotic distribution of the proposed tests. In Section 3 we will extend the analysis to cover the case of a time series of daily statistics for testing for jumps. In Section 4 we study how the jump tests behave in simulation studies, while in Section 5 we apply the theory to two exchange rate series. In Section 6 we discuss various additional issues, while Section 7 concludes. The proofs of the main results in the article are given in the appendix.

1 DEFINITIONS AND PREVIOUS WORK

1.1 Notation and Quadratic Variation

Let the log-price of a single asset be written as $Y_t$ for continuous time $t \geq 0$. $Y$ is assumed to be a semimartingale. For a discussion of economic aspects of this see Back (1991). Further, $Y^d$ will denote the purely discontinuous component of $Y$, while $Y^c$ will be the continuous part of the local martingale component of $Y$.

The quadratic variation (QV) process of $Y$ can be defined as

$$[Y]_t = p - \lim_{n \to \infty} \sum_{j=0}^{n-1} (Y_{t_{j+1}} - Y_{t_j})^2,$$  \hspace{1cm} (1)

[e.g., Jacod and Shiryaev (1987:55)] for any sequence of partitions $t_0 = 0 < t_1 < \ldots < t_n = t$ with $\sup \{t_{j+1} - t_j\} \to 0$ for $n \to \infty$. It is well known that

$$[Y]_t = [Y^c]_t + [Y^d]_t,$$

where $[Y^d]_t = \sum_{0 \leq u \leq t} \Delta Y^2_u$, (2)

and $\Delta Y_t = Y_t - Y_{t-}$ are the jumps in $Y$. We will test for jumps by asking if $[Y] = [Y^c]$.

We estimated $[Y]$ using a discretized version of $Y$ based on intervals of time of length $\delta > 0$. The resulting process, which we write as $Y_{\delta t}$, is $Y_{\delta \lfloor t/\delta \rfloor}$, for $t \geq 0$, recalling that $[x]$ is the integer part of $x$. This allows us to construct $\delta$-returns

$$y_j = Y_{\delta j} - Y_{\delta (j-1)}, \quad j = 1, 2, \ldots, \lfloor t/\delta \rfloor,$$

which are used in the realized quadratic variation process,

$$[Y_{\delta}]_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} y_j^2,$$

the QV of $Y_{\delta}$. Clearly the QV theory means that as $\delta \downarrow 0$, $[Y_{\delta}]_t \overset{L^2}{\to} [Y]_t$.

Our analysis of jumps will often be based on the special case where $Y$ is a member of the Brownian semimartingale plus jump (BSMJ) class:
\[ Y_t = \int_0^t a_s ds + \int_0^t \sigma_s dW_s + Z_t, \]

where \( Z \) is a jump process. In this article we assume that

\[ Z_t = \sum_{j=1}^{N_t} c_j, \]

with \( N_t \) being a simple counting process (which is assumed finite for all \( t \)) and the \( c_j \) are nonzero random variables. Further we assume \( a \) is càdlàg, the volatility \( \sigma \) is càdlàg, and \( W \) is a standard Brownian motion. When \( N \equiv 0 \), we write \( Y \in BSM \), which is a stochastic volatility plus drift model [e.g., Ghysels, Harvey, and Renault (1996)]. In this case, \( [Y^c]_t = \int_0^t \sigma_s^2 ds \) and \( [Y^d]_t = \sum_{j=1}^{N_t} c_j^2 \), so

\[ [Y]_t = \int_0^t \sigma_s^2 ds + \sum_{j=1}^{N_t} c_j^2. \]

### 1.2 Bipower Variation

The 1, 1-order BPV process is defined, when it exists, as

\[ \{Y\}^{[1,1]}_t = p - \lim_{\delta \downarrow 0} \sum_{j=2}^{[t/\delta]} |y_{j-1}| |y_j|. \quad (4) \]

Barndorff-Nielsen and Shephard (2004b) showed that if \( Y \in BSM \), \( a \equiv 0 \) and \( \sigma \) is independent from \( W \) then

\[ \{Y\}^{[1,1]}_t = \mu_1^2 \int_0^t \sigma_s^2 ds = \mu_1^2 [Y^c]_t, \]

where

\[ \mu_1 = E|u| = \sqrt{2}/\sqrt{\pi} \approx 0.79788 \quad (5) \]

and \( u \sim N(0, 1) \). Hence \( \mu_1^{-2} \{Y\}^{[1,1]}_t = [Y^c]_t \). This result is quite robust, as it does not depend on any other assumptions concerning the structure of \( N \), the distribution of the jumps, or the relationship between the jump process and the SV component. The reason for this is that only a finite number of terms in Equation (4) are affected by jumps, while each return that does not have a jump goes to zero in probability. Therefore, since the probability of jumps in contiguous time intervals goes to zero as \( \delta \downarrow 0 \), those terms do not impact the probability limit.

Clearly \( \{Y\}^{[1,1]}_t \) can be consistently estimated by the realized BPV process

\[ \{Y_\delta\}^{[1,1]}_t = \sum_{j=2}^{[t/\delta]} |y_{j-1}| |y_j|, \]
as \( \delta \to 0 \). One would expect these results on BPV to continue to hold when we extend the analysis to allow \( a \neq 0 \). This is indeed the case, as will be discussed in the next section.

Barndorff-Nielsen and Shephard (2004b) point out that

\[
[Y]_t - \mu_1^{-2} (Y_{\delta})^{[1,1]}_t = \sum_{j=1}^{N_t} c_j^2 = [Y^d]_t.
\]

This can be consistently estimated by \([Y_{\delta}^d]_t - \mu_1^{-2} (Y_{\delta})^{[1,1]}_t\). Hence, in theory, the realized BPV process can be used to consistently estimate the continuous and discontinuous components of QV or, if augmented with the appropriate asymptotic distribution theory, as a basis for testing the hypothesis that prices have continuous sample paths.

The only other work we know which tries to split QV into that due to the continuous and the jump components is Mancini (2004). She does this via the introduction of a jump threshold whose absolute value goes to zero as the number of observations within each day goes to infinity. Related work includes Coutin (1994). Following Barndorff-Nielsen and Shephard (2004b), Woerner (2004a) studied the robustness of realized power variation \( \delta^{1-r/2} \sum_{j=1}^{[\delta]} |y_j|^{r'} \) to an infinite number of jumps in finite time periods showing that the robustness property of realized power variation goes through in that case. A related article is Aït-Sahalia (2004), which shows that maximum-likelihood estimation can disentangle a homoskedastic diffusive component from a purely discontinuous infinite activity Lévy component of prices. Outside the likelihood framework, the article also studies the optimal combinations of moment functions for the generalized method of moments estimation of homoskedastic jump diffusions.

2 A THEORY FOR TESTING FOR JUMPS

2.1 Infeasible Tests

In this section we give the main contribution of the article, Theorem 1. It gives the asymptotic distribution for a linear jump statistic, \( G \), based on \( \mu_1^{-2} \{Y_{\delta}\}^{[1,1]}_t - [Y_{\delta}]_t \) and a ratio jump statistic, \( H \), based on \( 1 \mu_1^{-2} \{Y_{\delta}\}^{[1,1]}_t/[Y_{\delta}]_t \). Their distributions, under the null of \( Y \in BSM \), will be seen to depend upon the unknown integrated quarticity \( \int_0^t \sigma_u^4 du \), and so we will say the results of the theorem are statistically infeasible. We will overcome this problem in the next subsection.

Recall the definition \( \mu_1 = \sqrt{2}/\sqrt{\pi} \) in Equation (5) and let

\[
\vartheta = (\pi^2/4) + \pi - 5 \simeq 0.6090.
\]

1 Following Barndorff-Nielsen and Shephard (2004b), Huang and Tauchen (2005) have independently and concurrently used simulations to study the behavior of this type of ratio, although they do not provide the corresponding asymptotic theory.
Theorem 1  Let $Y \in BSM$ and let $t$ be a fixed, arbitrary time. Suppose the following conditions are satisfied:

(a) The volatility process $\sigma^2$ is pathwise bounded away from zero.

