Measuring CAPM beta and Empirical Analysis on the Effects of Jumps*

Hao Sun
Professor Tim Bollerslev, Faculty Advisor
Professor George Tauchen, Faculty Advisor
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1. Introduction

In recent years, the field of high frequency data research has been expanding at a very fast pace. Ever since the ARCH model (Engle, 1982) was introduced, there have been many other models for measuring volatility of stocks, including Realized Variance and Bipower Variation (Barndorff-Nielsen and Shephard, 2004). Here, we will use Realized Variance and Bipower Variation to introduce a new way to measure the beta in the Capital Asset Pricing Model (CAPM), with high frequency data.

The CAPM model, a single factor model measuring the risk of a single asset in terms of systemic and idiosyncratic risk, has been widely used today for estimating expected return and risk. According to Martin and Simin (2003), though there are many criticisms of the CAPM model, major Financial Information Services including Bloomberg, Dow Jones, and Standard and Poor’s use beta calculated from a simple Ordinary Least Square (OLS).

\[ R_t = \alpha + R_{m,t} \times \beta + \epsilon_t, \ t = 1, ..., T \]

Similarly, Andersen, Bollerslev, Diebold, and Wu (2005) argued that CAPM is a simple model that performs fairly well when compared against more complex models, such as multi-factor regressions. They pioneered the research in high frequency beta by using Pearson Correlation and Realized Variance of the market to calculate the CAPM beta.

\[ PCov_t(r_i, r_m) = \sum_{j=1}^{385/4} r_{i,j} r_{m,j} \]  \hspace{1cm} (1)

\[ R\beta_t = \frac{PCov(r_i, r_m)_t}{RV_{r_m}} \]

This paper is motivated by their research and their claim that a more realistic beta would be time-varying. We seek to find a jump robust estimator of the CAPM beta in order to assess the effects of jumps in stock prices on the CAPM beta, i.e. whether the jumps create any risks in addition to the systematic risks determined by the volatility of the market. As a minor goal, we will also look at the time-varying feature of beta, to see whether betas are significantly higher during
financial crisis. This idea is inspired by the higher betas of MS and NYX, for which we only have the data from 2006 to 2010. One possibility is that the recent financial crisis causes the betas to spike and the average betas for other stocks are dragged down by lower betas from other periods. Thus are the objectives of this paper.

In the next section, we will explore the theoretical framework underlying this research. In particular, we will derive Bipower Covariance from Bipower Variances, and dividing the Bipower Covariance between a stock and the market index by the (either Bipower or Realized) Variance of the market index will result in a jump robust estimator of the CAPM beta.

In section 3, we will describe our statistics methods in order to determine whether the contribution of jumps is significantly different from 0. And then in section 4 and 5, we will explain our dataset and provide empirical results of our research with the S&P 100 Futures index and stocks from S&P 100. Finally, in section 6, we will discuss the results and conclude our findings.
2. Theoretical Framework

2.1 Stochastic Model of Stock Prices

First, let’s define the model governing the movement of stock prices. Here, we use a simple continuous-time stochastic process, which states that log-prices of stocks are moved by a deterministic and a random drift.

\[ dp(t) = \mu(t)dt + \sigma(t)dw(t), \]  
\[ \text{(2)} \]

Where \( \mu(t) \) is the time-varying drift of log-price \( p(t) \) and \( \sigma(t)dw(t) \) is the time-varying volatility component, where \( \sigma(t) \) is the instantaneous volatility of the stock, and \( w(t) \) is the Wiener process, or Standard Brownian motion. However, this produces a continuous sequence of stock prices, which is contrary to the discontinuities observed empirically in actual stock prices. Therefore, we have to add a jump term (Poisson-driven) to incorporate these discontinuities:

\[ dp(t) = \mu(t)dt + \sigma(t)dw(t) + \kappa(t)df(t), \]  
\[ \text{(3)} \]

where \( \kappa(t) \) is the size of the jump and \( f(t) \) is number of jumps until time \( t \).

