More on Lévy Processes

1 The Types of Lévy Processes

Let $m(x)$ be the jump intensity function. We already know

$$\int_{|x| \geq a} m(x) \, dx < \infty$$

(1)

For any $a > 0$. Also, for the following limiting process (below) to work, it has to be the case that

$$\int_{|x| \leq 1} x^2 m(x) \, dx < \infty$$

(2)

Sometimes these two requirements are combined as

$$\int \min(x^2, 1) m(x) \, dx < \infty.$$  

(3)

The choice of 1 as the cut point is arbitrary; any positive number works. Also, recall that if $|x| < 1$ then $|x|^2 < |x| < 1$, so (2) does not imply $\int_{|x| \leq 1} |x|^p \, dx < \infty$ for $p < 2$.

If $\lambda = \int m(x) \, dx < \infty$ the process is said to be finitely active. Indeed, in this case $m(x)$ defines a compound Poisson process with jump density $f(x) = m(x)/\lambda$. Otherwise $m(x)$ defines a Lévy process by taking limits in distribution of compound Poisson processes. When $\int m(x) \, dx = \infty$ the process is infinitely active and takes a countably infinite number of jumps on any finite subinterval of $[0, T]$. So, finite versus infinite activity is determined by the integrability of $m(x)$.

A distinct, but closely related concept is Blumenthal-Getoor index defined as

$$\beta = \inf\{p : p \geq 0, \int |x|^p m(x) \, dx < \infty\}.$$  

(4)

If $0 \leq \beta \leq 1$ then the paths (trajectories) are of finite variation, $\sum_{0 \leq s \leq t} |\Delta_s X| < \infty$. Otherwise when $1 < \beta < 2$ the paths are of infinite variation $\sum_{0 \leq s \leq t} |\Delta_s X| = \infty$. If $\beta > 0$ then perforce $\int m(x) \, dx = \infty$ and the process is infinitely active. If $X$ is finitely active then $\beta = 0$; however, there are examples where $\beta = 0$ and the process is still infinitely.

2 Compensating Small Jumps to Define General Lévy Processes

Let $m(x)$ be the jump intensity function. We know

$$\int_{|x| \geq a} m(x) \, dx < \infty$$

(5)
\[
\int \min(x^2, 1) \, m(x) \, dx < \infty. \quad (6)
\]

We work with the small jumps and the big jumps. The small jumps are trickier. For each positive integer \( n \), let \( Y_{n,1,t} \) be a (compensated) compound Poisson process defined by \( m(x)1_{\{\frac{1}{n} \leq |x| \leq 1\}} \):

\[
Y_{1,n,t} = \sum_{j=1}^{N_{1,n,t}} X_{1,n,j} - t\lambda_n \mu_n, \quad (7)
\]

where \( X_{1,n,j} \sim f_{(1),n}(x) \),

\[
f_{(1),n} = \frac{1}{\lambda_n} \int_{\{\frac{1}{n} \leq |x| \leq 1\}} m(x) \, dx,
\]

\[
\lambda_n = \int_{\{\frac{1}{n} \leq |x| \leq 1\}} m(x) \, dx, \quad \mu_n = \int_{\{\frac{1}{n} \leq |x| \leq 1\}} x f_{(n),1}(x) \, dx,
\]

and \( N_{1,n,t} \) is a Poisson random variable with intensity \( \lambda_n \).

For the big jumps, consider the compound Poisson process defined by \( m(x)1_{\{|x| > 1\}} \):

\[
Y_{2,t} = \sum_{j=1}^{N_{2,i}} X_{2,j} X_{2,i} \sim f_{(2)}(x), \quad f_{(2)}(x) = \frac{1}{\lambda_2} \int_{|x| > 1} m(x) \, dx, \quad (8)
\]

Taken together we have

\[
Y_{n,t} = Y_{n,1,t} + Y_{2,t} \quad (9)
\]

Using advanced methods one can show that \( Y_{n,t} \) converges to a well defined Lévy process \( Y_t \). Because the small jumps are compensated is why the characteristic function of \( Y_t \) is of the form

\[
\exp \left[ t \int_{|x| \leq 1} (e^{iux} - 1 - iux) + t \int_{|x| > 1} (e^{iux} - 1) \right]. \quad (10)
\]