(b) The joint process $(a, \sigma)$ is independent of the Brownian motion $W$.

Then as $\delta \downarrow 0$

$$G = \frac{\delta^{-1/2} \left( \mu_1^2 \{Y_\delta\}^{[1,1]} - [Y_\delta]_t \right)}{\sqrt{\int_0^t g \sigma_u^4 du}} \xrightarrow{L} N(0,1),$$

and

$$H = \frac{\delta^{-1/2} \left( \mu_1^2 \frac{[Y_\delta]_t}{[Y_\delta]^{[1,1]}} - 1 \right)}{\sqrt{9 \left( \int_0^t \sigma_s^2 ds \right)^2}} \xrightarrow{L} N(0,1).$$

Further, if $Y \in BSM$ and (a) and (b) hold, then

$$\{Y\}^{[1,1]}_t = \mu_1^2 \int_0^t \sigma_s^2 ds.$$

Remark 1

(i) Condition (a) in Theorem 1 holds, for instance, for the square-root process (due to it having a reflecting barrier at zero) and the Ornstein-Uhlenbeck volatility processes considered in Barndorff-Nielsen and Shephard (2001). More generally (a) does not rule out jumps, diurnal effect, long memory, or breaks in the volatility process.

(ii) It is clear from the proof of Theorem 1 that in realized BPV we can replace the subscript $j - 1$ with $j - q$, where $q$ is any positive but finite integer.

(iii) Condition (b) rules out leverage effect [e.g., Black (1976), Nelson (1991), and Ghysels, Harvey, and Renault (1996)] and feedback between previous innovations in $W$ and the risk premium in $a_t$. This is an unfortunate important limitation of the result. This is often regarded to be empirically reasonable with exchange rules, but clashes with what we observe for equity data [e.g., Hansen and Lunde (2005)]. Simulation results in Huang and Tauchen (2005) suggest the behavior of the test statistic is not affected by leverage effects. The results in Barndorff-Nielsen et al. (2005) confirm this.
They show the central limit theorem of Equation (7) holds if \( a_t \) is a locally bounded, predictable process and \( \sigma^2 \) follows, for example,

\[
\sigma^2_t = \sigma^2_0 + \int_0^t a'_s dW_s + \int_0^t \nu'_s dL_s,
\]

where \( a' \) and \( \nu' \) are locally bounded, predictable processes and \( L \) is a Lévy process (which can include a Brownian component) independent of \( W \). The \( \sigma^2 \) process thus includes all diffusive models for volatility, but also allows for some types of jumps. This structure is relaxed in their paper even further to allow for more general forms of jumps, but the resulting conditions are highly technical and so we will not discuss them here.

(iv) Equation (9) means that under the alternative hypothesis of jumps

\[
\mu_1^{-2} \{ Y_{\delta} \}_t^{[1,1]} - \frac{P}{N_t} \sum_{j=1}^{N_t} \xi_j^2 \leq 0
\]

and

\[
\frac{\mu_1^{-2} \{ Y_{\delta} \}_t^{[1,1]}}{[Y_{\delta}]_t} - 1 \xrightarrow{p} - \frac{\sum_{j=1}^{N_t} \xi_j^2}{\int_0^t \sigma^2_s ds + \sum_{j=1}^{N_t} \xi_j^2} \leq 0.
\]

This implies the linear and ratio tests will be consistent.

(v) A by-product of the proof of Theorem 1 is Theorem 3, given in the appendix, which is a joint central limit theorem for scaled realized BPV and QV processes. This is proved under the assumption that \( Y \notin BSM \) and shows that, of course, both estimate \( \int_0^t \sigma^2_0 ds \), with the efficient realized QV having a slightly smaller asymptotic variance. Thus we can think of Equation (7) as a Hausman (1978)-type test, a point first made by Huang and Tauchen (2005) following the initial draft of this article.

### 2.2 Feasible Tests

To construct computable linear and ratio jump tests we need to estimate the integrated quarticity \( \int_0^t \sigma^4_u du \) under the null hypothesis of \( Y \in BSM \). However, in order to ensure the test has power under the alternative, it is preferable to have an estimator of integrated quarticity that is also consistent under the alternative \( BSM^J \). This is straightforward using realized quadpower variation,

\[
\{ Y_{\delta} \}_t^{[1,1,1]} = \delta^{-1} \sum_{j=4}^{t/\delta} |y_{j-3}| |y_{j-2}| |y_{j-1}| |y_{j}| \xrightarrow{P} \mu_1^4 \int_0^t \sigma^4_s ds.
\]

The validity of this result when there are no jumps is discussed in Barndorff-Nielsen et al. (2005). The robustness to jumps is straightforward so long as there are a finite number of jumps. It also holds in cases where the jump component \( Z \) of the \( BSM^J \) model is a Lévy or non-Gaussian OU model [see Barndorff-Nielsen, Shephard, and Winkel (2004)].
The above discussion allows us to define the feasible linear jump test statistic, \( \hat{G} \), which has the asymptotic distribution
\[
\hat{G} = \frac{\delta^{-1/2} \left( \mu_i^{-2} \{ Y_{\delta_i} \}_i^{[1,1]} - [Y_{\delta_i}]_i \right) }{\sqrt{9 \mu_i^{-4} \{ Y_{\delta_i} \}_i^{[1,1]}{[1,1,1]}}} \overset{L}{\rightarrow} N(0,1), \tag{10}
\]
where we would reject the null of a continuous sample path if Equation (10) is significantly negative. Likewise, the ratio jump test statistic, \( \hat{H} \), defined as
\[
\hat{H} = \frac{\delta^{-1/2} }{\sqrt{9 \{ Y_{\delta_i} \}_i^{[1,1,1,1]}/\{ Y_{\delta_i} \}_i^{[1,1,1,1]}}} \left( \frac{\mu_i^{-2} \{ Y_{\delta_i} \}_i^{[1,1]} }{[Y_{\delta_i}]_i} - 1 \right) \overset{L}{\rightarrow} N(0,1), \tag{11}
\]
rejects the null if significantly negative.

The ratio \( \{ Y_{\delta_i} \}_i^{[1,1,1,1]}/\{ Y_{\delta_i} \}_i^{[1,1,1,1]} \) is asymptotically equivalent to the realized correlation between \(|y_{t+1} - y_t|\) and \(|y_t|\) [e.g., Barndorff-Nielsen and Shephard (2004a)]. It converges to \( \mu_1^2 \approx 0.6366 \) under BSM. Estimates below \( \mu_1^2 \) provide evidence for jumps. Its asymptotic distribution under the null follows trivially from Equation (11). Further, in Equation (8), clearly the ratio
\[
\int_0^t \sigma_s^4 ds/ \left( \int_0^t \sigma_s^2 ds \right)^2 \geq 1/t, \text{ with equality obtained in the homoskedastic case.}
\]
This suggests replacing \( \hat{H} \) by the adjusted ratio jump test
\[
\hat{J} = \frac{\delta^{-1/2} }{\sqrt{9 \max \left( t^{-1} \{ Y_{\delta_i} \}_i^{[1,1,1,1]}/\{ Y_{\delta_i} \}_i^{[1,1,1,1]} \right) } \left( \frac{\mu_i^{-2} \{ Y_{\delta_i} \}_i^{[1,1]} }{[Y_{\delta_i}]_i} - 1 \right) \overset{L}{\rightarrow} N(0,1). \tag{11}
\]