2.2 Realized Variance and Covariance

Since we are using 5-minute prices to calculate Realized Variance, let log-stock prices be \( p(t) \), where \( t \) time:

\[ r_{t,i} = p\left(t - 1 + \frac{i}{M}\right) - p\left(t - 1 + \frac{i - 1}{M}\right). \]

And then, the realized variance can be defined as:

\[ RV_t = \sum_{i=1}^{M=385/\Delta} r_{t,i}^2 \]  
\[ \text{(4)} \]

where \( \Delta = 5 \), since we are using 5-minute frequencies in a trading day with a total of 385 minutes (i.e. 9:35-15:59). Since we only use the returns from each day to calculate the Realized Variance, we are only calculating the intraday Realized Variance, meaning we don’t take the overnight return into consideration. Since majority of the jumps happen overnight as the closing price from previous day is very different from the opening price on the following day because of
overnight announcements or macroeconomics policies, our method already excludes those jumps from our Realized Variance calculations. However, we still have to deal with the minor intraday jumps.

When delta goes to 0, Realized Variance converges to integrated variance plus the jumps:

$$\lim_{\Delta \to 0} RV_t = \int_{t-1}^{t} \sigma^2(s)ds + \sum_{i=1}^{M} \kappa^2(t_i).$$

Now we introduce a new method to calculate covariance using Realized Variances. First, let us construct a portfolio with equal weights of stock $i$ and S&P 100 market index $m$.

$$r_p = \frac{1}{2} r_i + \frac{1}{2} r_m,$$  \hspace{1cm} (5)

where $r_i$ is the return on stock $i$ and $r_m$ is the return on the S&P 100 index.

Then,

$$Var(r_p) = Var\left(\frac{1}{2} r_i + \frac{1}{2} r_m\right)$$

$$= \left(\frac{1}{2}\right)^2 Var(r_i) + \left(\frac{1}{2}\right)^2 Var(r_m) + 2 \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) Cov(r_i, r_m)$$

$$= \frac{1}{4} Var(r_i) + \frac{1}{4} Var(r_m) + \frac{1}{2} Cov(r_i, r_m).$$

The expansion of the Variance function can be easily proved by the applying the definitions of Variance and Covariance in terms of Expectation (i.e. $Var(r) = E[r^2] - E[r]^2$, $Cov(r, s) = E[rs] - E[r]E[s]$).

To calculate, we can rearrange the terms and get:

$$Cov(r_i, r_m) = 2 \times \left[ Var(r_p) - \frac{1}{4} Var(r_i) + \frac{1}{4} Var(r_m) \right].$$  \hspace{1cm} (6)
Since we have $r_i$ and $r_m$, we can easily calculate $r_p$ for each 5-minute interval. And then we can get:

$$RV_{p,t} = \sum_{j=1}^{M} r_{p,t,j}^2,$$

Similarly, we have

$$RV_{i,t} = \sum_{j=1}^{M} r_{i,t,j}^2,$$

$$RV_{m,t} = \sum_{j=1}^{M} r_{m,t,j}^2.$$

And finally we can get Realized Covariance, which is simply replacing all the variances in equation (6) with Realized Variances:

$$RCov_t(r_i, r_m) = 2 \times \left[ RV_{p,t} - \frac{1}{4} RV_{i,t} + \frac{1}{4} RV_{m,t} \right]. \tag{7}$$

Note that when stock $i$ and stock $m$ receive equal weights (i.e. $\frac{1}{2}$) in the portfolio, the Realized Covariance calculated by equation (7) is exactly the same as Pearson Correlation in equation (1). The proof is as the following:

$$RCov_t(r_i, r_m) = 2 \times \left[ RV_{p,t} - \frac{1}{4} RV_{i,t} + \frac{1}{4} RV_{m,t} \right]$$

$$= 2 \times \left[ \sum_{j=1}^{M} r_{p,t,j}^2 - \frac{1}{4} \sum_{j=1}^{M} r_{i,t,j}^2 - \frac{1}{4} \sum_{j=1}^{M} r_{m,t,j}^2 \right]$$

$$= 2 \times \left[ \sum_{j=1}^{M} \left( \frac{1}{2} r_{i,t,j} + \frac{1}{2} r_{m,t,j} \right)^2 - \frac{1}{4} \sum_{j=1}^{M} r_{i,t,j}^2 - \frac{1}{4} \sum_{j=1}^{M} r_{m,t,j}^2 \right]$$