3 TIME SERIES OF REALIZED QUANTITIES

To ease the exposition we will use \( t = 1 \) to denote the period of a day. Then we define
\[
\hat{v}_i = \sum_{j=1}^{1/\delta} (Y_{\delta_{i+(j-1)}} - Y_{\delta_{(j-1)+(i-1)}})^2 = [Y_{\delta_i}]_i - [Y_{\delta_i}](i-1), \quad i = 1, 2, \ldots, T,
\]
the daily increments of realized QV, so as \( \delta \downarrow 0 \) then \( \hat{v}_i \overset{p}{\rightarrow} [Y]_i - [Y](i-1) \). The \( \hat{v}_i \) and \( \sqrt{\hat{v}_i} \) are called the daily realized variance and volatility in financial economics, respectively. Here we give the corresponding results for realized BPV and then discuss the asymptotic theory for a time series of such sequences. These results follow straightforwardly from our previous theoretical results.
Clearly we can define a sequence of $T$ daily realized BPVs,
\[ \tilde{\sigma}_i = \sum_{j=2}^{[1/\delta]} |y_{j-1,i}| |y_{j,i}|, \quad i = 1, 2, ..., T, \]
where we assume $\delta$ satisfies $[1/\delta] = 1$ for ease of exposition and
\[ y_{j,i} = Y_{\delta j+(i-1)} - Y_{\delta j+(i-1)}. \]
Clearly $\mu_\delta^{-2} \tilde{\sigma}_i \sim \mathcal{N}[0, 1]$. In order to develop a feasible limit theory it will be convenient to introduce a sequence of daily realized quadpower variations,
\[ \tilde{q}_i = \delta^{-1} \sum_{j=4}^{[1/\delta]} |y_{j-3,i}| |y_{j-2,i}| |y_{j-1,i}| |y_{j,i}|, \quad i = 1, 2, ..., T. \]

The above sequence of realized quantities suggest constructing a sequence of nonoverlapping, feasible daily jump test statistics:
\begin{align*}
\hat{G}_{\delta i} &= \frac{\delta^{-1/2} \left( \mu_\delta^{-2} \tilde{\sigma}_i - \hat{\sigma}_i \right)}{\sqrt{9 \mu_1^{-4} \tilde{q}_i}}, \\
\hat{H}_{\delta i} &= \frac{\delta^{-1/2} \left( \mu_\delta^{-2} \tilde{\sigma}_i - 1 \right)}{\sqrt{9 \{ \tilde{q}_i / \hat{\sigma}_i \}^2}}, \\
\hat{J}_{\delta i} &= \frac{\delta^{-1/2}}{\sqrt{9 \max \left( 1, \tilde{q}_i / \hat{\sigma}_i \right)^2}} \left( \mu_\delta^{-2} \tilde{\sigma}_i - 1 \right). 
\end{align*}

By inspecting the proof of Theorem 1, it is clear that as well as each of these individual tests converging to $\mathcal{N}(0, 1)$ as $\delta \downarrow 0$, the converge as a sequence in time jointly to a multivariate normal distribution. For example, define a sequence of feasible ratio tests $\hat{H}_{\delta} = (\hat{H}_{\delta 1}, \ldots, \hat{H}_{\delta T})'$, then as $\delta \downarrow 0$ so $\hat{H}_{\delta} \Rightarrow \mathcal{N}(0, I_T)$. Likewise $\hat{G}_{\delta} \Rightarrow \mathcal{N}(0, I_T)$ and $\hat{J}_{\delta} \Rightarrow \mathcal{N}(0, I_T)$. Thus each of these tests have the property that they are asymptotically serially independent through time, under the null hypothesis that there are no jumps.

4 SIMULATION STUDY

4.1 Simulation Design

In this section we document some Monte Carlo experiments that assess the finite sample performance of our asymptotic theory for the feasible tests for jumps. Throughout, we assume $Y \in \text{BSMJ}$, but set $a \equiv 0$ and the component processes $\sigma$, $W$, $N$, and $c$ to be independent. Before we start we should
mention that in independent and concurrent work Huang and Tauchen (2005) also studied the finite sample behavior of our central limit theory using an extensive simulation experiment. Their conclusions are broadly in line with the ones we reach here.²

Our model for $\sigma$ is derived from some empirical work reported in Barndorff-Nielsen and Shephard (2002), who used realized variances to fit the spot variance of the deutsch mark (DM) dollar rate from 1986 to 1996 by the sum of two uncorrelated, stationary processes, $\sigma^2 = \sigma_1^2 + \sigma_2^2$. Their results are compatible with using Cox, Ingersoll and Ross (CIR) processes for the $\sigma_1^2$ and $\sigma_2^2$ processes. In particular, we will write these, for $s = 1, 2$, as the solution to

$$d\sigma_{t,s}^2 = -\lambda_s \{\sigma_{t,s}^2 - \xi_s\}dt + \omega_s \sigma_{t,s} dB_{t,s}, \quad \xi_s \geq \omega_s^2/2,$$

(15)

where $B = (B_1, B_2)'$ is a vector standard Brownian motion, independent from $W$.

Equation (15) has a gamma marginal distribution,

$$\sigma_{t,s}^2 \sim \text{Ga}(2\omega_s^{-2} \xi_s, 2\omega_s^{-2}) = \text{Ga}(v_s, a_s), \quad v_s \geq 1,$$

with a mean of $v_s/a_s$ and a variance of $v_s/a_s^2$ [Cox, Ingersoll, and Ross (1985)]. The parameters $\omega_s$, $\lambda_s$, and $\xi_s$ were calibrated by Barndorff-Nielsen and Shepard (2002) as follows. Setting $p_1 + p_2 = 1$, they estimated

$$E(\sigma_s^2) = p_10.509, \quad \text{var}(\sigma_s^2) = p_10.461, \quad s = 1, 2,$$

with $p_1 = 0.218$, $p_2 = 0.782$, $\lambda_1 = 0.0429$, and $\lambda_2 = 3.74$, which means the first, smaller component of the variance process is slowly reverting with a half-life of about 16 days, while the second has a half-life of about 4 hours.

All jumps will be generated by taking $N$ as $K$ jumps uniformly scattered in each unit of time. We call this a stratified Poisson process, using the terminology of sample survey theory, [e.g., Kish (1965)]. This setup means that when $K > 0$, we can view the power of the test conditionally as the probability of rejection in time units where there actually were jumps. We specify $c_i^{j,i,d,N}(0, \sigma^2_c)$, so the variance of $Y_t^i$ and $Y_t^j$ are $tK\sigma^2_c$ and $0.509$, respectively. We will vary $K$ and $\sigma^2_c$, which allows us to see the impact of the frequency of jumps and their size on the behavior of the realized bipower variation process. To start off, we will fix $K = 2$ and $\sigma^2_c = 0.2 \times 0.509$, which means that the jump process will account for 28% of the variation of the process. Clearly this is a high proportion. Later we will study the cases when $K = 1$ and $\sigma^2_c = 0.1 \times 0.509$ and $0.05 \times 0.509$.

Finally, the results will be indexed by $n = 1/\delta$, the number of observations per unit of time.

² Huang and Tauchen (2005) also report results on the finite sample behavior of the test when it is carried out over long stretches of data, such as 1 year or 10 years. In this case, the results differ from the ones given here with significant size distortions. As Huang and Tauchen (2005) explain, this is not surprising, and this effect is also present when we look at the behavior of the asymptotic theory for realized quadratic variation. See also the work of Corradi and Distaso (2004).
4.2 Null Distribution

We will use 5000 simulated days to assess the finite sample behavior of the jump tests given in Equations (12), (13), and (14). We start by looking at their null distributions when $N \equiv 0$.

The left-hand side of Figure 1 shows the results from the first 300 days in the sample. The crosses depict the linear difference $\frac{\mu_1^2 \hat{v}_1}{\mu_0 \hat{v}_i}$, while the feasible 99% one-sided critical values [using Equation (12)] of the difference are given by the solid line. As we go down the graph, $n$ increases, and so, as the null hypothesis is true, the magnitude of $\frac{\mu_1^2 \hat{v}_1}{\mu_0 \hat{v}_i}$ and corresponding critical values tend to fall toward zero. The most important aspect of these graphs is that the critical values of the tests change dramatically through time, reflecting the volatility clustering in the data.

The second column of Figure 1 repeats this analysis, but now using the jump ratio $\frac{\mu_1^2 \hat{v}_1}{\mu_0} - 1$, which tends to fall as $n$ increases. The feasible critical values of this ratio are more stable through time, reflecting the natural scale invariance of the denominator for the ratio jump tests. The third column shows the results for the adjusted critical values. This has some differences from the unadjusted version when $n$ is small. The right-hand side of Figure 1 shows the QQ plots of the $t$-tests of Equations (12), (13), and (14). On the $y$-axis are the ranked values of the simulated $t$-tests, while on the $x$-axis are the corresponding expected values under Gaussian sampling. We see a very poor QQ plot for the linear test, even when

![Figure 1](simulated_null_distribution)

Figure 1 Simulation from the null distribution of the feasible limit theory for the linear difference $\frac{\mu_1^2 \hat{v}_1}{\mu_0} - 1$, and jump ratios $\left(\frac{\mu_1^2 \hat{v}_1}{\mu_0} - 1\right)$ for a variety of values of $n$. Also given are the corresponding results for the adjusted version. Also shown is the 99% one-sided critical value of the statistics. Right-hand side gives the QQ plots of the three sets of $t$-statistics.
For larger values of $n$, the asymptotics seem to have some substantial bite. The ratio test has quite good QQ plots for $n \geq 72$.

In the upper part of Table 1 we show the means, standard deviations and acceptance rates (defined as the probability of not rejecting the null hypothesis) of Equations (12), (13), and (14). Under the asymptotics, they should ideally have values, 0, 1, .95, and .99, respectively. All three statistics have a negative mean, leading to overrejection of the null due to the one-sided nature of the test. Even when $n = 288$, the linear test rejects the null around 8%, rather than 5%, of the time. The small sample performance of the adjusted ratio test is better for a range of values of $n$.

As a final check on the null distribution of the jump tests, we repeat the above analysis, but increasing $\lambda_2$, the mean reversion parameter of the fast decaying volatility process, by a factor of five. This reduces its half-life down to 20 minutes. This case of an extremely short half-life is quite a challenge, as a number of econometricians view very short memory SV models as being good proxies for processes with jumps. Table 1 shows the results. The linear test has a negative bias that reduces as $n$ becomes very large. The ratio test has a smaller negative bias and overreject less than the linear test. The degree of overrejection is modest, but more important than in the first simulation design. Hence this testing procedure can be challenged by very fast reverting volatility components.

### 4.3 Impact of Jumps: The Alternative Distribution

We now introduce some jumps into the process and see how the tests react. The stratified Poisson process is set up to have either one or two jumps per day, while the variance of the jumps is either 5%, 10%, or 20% of $E(\sigma_t^2)$.

The results are given in Table 2 and they are in line with expectations. There is little difference in the nominal power of the linear and adjusted ratio tests. As the number of jumps or the variance of the jumps increases, so the rate of accepting the null falls. In the case where there is a single jump per day and the jump is 5% of the variability of the continuous component of prices, we reject the null 20% of the time when $n = 288$.

One of the interesting features of Table 2 is that the probability of accepting the null is roughly similar if $N = 2$ and each jump is 10% of the variation of $\sigma_t^2$ compared to the case where $N = 1$ and we look at the 20% example. This is repeated when we move to the $N = 2$ and 5% case and compare it to the $N = 1$ and 10% case. This suggests the rejection rate is heavily influenced by the variance of the jump process, not just the frequency of the jumps or the size of the individual jumps.

### 5 TESTING FOR JUMPS EMPIRICALLY

#### 5.1 Dataset

We now turn our attention to using the adjusted ratio jump test of Equation (14) on economic data. We use the bivariate German DM/U.S. dollar and Japanese yen/U.S. dollar exchange rate series, which covers the 10-year period from December 1, 1986, until November 30, 1996. Hence if the dollar generally
Table 1 Finite sample behavior of the feasible linear ratio and adjusted ratio t-tests based on 5000 separate days under the null hypothesis. Here .95 and .99 denote the designed acceptance rate. (Top) Standard setup with $\lambda_2 = 3.74$. (Bottom). Changes $\lambda_2$ to $5 \times 3.74$. The last two columns give the sample average of the daily realized volatility and realized bipower variation. Throughout, common random numbers are used.

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strengthens, both of these rates would be expected to rise. The original dataset records every five minutes the most recent midquote to appear on the Reuters screen. We have multiplied all returns by 100 in order to make them easier to present. The database has been kindly supplied to us by Olsen and Associates, Zurich, Switzerland, who document their ground breaking work in this area in Dacorogna et al. (2001).

5.2 Ratio Jump Statistic

Figure 2 plots the ratio statistic $\mu_i^2 \tilde{v}_i / \hat{v}_i$ and its corresponding 99% critical values, computed under the assumption of no jump using the adjusted theory given in Equation (14), for each of the first 250 working days in the sample for $n = 12$ and $n = 72$. We reject the null if the ratio is significantly less than one. The values of $n$ are quite small, corresponding to 2-hour and 20-minute returns, respectively. Results for larger values of $n$ will be reported in a moment. Of importance is that the critical values do not change very much between different days.

Figure 2 shows quite a lot of rejections of the null of no jumps, although the times where the rejections are statistically significant sometimes change with $n$. When $n$ is small, the rejections are marginal (note the Monte Carlo results suggest one should not trust the decisions based on the test with such small samples unless the test is absolutely overwhelming, which is not the case here), but by the

![Figure 2](image-url)
Table 2  Effect of jumps on the linear and adjusted ratio tests. On the right-hand side we show results for the case where there are two jumps per day. On the left hand side, there is a single jump per day. The variances of the jumps are 20%, 10%, and 5%, respectively, of the expectation of $\sigma^2$, with the results for the 20% case given at the top. The last two columns give the sample average of the daily realized volatility and realized bipower variation. Throughout, common random numbers are used.

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### 20%, $N_1=2$

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<td>-0.11 1.07</td>
<td>.912 .970</td>
<td>.507 .495</td>
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<tr>
<td>72</td>
<td>-0.51 1.49</td>
<td>.823 .896</td>
<td>-0.30 1.19</td>
<td>.868 .945</td>
<td>-0.28 1.15</td>
<td>.875 .951</td>
<td>.506 .492</td>
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<tr>
<td>288</td>
<td>-0.97 2.12</td>
<td>.749 .836</td>
<td>-0.74 1.59</td>
<td>.776 .868</td>
<td>-0.74 1.58</td>
<td>.778 .870</td>
<td>.505 .488</td>
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<tr>
<td>1152</td>
<td>-2.19 3.96</td>
<td>.607 .706</td>
<td>-1.78 2.81</td>
<td>.623 .723</td>
<td>-1.78 2.81</td>
<td>.623 .723</td>
<td>.504 .483</td>
<td></td>
<td></td>
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</table>
time \( n = 72 \), there is strong evidence for the presence of some specific jumps. In both cases and for both series, the average ratio is less than one. When \( n = 12 \), the percentage of ratios less than one is 70% and 73%, while when \( n \) increases to 72 these percentages become 71% in both cases.