$$= 2 \times \left[ \sum_{j=1}^{M} \left( \frac{1}{4} r_{i,t,j}^2 + \frac{1}{4} r_{m,t,j}^2 + \frac{1}{2} r_{i,t,j} r_{m,t,j} \right) - \frac{1}{4} \sum_{j=1}^{M} r_{i,t,j}^2 - \frac{1}{4} \sum_{j=1}^{M} r_{m,t,j}^2 \right]$$

$$= 2 \times \sum_{j=1}^{M} \frac{1}{2} r_{i,t,j} r_{m,t,j}$$
\[ = \sum_{j=1}^{M} r_{i,t,j} r_{m,t,j} = PCov_t(r_i, r_m) \]

### 2.3 Bipower Variance and Covariance

Bipower Variance, introduced by Barndorff-Nielsen and Shephard (2003), is as the following:

\[
BV_t \equiv \mu_1^{-2} \frac{M}{M - 1} \sum_{j=2}^{M=385/\Delta} |r_{t,j-1}| |r_{t,j}|, \tag{8}
\]

where \( \mu_1^{-2} = \frac{\pi}{2} \), and \( \Delta = 5 \) (again for 5-minute sampling). And Bipower Variance is a jump robust estimator of the integrated variance, since Barndorff-Nielsen and Shephard (2003) has shown that

\[
BV_t \equiv \mu_1^{-2} \frac{M}{M - 1} \sum_{j=2}^{M=385/\Delta} |r_{t,j-1}| |r_{t,j}| \to \int_{t-1}^{t} \sigma^2(s) ds.
\]

With the same construction as equation (5) in section 2.2, we can calculate Bipower Covariance in terms of Bipower Variance. Let \( BV_{p,t} \) be the Bipower Variance for portfolio \( p \), \( BV_{i,t} \) be the Bipower Variance for stock \( i \), and \( BV_{m,t} \) be the Bipower Variance for S&P 100 index at time \( t \). Then we define Bipower Covariance as

\[
BCov_t(r_i, r_m) = 2 \times \left[ BV_{p,t} - \frac{1}{4} BV_{i,t} + \frac{1}{4} BV_{m,t} \right]. \tag{9}
\]

We shall use the Realized Covariance and Bipower Covariance that we have constructed in section 2.2 and 2.3 to calculate Realized Beta and Bipower Beta. But first, let us look at CAPM model in more detail.

### 2.4 Capital Asset Pricing Model

Under the CAPM model, we have:

\[
R_i = \beta_i R_m + \epsilon_i,
\]
where \( R_i = r_i - r_f \) is the excess return on stock \( i \) (i.e. return on stock \( i \) minus the risk-free rate), \( R_m \) is the excess return of the market index, and \( \epsilon_i \) is the idiosyncratic risk of stock \( i \), which is uncorrelated with \( R_m \) and idiosyncratic risk of any other stocks under CAPM assumptions (i.e. 
\[
0 = Cov(\epsilon_i, R_m) = Cov(\epsilon_i, r_m - r_f) = Cov(\epsilon_i, r_m).
\]

Since the variation in the risk free rate relative to that of the individual stock return is so small over the time horizon that we are looking at, we can safely disregard \( r_f \).

So we have

\[
r_i = \beta_i r_m + \epsilon_i. \tag{10}
\]

Now, we can calculate covariance of stock \( i \) and market \( m \) in the CAPM setting:

\[
Cov(r_i, r_m) = Cov(\beta_i r_m + \epsilon_i, r_m)
\]

\[
= \beta_i Cov(r_m, r_m) + Cov(\epsilon_i, r_m)
\]

\[
= \beta_i Var(r_m)
\]

Dividing both sides by variance of the market return gives us a formula for beta:

\[
\beta_i = \frac{Cov(r_i, r_m)}{Var(r_m)} \tag{11}
\]

### 2.5 Realized Beta and Bipower Beta

As derived in the CAPM model, we can find the beta for stock \( i \) by dividing covariance between stock \( i \) and market \( m \) by the variance of returns on market \( m \). From section 2.2 and 2.3, we have derived Realized Covariance and Bipower Covariance, which can be readily substituted into equation (11) to calculate the Realized Beta and Bipower Beta. However, we still have to discuss the choice of the Variance of returns on the market \( m \). We can use Realized Variance or Bipower Variance for both, or use Realized Variance for Realized Beta and Bipower Variance for the Bipower Beta.