Table 3 reports the corresponding results for the whole 10-year sample. This table, which provides a warning of the use of too high a value of \( n \), shows the sum, denoted \( r_5 \), of the first to fifth serial correlation coefficients of the high-frequency data. We see that in the DM/dollar series, as \( n \) increases, this correlation builds up, probably due to bid/ask bounce effects. By the time \( n \) has reached 288, the summed correlation has reached nearly -0.1, which means the realized variance overestimates the variability of prices by around 20%. Of course, this effect could be removed by using a further level of prefiltering before we analyse the data. The situation is worse for the yen/dollar series, which has a moderate amount of negative correlation among the high-frequency returns even when \( n \) is quite small. We will ignore these market microstructure effects here.

Table 3 shows the average value of \( \mu^{2}_{\delta} \hat{v}_{t} \) and \( \hat{v}_{t} \) as well as the proportion of times the null is rejected using 95% and 99% asymptotic tests. These values are given for a variety of values of \( n \) and for both exchange rates. The results are reasonably stable with respect to \( n \), although the percentage due to jumps does drift as \( n \) changes.

The table shows that for the DM/dollar series, the variation of the jumps is estimated to contribute between about 5% and 20% of the QV. On 20% of days, the hypothesis of no jumps is rejected at the 5% asymptotic level, while at the 1% asymptotic level, this falls to 10%. The results for the yen/dollar are rather similar. These results should be viewed tentatively, as the Monte Carlo results suggest there are finite sample biases in the critical values, even when we ignore market microstructure effects. However, the statistical evidence does push us toward believing there are jumps in the price processes. Interestingly, the percentage of rejections and proportions due to jumps seem rather stable as we move between the two exchange rates.

### 5.3 Case Studies

To illustrate this methodology we will apply the jump test to the DM/dollar rate, asking if the hypothesis of a continuous sample path is consistent with the data we have. To start, we will give a detailed discussion of an extreme day—Friday, January 15, 1988—which we will put in context by analyzing it together with a few days before the extreme event. In Figure 3 we plot 100 times the discretized \( Y_{\delta} \), so a one-unit uptick represents a 1% change, for a variety of values of \( n = 1/\delta \), as well as giving the ratio jump statistics \( \mu^{2}_{\delta} \hat{v}_{t} / \hat{v}_{t} \) with their corresponding 99% critical values.

In Figure 3 there is a large uptick in the DM against the dollar, with a movement of nearly 2% in a five-minute period. This occurred on Friday and was a response to the news of a large decrease in the U.S. balance of payment
deficit, which led to a large strengthening of the dollar. The Financial Times reported on its front page the next day:

The dollar and share prices soared in hectic trading on world financial markets yesterday after the release of official figures showing that the US

Table 3  \( r. \) denotes the sum of the first five serial correlation coefficients of the high-frequency data. BPV denotes the average value of \( \mu_i^2 \tilde{\omega}_i \) over the sample. QV gives the corresponding result for \( \tilde{\omega}_i \). Jump % is the percentage of the quadratic variation due to jumps in the sample. 5% rej. and 1% rej. show the proportion of rejections at the 5% and 1% levels, respectively.

<table>
<thead>
<tr>
<th></th>
<th>Dollar/DM</th>
<th></th>
<th>Dollar/yen</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>( r. )BPV</td>
<td>QV</td>
<td>Jump %</td>
</tr>
<tr>
<td>72</td>
<td>-.001 .437 .487</td>
<td>10.2</td>
<td>.225</td>
</tr>
<tr>
<td>144</td>
<td>-.056 .471 .510</td>
<td>7.6</td>
<td>.220</td>
</tr>
<tr>
<td>288</td>
<td>-.092 .502 .531</td>
<td>5.4</td>
<td>.181</td>
</tr>
</tbody>
</table>

Figure 3  (Left) Discretized sample paths of the exchange rate, centered at zero on Monday, January 11, 1988, and running until Friday of that week. Drawn every 20 and 5 minutes. An up-tick of one indicates strengthening of the dollar by 1%. (Right) shows an index plot, starting at one, of \( \frac{\mu_i^2 \tilde{\omega}_i}{\hat{\omega}_i} \), which should be about one if there are no jumps. Test is one sided, with critical values also drawn as a line.
trade deficit had fallen to $13.22 bn in November from October’s record level of $17.63 bn. The US currency surged 4 pfennigs and 4 yen within 10 minutes of the release of the figures and maintained the day’s highest levels in late New York business . . .“

The data for Friday had a large realized variance, but a much smaller estimate of the integrated variance. Hence the statistics are attributing a large component of the realized variance to the jump, with the adjusted ratio statistic being larger than the corresponding 99% critical value. When \( \delta \) is large, the statistic is on the borderline of being significant, while the situation becomes much clearer as \( \delta \) becomes small.

An important question is whether this day is typical of extreme days on the foreign exchange market. Here the focus will be on days where the ratio statistic is small and the realized variance is quite large. Throughout \( n = 288 \) is used.

Figure 4 plots results for all the eight days when the ratio statistic \( \mu_i^{-2} \hat{\delta}_i / \hat{\theta}_i \) is less than 0.6, suggesting a jump, and where the realized variance is greater than 1.2. On each day, the figure shows a single big movement that is much larger in magnitude than the others on that day. These large changes are listed in Table 4.

Most U.S. macroeconomic announcements are made at 8:30 EST, which is 12:30 GMT, from early April to late October, and 13:30 GMT otherwise. This means that all the jumps observed in Figure 4 correspond to macroeconomic

**Figure 4** Days on which \( \hat{\delta}_i \) is high and the jump ratio \( \mu_i^{-2} \hat{\delta}_i / \hat{\theta}_i \) found a jump using \( n = 288 \). Depicted is \( Y_{\delta} - Y_i \), the corresponding ratios \( \mu_i^{-2} \hat{\delta}_i / \hat{\theta}_i \), and 99% critical values.
announcements. There is substantial economic literature trying to relate movements in exchange rates to macroeconomic announcements [e.g., Ederington and Lee (1993) and Andersen, Bollerslev, Diebold, and Vega (2003b)]. Generally this concludes that such news is quickly absorbed into the market, moving the rates vigorously, but with little long-term impact on the subsequent volatility of the rates, which is very much in line with what we saw in Figure 3.

6 Conclusion

In this article we provide a test to ask, for a given time series of prices recorded every \( \delta \) time periods, if it is statistically satisfactory to regard the data as if it had a continuous sample path. We derive the asymptotic distribution of the testing procedure as \( \delta \downarrow 0 \) under the null of no jumps and ignoring the possible impact of market microstructure effects. Monte Carlo results suggest an adjusted ratio jump statistic can be reliably used to test for jumps if \( \delta \) is moderately small and the test is carried out over relatively short periods, such as a day. We applied this test to some exchange rate data and found many rejections of the null of no jumps. In some case studies we related the rejections to macroeconomic news.

The article opens up a number of technical questions. Can multivariate versions of these ideas be developed so one can detect common jumps across assets? How robust is bipower variation to market microstructure effects and can these effects be moderated in some way? We are currently researching on these topics with various coauthors and hope to report results on them soon. Another question is whether bipower variation is robust to infinite activity jump processes—that is, jump processes with an infinite number of jumps in a finite time interval? This is a technically demanding question and is addressed in recent work by Woerner (2004b) and Barndorff-Nielsen, Shephard, and Winkel (2004).

The article also naturally points to a number of economic issues. Can specific types of economic news be formally linked to the jumps indicated by these tests? Can the tests for jumps be used to improve volatility forecasts? Research on the second of these points has been recently reported by Andersen, Bollerslev, and Diebold (2003) and Forsberg and Ghysels (2004).

<table>
<thead>
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<th>Sequence</th>
<th>Day</th>
<th>GMT</th>
<th>Move</th>
</tr>
</thead>
<tbody>
<tr>
<td>173th</td>
<td>Friday, September 11, 1987</td>
<td>12.35</td>
<td>-.967</td>
</tr>
<tr>
<td>234th</td>
<td>Thursday, December 10, 1987</td>
<td>13.35</td>
<td>-1.44</td>
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<tr>
<td>253th</td>
<td>Friday, January 15, 1988</td>
<td>13.35</td>
<td>2.03</td>
</tr>
<tr>
<td>273th</td>
<td>Friday, February 12, 1988</td>
<td>13.35</td>
<td>1.16</td>
</tr>
<tr>
<td>312th</td>
<td>Thursday, April 14, 1988</td>
<td>12.35</td>
<td>-1.65</td>
</tr>
<tr>
<td>333th</td>
<td>Tuesday, May 17, 1988</td>
<td>12.35</td>
<td>1.14</td>
</tr>
<tr>
<td>416th</td>
<td>Wednesday, September 14, 1988</td>
<td>12.35</td>
<td>0.955</td>
</tr>
</tbody>
</table>
APPENDIX A: PROOF OF THEOREM 1

A.1 Assumptions and Statement of Two Theorems

In this appendix we prove two results we state in this subsection: (i) Theorem 2, which shows consistency of realized BPV; (ii) Theorem 3, which gives a joint central limit theory for realized BPV and QV under BSM. These two results then deliver Theorem 1 immediately. Sometimes we will refer to the unnormalized version of the $r, s$ order of realized BPV,

$$[Y_\delta]_t^{[r,s]} = \sum_{j=2}^{n} |y_{j-1}|^r |y_j|^s, r,s > 0.$$  

Clearly

$$[Y_\delta]_t^{[1,1]} = \{Y_\delta\}_t^{[1,1]},$$  

which is of central interest.

We will derive the limit results for a fixed value of $t$, and without loss of generality we assume that $[t/\delta]$ is integer, writing $t = \delta n$. So as $\delta \downarrow 0$, then necessarily $n \rightarrow \infty$. The general approach in our proofs is to study the limit theory conditionally on $(a, \sigma)$. The unconditional limit results then follow trivially, as, in the present circumstances, conditional convergence implies global convergence.

Recall the two assumptions we use in Theorem 1.

(a) The volatility process $\sigma$ is pathwise bounded away from zero.

(b) The joint process $(a, \sigma)$ is independent of the Brownian motion $W$.

Note also that our general precondition that $\sigma$ is càdlàg implies that $\sigma$ is pathwise bounded away from $\infty$.

**Theorem 2** Let $Y \in BSMJ$ and suppose conditions (a) and (b) hold, then

$$\{Y\}_t^{[1,1]} = \mu_1^2 \int_0^t \sigma_s^2 ds.$$  

**Theorem 3** Let $Y \in BSM$ and suppose conditions (a) and (b) hold. Then conditionally on $(a, \sigma)$, the realized QV and BPV processes

$$[Y_\delta]_t \quad \text{and} \quad \mu_1^2 \{Y_\delta\}_t^{[1,1]}$$

follow asymptotically, as $\delta \downarrow 0$, a bivariate normal law with common mean $\int_0^t \sigma_s^2 ds$. The asymptotic covariance of

$$\delta^{-1/2} \left[ \begin{pmatrix} [Y_\delta]_t \\ \mu_1^{-2} \{Y_\delta\}_t^{[1,1]} \end{pmatrix} - \left( \int_0^t \sigma_s^2 ds \right) \right]$$
is

$$\Pi \int_0^t \sigma_s^4 \, ds$$

where

$$\Pi = \begin{pmatrix} \text{var}(u^2) & 2\mu_2 \text{Cov}(u^2, |u||u'|) \\ 2\mu_2 \text{Cov}(u^2, |u||u'|) & \mu_4 \{\text{var}(|u||u'|) + 2\text{Cov}(|u||u'||u'')\} \end{pmatrix}$$

$$= \left( \frac{2}{2} \right) \frac{2}{(\pi^2/4) + \pi - 3} \approx \left( \frac{2}{2} \right) \frac{2}{2.6090}$$

with $u, u', u''$ being independent standard normals.

A.2 Consistency of Realized BPV: Theorem 2

Once the theorem is proved in the no jumps case, the general result follows trivially using the argument given in Barndorff-Nielsen and Shephard (2004b). Here we therefore assume $Y \in BSM$. The proof goes in three stages. We provide some preliminary results on discretization of integrated variance. Then we recall the consistency of BPV when $a = 0$, and finally we show that allowing $a \neq 0$ has negligible impact.

For the latter conclusion we only need to establish that the impact of $a$ is of order $O_p(1)$. However, for the proof of Theorem 3 we require order $O_p(\delta^{1/2})$. That this holds is verified separately in the next subsection.

We first recall a result, which is obtained in the course of the proof of Theorem 2 of Barndorff-Nielsen and Shephard [2004b, cf. Equation (13)].

**Proposition 1** [Barndorff-Nielsen and Shephard (2004b)] Under (a) we have for any $r > 0$ and $\sigma_j^2 = \int_0^t \sigma_s^2 \, ds$ that

$$\delta^{3-r} \left\{ \sum_{j=2}^{n} \sigma_{j-1}^r \sigma_j^r - \sum_{j=1}^{n} \sigma_{j-1}^{2r} \right\} = O_p(\delta).$$

**Corollary 1** Under (a) we have that

$$\sum_{j=2}^{n} \sigma_{j-1} \sigma_j - \int_0^t \sigma_s^2 \, ds = O_p(\delta).$$

This corollary is a special case of the previous proposition and follows from the fact that $\sum_{j=1}^{n} \sigma_j^2 = \int_0^t \sigma_s^2 \, ds$.

The following is a restatement of Theorem 2 in Barndorff-Nielsen and Shephard (2004b) in the case where $r = s = 1$. It will be used to prove Theorem 3.
**Theorem 4** Suppose $Y \in BSM$ and in addition (a), (b), and $a = 0$, then

$$\left( \sum_{j=2}^{n} |y_{j-1}| |y_j| \right) - \mu_1^2 \int_0^t \sigma_s^2 ds = o_P(1).$$

To complete the proof of Theorem 2 we need to show that the impact of the drift is negligible. As already mentioned, this follows from the stronger result, Proposition 2 which we derive in the next subsection.

### A.3 Negligibility of Drift

For simplicity of notation we now write $M_t = \int_0^t \sigma_s dW_s$, which, conditional on $\sigma$, has a Gaussian law with a zero mean and variance of $\int_0^t \sigma_s^2 ds$. To establish that the effect of the drift is negligible in the contexts of Theorems 2 and 3 it suffices to show that, under conditions (a) and (b),

$$[Y_\delta]_{t}^{[1,1]} - [M_{\delta}]_{t}^{[1,1]} = o_P(\delta^{1/2}).$$

In fact, we shall prove the following stronger result, which covers a variety of versions of realized BPV. To do this we will use the notation

$$h_r(u; \rho) = |\rho \delta^{1/2} + u|^r - |u|^r,$$

$$h_{r,s}(u,v; \rho_1, \rho_2) = |\rho_1 \delta^{1/2} + u|^r |\rho_2 \delta^{1/2} + v|^s - |u|^r |v|^s.$$

For reasons of compactness we often write $h_{r,s}(u,v; \rho_1, \rho_2)$ as $h_{r,s}(u,v; \rho)$, where $\rho = (\rho_1, \rho_2)^T$.