For the sake of comparing Realized Beta and Bipower Beta, We will use Realized Variance for the denominator to calculate both the Realized and Bipower Beta. But in Section 3, we will also provide Bipower Beta calculated from Bipower Variance of the market return to show there is no
statistically significant difference between using Bipower Variance and using Realized Variance in the denominator.

So now after substituting equation (7) and (9) into equation (11), we get Realized Beta:

$$ R\beta_{i,t} = \frac{RCov_t(r_i, r_m)}{RV_{m,t}} = \frac{2 \times \left[ R\nu_{p,t} \frac{1}{4} RV_{i,t} + \frac{1}{4} RV_{m,t} \right]}{RV_{m,t}} $$

(12)

And Bipower Beta:

$$ B\beta_{i,t} = \frac{BCov_t(r_i, r_m)}{RV_{m,t}} = \frac{2 \times \left[ BV_{p,t} \frac{1}{4} BV_{i,t} + \frac{1}{4} BV_{m,t} \right]}{RV_{m,t}} $$

(13)

2.5 Jumps

Since Bipower Variances are jump-robust, we can conclude that Bipower Covariances are jump-robust, and so is Bipower Beta. Thus, to measure the contribution of jump in the Realized Beta, we define:

$$ Jump \equiv R\beta_{i,t} - B\beta_{i,t}. $$

(14)

Thus, concludes our theoretical framework. We will then provide the empirical results calculated based on equation (12) and (13).
3. Statistical Tests

For the testing of contribution of jumps, we will mainly use t-test:

\[ H_0: \text{Jump} = 0 \]
\[ H_a: \text{Jump} \neq 0 \]

and

\[ t - \text{stat} = \frac{\text{Jump} - 0}{\hat{SE}} \]

(15)

and the df = n - 1, where n is the sample size, and \( \hat{SE} = \frac{\hat{\sigma}}{\sqrt{n}} \).
4. Data

We used high frequency (1-minute) prices of 15 stocks from S&P 100 index, sampled at 5-minute frequency. As indicated in Graph 1 (Volatility Signature Plot), for Realized Variance, sampling at any finer frequencies would contaminate the results with microstructure noise. Among the 14 stocks we choose, there are 7 from the Financial sector, 4 from Food, and 3 from Pharmaceutical. They include: Bank of America Corp. (BAC), The Bank of New York Mellon Corp. (BK), CVS Caremark Corp. (CVS), Goldman Sachs Group, Inc. (GS), H.J. Heinz Company (HNZ), JP Morgan Chase & Co. (JPM), The Coca-Cola Company (KO), Kraft Foods Inc. (KFT), Morgan Stanley (MS), NYSE Euronext (NYX), PepsiCo, Inc. (PEP), Pfizer Inc. (PFE), Walgreens Co. (WAG), Wells Fargo & Co. (WFC). To draw conclusion about contribution of jump to Realized Beta measure, we examine these stocks from different sectors to do a comprehensive study and to eliminate the potential bias from a single stock or a single sector. Here we have substantially high number of financial stocks because we want look closer at how the betas for financial stocks vary over times, especially during times of great financial stress. And we also choose a number of firms from the Food and Pharmaceutical industry, because they tend to be relativity stable during financial stress as food and over the country drugs are necessities. Hence, by comparing and contrasting the Financial Stocks and Food/Pharmaceutical, we might get some insights in the overall market during times of financial stress.

The datasets contain prices for varying periods, with the longest from April 1997 to December 2010, and shortest from Jan. 2006 to Dec. 2010. But we will put everything on the same times scale for the sake of cross-stock comparison.
5. Empirical Results

5.1 Average Realized, Bipower Beta and Jump Contribution

Table 1 summarizes most of the important results from our research. The columns 2-4 display the mean and standard error of the Realized Beta, Bipower Beta and Jump for the 14 different stocks calculated with equation (12), (13) and (14) with 5-minute sampling frequency. At first glance, the mean for the Realized Beta isn’t too different from the mean for the Bipower beta. To better show the relationship between the Realized Beta and Bipower Beta, in Graph 2, we plotted the Realized Beta against Bipower Beta for JPM. However, as we can see, because of our vast sample size, the standard error for each stock’s Jump contribution is very small relative to the mean. Thus, as described in section 3, we will use t-test to determine whether the jump contribution to beta is significantly different from 0. In column 6, we record the t-stat of the Jumps, and in column 7, we give the critical value for 0.1% significance level for each stock. And finally in the last column, we report whether the jumps are significantly different from 0.

And to assess the magnitude of the jump contribution relative to Realized Beta, we divide the mean Jump for each stock by its respective mean Realized Beta in column 5.

5.2 Time-Varying Beta

In graph 3, we provided the 3-month moving average of JPM’s Bipower Beta from April 1999 to December 2010. The reason we use 3-month moving average is because 3-month moving average washes out most of the noise from the daily Bipower Beta while maintaining enough variation of beta across time to give us a good picture of the time varying beta. Here we only displayed the Bipower Beta, because it’s jump-robust. In the graph, we also have a red line at \( y=1 \) to show how JPM’s Realized Beta compare to 1 throughout different periods in the past.

In Graph 4, we did the same for BAC, GS, MS and WFC.

In Graph 5 and 6, we put together the similar graphs for HNZ, KFT, KO, PEP, CVS, PEF and WAG, with an additional red line at the mean of each Bipower Beta, since their means are relatively far from 1.
6. Interpretation and Conclusion

6.1 Jump Contribution

As Table 1 indicates, for more than half of the stocks we chose as our sample, we reject the null hypothesis even at 0.1% significance level. Thus, we conclude that the jump contributions are statistically significantly different from 0. However, another issue is the magnitude of the effect. As we can see in the fifth column of Table 1, the magnitude of the relative contribution of jumps stays below 5% for the Financial stocks, and mostly below 10% for the Food and Pharmaceuticals. Thus, even when the jumps are statistically significant, they do not have major effects on the Realized Beta over time. Intuitively, if there are jumps within a single stock, it’s not going to affect the entire market much. And since, beta is a measure of sensitivity to the market risks, it is not going to change much, when the market doesn’t change much. Thus, intuitively, it also make sense that jumps don’t have much impact on the betas.

If we look at the Financial stocks alone, they are riskier since they have higher betas in general. However, jumps only contribute to less than 5% of their realized beta, which is “economically” insignificant when considering their risks. Thus, we can safely conclude, at least for Financial stocks, that the contribution of jumps to Realized is small enough that they don’t hold any “economic” significance. Therefore, as we can diversify away idiosyncratic risks with appropriate portfolio choice, we can say that the only risks investors face are purely systematic risks, and the jumps of individual stocks due to various reasons do not hold any additional risks or potential risks.

6.2 Time-Varying Betas

Based on our method, we implicitly assume that betas are time varying. From equation (12) and (13), we calculated beta for each trading day. Though we report the mean of the betas in Table 1, the betas for each stock can’t just be summarized by their mean. If we look at Graph 3-6, we can clearly see the betas are far from uniform throughout historical periods. Specifically in Graph 3 and 4, we can see that for all 5 stocks, the Bipower Beta peaks over 1 around the year 2008. This suggests that betas increase rapidly during financial crisis. However, if we look at Graph 5 and 6, we do not see the same trend. In fact, there doesn’t seem to be any pattern at all. Perhaps it’s
unique to stocks in the Financial sector that the betas peak (above 1) during financial crisis. However, with our current results, we cannot make any conclusions. We would need further statistical analysis to test with the Bipower Betas are statistically significantly different between different periods in history. That may be the next steps of our research. But this paper concludes here on the note that jumps do not significantly affect betas.
### 7. Tables