**Proposition 2** Under conditions (a) and (b), for any $r,s > 0$ and for every $\varepsilon \in \left( 0, \frac{1}{4} \right)$

$$[Y_\delta]_{t}^{[r,s]} - [M_{\delta}]_{t}^{[r,s]} = O_P(\delta^{(r+s-1)/2 + \varepsilon}).$$

**Proof.** Let $\sigma^2 = \inf_{0 \leq s \leq t} \sigma_s^2$ and $\overline{\sigma^2} = \sup_{0 \leq s \leq t} \sigma_s^2$, $m_j = M_{j\delta} - M_{(j-1)\delta}$ and $\sigma_j^2 = \int_{(j-1)\delta}^{j\delta} \sigma_s^2 ds$,

$$\gamma_j = \delta^{-1} a_j, \ a_j = \int_{(j-1)\delta}^{j\delta} a_s ds, \ j = 1,2, \ldots, n.$$

Note that (pathwise for $(a, \sigma)$), by assumption (a), $0 < \sigma^2 \leq \overline{\sigma^2} < \infty$, implying if $\theta_j \delta = \sigma_j^2$, then $0 < \min \theta_j \leq \max \theta_j < \infty$, while, due to $a$ being càdlàg, there exists (pathwise) a constant $c$ for which $\max_j |\gamma_j| \leq c$, whatever the value of $n$. 


We have, using (b) and writing now $m_j \triangleq \sigma_j |u_j|$, where the $u_j \sim i.i.d. N(0, 1)$, that

$$
[Y_{ij}]^{(rs)} - [M_\delta]^{(rs)} = \sum_{j=2}^{n} (|a_{j-1} + m_{j-1}|^q |a_j + m_j|^r - |m_{j-1}|^q |m_j|^r)
$$

$$
= \sum_{j=2}^{n} (|\delta \gamma_{j-1} + \delta^{1/2} \theta^{1/2}_j u_{j-1}|^q |\delta \gamma_j + \delta^{1/2} \theta^{1/2}_j u_j|^r
$$

$$
- |\delta^{1/2} \theta^{1/2}_{j-1} u_{j-1}|^q |\delta^{1/2} \theta^{1/2}_j u_j|^r)
$$

$$
= \delta^{s/2} \theta^{s/2} \sum_{j=2}^{n} \theta_j^{s/2} \theta_j^{1/2} \left\{ |(\gamma_{j-1}/\theta_j^{1/2})| \delta^{1/2} + u_j|^r - |u_{j-1}|^q |u_j|^r \right\}
$$

and hence

$$
\delta^{-(r+s)/2} \left\{ [Y_{ij}]^{(rs)} - [M_\delta]^{(rs)} \right\} = \sum_{j=2}^{n} \theta_j^{s/2} \theta_j^{1/2} \left\{ |u_{j-1}|, u_j; \gamma_{j-1}, \theta_{j-1}, \gamma_j, \theta_j \right\}.
$$

As $|\gamma_j/\theta_j^{1/2}|$ is bounded for all $j$, the conclusion of Proposition 2 now follows from Corollary 2, below.

To obtain that corollary, we establish three lemmas—1, 2, and 3. Lemma 1 collates several results from Barndorff-Nielsen and Shephard (2003) which are used to prove Lemmas 2 and 3.

**Lemma 1** [Barndorff-Nielsen and Shephard (2003)] For any $r > 0$ and $\rho \geq 0$, we have

$$
E\{h_r(u; \rho)\} = O(\delta), \quad E\{|u|^r h_r(u; \rho)\} = O(\delta^{(1+1/\rho)/2}),
$$

$$
E\{h_r^2(u; \rho)\} = O(\delta^{(1+1/\rho)/2}), \quad \text{var}\{h_r(u; \rho)\} = O(\delta^{(1+1/\rho)/2}).
$$

The results given in Lemma 1 are derived in the course of the proof of Proposition 3.3 in Barndorff-Nielsen and Shephard (2003), so a separate proof will not be given here.

We proceed to state and prove Lemmas 2 and 3.

**Lemma 2** For any $r, s > 0, u, v \sim i.i.d. N(0, 1)$ and $\rho_1$ and $\rho_2$ nonnegative constants, we have

$$
E\{h_{r,s}(u,v; \rho_1, \rho_2)\} = O(\delta).
$$

**Proof.** The independence of $u, v$ together with the first equation in Lemma 1 implies
Furthermore, so, by Lemma 2 and the independence of $u$ nonnegative constants, we have

$$E\{ h_{r,s}(u,v;\rho) \} = E\left\{ |\rho_1 \delta^{1/2} + u|^r \right\} E\left\{ |\rho_2 \delta^{1/2} + u|^s \right\} - E\{|u|^r\} E\{|v|^s\}
= E\{ h_r(u;\rho_1) \} E\{ h_s(v;\rho_2) \} + E\{ h_r(u;\rho_1) \} E\{|v|^s\}
+ E\{ h_s(v;\rho_2) \} E\{|u|^r\} = O(\delta).$$

**Lemma 3** For $u, v$ independent standard normal random variables and $\rho_1$ and $\rho_2$ nonnegative constants, we have

$$E\{ h_{r,s}^2(u,v;\rho_1,\rho_2) \} = O\left( \delta^{(1+1\wedge r\wedge s)/2} \right).$$

**Proof.** Clearly

$$h_{r,s}^2(u,v;\rho) = |\rho_1 \delta^{1/2} + u|^{2r} |\rho_2 \delta^{1/2} + v|^{2s} + |u|^{2r}|v|^{2s} - 2|\rho_1 \delta^{1/2} + u|^r |\rho_2 \delta^{1/2} + v|^s |u|^r |v|^s
= h_{2r,2s}(u,v;\rho) + 2|u|^{2r}|v|^{2s} - 2|\rho_1 \delta^{1/2} + u|^r |\rho_2 \delta^{1/2} + v|^s |u|^r |v|^s,$$
so, by Lemma 2 and the independence of $u$ and $v$,

$$E\{ h_{r,s}^2(u,v;\rho) \} = E\{ h_{2r,2s}(u,v;\rho) \} + 2E\{|u|^{2r}\} E\{|u|^{2s}\}
- 2E\{|u|^r|\rho_1 \delta^{1/2} + u|^r\} E\{|u|^s|\rho_2 \delta^{1/2} + u|^s\}
= O(\delta) - 2\left( E\{|u|^r|\rho_1 \delta^{1/2} + u|^r\} E\{|u|^s|\rho_2 \delta^{1/2} + u|^s\}
- E\{|u|^{2r}\} E\{|u|^{2s}\}\right).$$

Furthermore,

$$E\left\{ |u|^r|\rho_1 \delta^{1/2} + u|^r \right\} E\left\{ |u|^s|\rho_2 \delta^{1/2} + u|^s \right\} - E\left\{ |u|^{2r}\right\} E\left\{ |u|^{2s}\right\}
= E\left\{ |u|^r|\rho_1 \delta^{1/2} + u|^r - |u|^{2r}\right\} E\left\{ |u|^s|\rho_2 \delta^{1/2} + u|^s \right\}
+ E\left\{ |u|^{2r}\right\} E\left\{ |u|^s|\rho_2 \delta^{1/2} + u|^s \right\} - E\left\{ |u|^{2r}\right\} E\left\{ |u|^{2s}\right\}
= E\left\{ |u|^r|\rho_1 \delta^{1/2} + u|^r - |u|^{2r}\right\} E\left\{ |u|^s|\rho_2 \delta^{1/2} + u|^s - |u|^{2s}\right\}
+ E\left\{ |u|^{2r}\right\} E\left\{ |u|^s|\rho_2 \delta^{1/2} + u|^s - |u|^{2r}\right\}
+ E\left\{ |u|^{2r}\right\} E\left\{ |u|^s|\rho_2 \delta^{1/2} + u|^s \right\} - E\left\{ |u|^{2r}\right\} E\left\{ |u|^{2s}\right\}
= E\{ h_r(u;\rho_1) \} E\{ h_s(u;\rho_2) \}
+ E\left\{ |u|^{2r}\right\} E\left\{ |u|^r h_r(u;\rho_1) \right\} + E\left\{ |u|^{2r}\right\} E\left\{ |u|^s h_s(u;\rho_2) \right\}.$$
\[
E\{h_{r,s}^2(u,v;\rho_1,\rho_2)\} = O(\delta) + O\left(\delta^{(1+1\lambda\rho)}\right)O\left(\delta^{(1+1\lambda\sigma)}\right)
+ O\left(\delta^{(1+1\lambda\rho)}\right) + O\left(\delta^{(1+1\lambda\sigma)}\right) = O\left(\delta^{(1+1\lambda\rho\sigma)}\right). \]