#### Table 1

| Stocks | Realized Beta | Bipower Beta | Jump | Relative Jump† | |-stat| t-critical‡ | Reject null? |
|--------|---------------|--------------|------|----------------|------|-------------|-------------|
| **Financial Services** | | | | | | | |
| BAC    | 0.8715 (0.0083) | 0.8681 (0.0089) | 0.0034 (0.0043) | 0.39% | 0.7949 | 3.2934 | No |
| BK     | 0.8482 (0.0065) | 0.8722 (0.0072) | -0.0239 (0.0050) | -2.82% | 4.8126 | 3.2934 | Yes |
| GS     | 0.8807 (0.0072) | 0.8575 (0.0079) | 0.0232 (0.0050) | 2.63% | 4.6339 | 3.2939 | Yes |
| JPM    | 0.9442 (0.0063) | 0.9188 (0.0072) | 0.0253 (0.0044) | 2.68% | 5.7729 | 3.2934 | Yes |
| MS     | 1.2513 (0.0149) | 1.2049 (0.0165) | 0.0461 (0.0093) | 3.68% | 4.9483 | 3.2985 | Yes |
| NYX    | 1.0328 (0.0149) | 1.0001 (0.0178) | 0.0328 (0.0113) | 3.18% | 2.8926 | 3.2987 | No |
| WFC    | 0.7906 (0.0074) | 0.7910 (0.0081) | -3.9334e-4 (0.0043) | -0.05% | 0.0907 | 3.2934 | No |
| **Food** | | | | | | | |
| HNZ    | 0.4277 (0.0039) | 0.4676 (0.0047) | -0.0400 (0.0035) | -9.35% | 11.3402 | 3.2934 | Yes |
| KFT    | 0.3173 (0.0057) | 0.3515 (0.0067) | -0.0342 (0.0046) | -10.78% | 7.4348 | 3.2955 | Yes |
| KO     | 0.5645 (0.0047) | 0.5715 (0.0056) | -0.0070 (0.0031) | -1.24% | 2.2581 | 3.2934 | No |
| PEP    | 0.4974 (0.0045) | 0.5461 (0.00570 | -0.0487 (0.0040) | -9.79% | 12.1097 | 3.2934 | Yes |
| **Pharmaceutical** | | | | | | | |
| CVS    | 0.5783 (0.0058) | 0.6289 (0.0066) | -0.0507 (0.0049) | -8.77% | 10.3850 | 3.2934 | Yes |
| PFE    | 0.6702 (0.0054) | 0.6957 (0.0064) | -0.0254 (0.0041) | -3.79% | 6.1375 | 3.2934 | Yes |
| WAG    | 0.6241 (0.0054) | 0.6809 (0.0064) | -0.0568 (0.0050) | -9.10% | 11.3723 | 3.2935 | Yes |

† Jump/Realized Beta  
‡ At 0.1% significance leve
Figures

Graph 1: Volatility Signature Plot

Introduced in Andersen, Bollerslev, Diebold and Labys (2001), this plot describes the relationship between average Realized Variance of the stock JPM from Apr. 1997 to Dec. 2010 and the sampling frequency in minutes on the x-axis. As the plot indicates, the higher the sampling frequency, the higher the average Realized Variance as more microstructural noise contributes to the Realized Variance measurement at higher sampling frequency. So an ideal sampling frequency, which captures enough variation but does not include so much microstructural noise, would be 5-minute (or 10-minute).
The red dot in the middle of the cluster is the mean of Realized Beta plotted against the mean of the Bipower Beta. And the red line is the 45 degree line. Cluster of points seems to center around the 45 degree line and the red dot is right below the line. At first glance, we would have suggested that the Realized Beta is not significantly different from the Bipower Beta, but after further statistical testing as shown in table 1, the contribution of jump is significantly different from 0 at even 0.1% significance level.
As we can see, the Bipower Beta peaks around 1999, 2002, and 2008 (i.e. during times of major financial crisis). This indicates the relationship between beta and current economic condition. To further illustrate this property, we also include 4 more stocks from the Financial sector in Graph 4. In Graph 5 and 6, we have stocks from Food and Pharmaceuticals, which doesn’t seem to share the same story.
Notice the sudden drops of Bipower Beta to 0, those where adjustments so we can look at the betas of different stocks on the same time scale. These adjustments are necessary because as mentioned in Section 4, we don’t have the prices over the same time period for all of the stocks, so we have to set the beta to 0 on the days where we don’t have any data.
References