Lemmas 2 and 3 and the Cauchy-Schwarz inequality together imply

**Corollary 2** For \(u, v, v'\) independent standard normal random variables and \(\rho_1, \rho_2, \rho_1', \rho_2'\) nonnegative constants, we have

\[
\text{var}\{h_{r,s}(u,v;\rho_1,\rho_2)\} = O\left(\delta^{(1+1\lambda\rho\sigma)}\right)
\]

and

\[
\text{cov}\{h_{r,s}(u,v;\rho_1,\rho_2),h_{r,s}(u,v';\rho_1',\rho_2')\} = O\left(\delta^{(1+1\lambda\rho\sigma)}\right).
\]

As already mentioned, the conclusion of Proposition 2 follows from Corollary 2.

**Remark 2** From the final equation in the proof of Lemma 3, one sees that in the special case when \(r = s = 1\), then

\[
\text{var}\{h_{1,1}(u,v;\rho_1,\rho_2)\} = O(\delta),
\]

and hence the conclusion of Proposition 2 may be sharpened to \([Y_\delta]^{[1,1]} - [M_\delta]^{[1,1]} = O_p(\delta).\)

**A.4 Asymptotic Distribution of BPV: Theorem 3**

Given Proposition 2, what remains is to prove Theorem 3 when \(Y_{BSM}\) and the additional conditions (a), (b), and \(a = 0\) hold. The key feature is that, ignoring the asymptotically negligible \(y_1^2\) and conditioning on the \(\sigma\) process, we have that

\[
\left(\frac{\sum_{j=2}^n y_j^2}{\sum_{j=2}^n |y_{j-1}| y_j}\right) - \left(\frac{j_0^l \sigma_z^2 ds}{\mu_1^2 j_0^l \sigma_z^2 ds}\right)
\]

is asymptotically equivalent in law to

\[
\sum_{j=2}^n \left(\frac{\sigma_j^2 v_j}{\sigma_{j-1} \sigma_j v_j}\right)
\]
where \( v_j = u_j^2 - 1, w_j = |u_{j-1}| |u_j| - \mu_1^2, \) and \( u_j \overset{i.i.d.}{\sim} N(0, 1) \). The sequences \( \{v_j\} \) and \( \{w_j\} \) have zero means, with the former being i.i.d., while the latter satisfy \( w_j \perp w_{j+s} \) for \( |s| > 1 \). Then the theorem follows if we can show that

\[
\delta^{-1/2} \sum_{j=2}^{n} \left( \frac{\sigma_j^2 v_j}{\sigma_{j-1} \sigma_j w_j} \right) \overset{d}{\rightarrow} N\left( 0, \int_0^t \sigma_s^4 ds \left( \text{var}(v_1) + 2 \text{cov}(v_1, w_1) \right) \right).
\]

(19)

Our strategy for proving this is to show\(^3\) the limiting Gaussian result using any real constants \( c_1 \) and \( c_2 \),

\[
\delta^{-1/2} \sum_{j=2}^{n} \left( c_1 \sigma_j^2 v_j + c_2 \sigma_{j-1} \sigma_j w_j \right) \overset{d}{\rightarrow} N\left( 0, \int_0^t \sigma_s^4 ds \left[ c_1^2 \text{Var}(v_1) + 4 c_1 c_2 \text{cov}(v_1, w_1) + c_2^2 \left( \text{var}(w_1) + 2 \text{cov}(w_1, w_2) \right) \right] \right).
\]

The asymptotic Gaussianity follows from standard calculations from the classical central limit theorem for martingale sequences due to Lipster and Shiryaev [e.g., Shiryaev (1981: 216)].

What remains is to derive the asymptotic variance of this sum. Let use define

\[
\psi_j = \sqrt{\delta^{-1} \int_{\delta(j-1)}^{\delta j} \sigma_s^2 ds}.
\]

Clearly

\[
\delta^{-1/2} \sum_{j=2}^{n} \left( c_1 \sigma_j^2 v_j + c_2 \sigma_{j-1} \sigma_j w_j \right) = \delta^{1/2} \sum_{j=2}^{n} \left( c_1 \psi_j^2 v_j + c_2 \psi_{j-1} \psi_j w_j \right)
\]

has the variance

\[
\delta \sum_{j=2}^{n} \text{var} \left( c_1 \psi_j^2 v_j + c_2 \psi_{j-1} \psi_j w_j \right) + 2 \delta \sum_{j=3}^{n} \text{cov} \left( c_2 \psi_{j-2} \psi_{j-1} w_j, c_2 \psi_{j-1} \psi_{j-2} w_{j-1} \right) + 2 \delta \sum_{j=2}^{n} \text{cov} \left( c_1 \psi_{j-1}^2 v_{j-1}, c_2 \psi_{j-1} \psi_j w_j \right).
\]

\(^3\) Recall that if \( z_n = (z_n^1, ..., z_n^n) \) is a sequence of random vectors having mean zero, then to prove that \( z_n \overset{d}{\rightarrow} N_q(0, \Psi) \) for some nonnegative definite matrix \( \Psi \), it suffices to show that for arbitrary real constants \( c_1, ..., c_q \) we have \( c_n^t \overset{d}{\rightarrow} N_q(0, c^t \Psi c) \), where \( c = (c_1, ..., c_q)^t \). (This follows directly from the characterization of convergence in law in terms of convergence of the characteristic functions.)
Now using Riemann integrability

\[
\delta \sum_{j=2}^{n} \text{var} \left( c_1 \psi_j^2 v_j + c_2 \psi_{j-1} \psi_j w_j \right) \\
= \text{var}(v_1) c_1^2 \delta \sum_{j=2}^{n} \psi_j^4 + \text{var}(w_1) c_2^2 \delta \sum_{j=2}^{n} \psi_{j-1}^2 \psi_j^2 \\
+ 2 \delta \text{cov}(v_1, w_1) c_1 c_2 \sum_{j=2}^{n} \psi_{j-1} \psi_j^3 \\
\rightarrow \int_0^t \sigma_s^4 \text{d}s \left\{ c_1^2 \text{var}(v_1) + c_2^2 \text{var}(w_1) + 2 c_1 c_2 \text{cov}(v_1, w_1) \right\},
\]

using the fact that \( \text{cov}(v_1, w_2) = \text{cov}(v_1, w_1) \). Likewise

\[
\delta \sum_{j=3}^{n} \text{cov} \left( c_2 \psi_{j-1} \psi_j w_j, c_2 \psi_{j-2} \psi_{j-1} w_{j-1} \right) \\
= c_2^2 \text{cov}(w_1, w_2) \delta \sum_{j=3}^{n} \psi_{j-2} \psi_{j-1} \psi_j \\
\rightarrow c_2^2 \text{cov}(w_1, w_2) \int_0^t \sigma_s^4 \text{d}s,
\]

while

\[
\delta \sum_{j=2}^{n} \text{cov} \left( c_1 \psi_{j-1}^2 v_{j-1}, c_2 \psi_{j-1} \psi_j w_j \right) \rightarrow c_1 c_2 \text{cov}(v_1, w_1) \int_0^t \sigma_s^4 \text{d}s.
\]

This confirms the required covariance pattern stated in Equation (19).

Received February 22, 2005; revised May 13, 2005; accepted June 9, 2005.

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