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Realized range-based estimation of integrated variance ☆

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Abstract

We provide a set of probabilistic laws for estimating the quadratic variation of continuous semimartingales with the realized range-based variance—a statistic that replaces every squared return of the realized variance with a normalized squared range. If the entire sample path of the process is available, and under a set of weak conditions, our statistic is consistent and has a mixed Gaussian limit, whose precision is five times greater than that of the realized variance. In practice, of course, inference is drawn from discrete data and true ranges are unobserved, leading to downward bias. We solve this problem to get a consistent, mixed normal estimator, irrespective of non-trading effects. This estimator has varying degrees of efficiency over realized variance, depending on how many observations that are used to construct the high–low. The methodology is applied to TAQ data and compared with realized variance. Our findings suggest that the empirical path of quadratic variation is also estimated better with the realized range-based variance.

JEL classification: C10; C80

Keywords: Central limit theorem; Continuous semimartingales; Integrated variance; Realized range-based variance; Realized variance

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1. Introduction

The volatility of asset prices is a key ingredient in several areas of financial economics. Not long ago, academic studies routinely used constant volatility models (e.g., Black and Scholes, 1973), despite empirical evidence in the data suggesting that the conditional variance is both time-varying and highly persistent. These facts were uncovered by the development and application of parametric models, such as ARCH (see, e.g., Bollerslev et al., 1994), through stochastic volatility models (e.g., Ghysels et al., 1996), and more recently non-parametric methods based on high-frequency data, the most conspicuous idea being *realized variance* (RV), see, e.g., Andersen et al. (2001b) or Barndorff-Nielsen and Shephard (2002); henceforth ABDL and BN–S.

RV is the sum of squared returns over non-overlapping intervals within a sampling period. Given weak regularity conditions, RV converges in probability to the *quadratic variation* (QV) of all semimartingales as the sampling frequency tends to infinity.

In practice, the consistency of RV breaks down as data limitations prevent the sampling frequency from rising without bound. Most notably, market microstructure noise contaminates high-frequency asset prices. This invalidates the asymptotic theory, and RV is known to be inconsistent in the presence of noise (e.g., Bandi and Russell, 2005, 2006, and Hansen and Lunde, 2006). Therefore, it is common in applied work to construct RV at a moderate frequency, where the impact of noise is small enough to be ignored, but this leads to loss of information. Though current research seeks to make RV robust against microstructure noise (e.g., Zhang et al., 2004 or Barndorff-Nielsen et al., 2006b), the most accurate estimator of QV remains unknown. Set against this backdrop, we suggest the *realized range-based variance* (RRV).

Range-based estimation of volatility (developed in, e.g., Feller, 1951; Garman and Klass, 1980; Parkinson, 1980; Rogers and Satchell, 1991; Kunitomo, 1992; Alizadeh et al., 2002) reveals more information than returns sampled at fixed intervals, because the extremes are formed from the entire price process. The daily squared range, for example, is about five times more efficient at estimating the scale of Brownian motion than the daily squared return. But, as noted in Andersen and Bollerslev (1998), the accuracy of the high–low estimator is only around that afforded by RV based on 2- or 3-h returns, and the range has largely been neglected in the recent literature.

Intraday range-based estimation of volatility, however, has the potential of achieving smaller sampling errors than a sparsely sampled RV, because we can replace every squared return of RV with a squared range and extract most of the information about volatility contained in the intermediate data points. No prior studies have explored the properties of such an estimator. Indeed, it is not clear what to expect from sampling, properly transformed, high-frequency ranges. Extrapolating from the daily interval would suggest that hourly ranges, say, achieve the accuracy of RV based on 5- or 10-min returns, but the comparison is more complicated as each intraday range is constructed from less data.

We propose to sample and sum intraday price ranges to construct more efficient estimates of QV. Our contributions are four-fold. First, we develop a non-parametric method for measuring QV with the RRV. Second, and unlike the existing time-invariant theory for the high–low, we deal with estimation of time-varying volatility, when the driving terms of the price process are (possibly) continuously evolving random functions. Third, we derive a set of probabilistic laws for sampling intraday high–lows. Fourth, we remove the problems with downward bias reported in the previous range-based literature.

The new estimator is defined as

$$RRV_m^{\Delta} = \frac{1}{\lambda_{2,m}} \sum_{i=1}^{n} s_{p_{i\Delta,\Delta},m}^2, \tag{1.1}$$

where $s_{p_{i\Delta,\Delta},m} = \max_{0\leqslant s,t\leqslant m}\{p_{(i-1)/n+t/mn} - p_{(i-1)/n+s/mn}\}$ is the observed range of a price process p over the interval $[(i-1)/n, i/n]$, $i = 1, \dots, n$. m is the number of high-frequency returns used to construct $s_{p_{i\Delta,\Delta},m}$ and $\lambda_{2,m}$ is a constant. We prove that RRV_m^{Δ} is consistent for the *integrated variance* (IV) and that $\sqrt{n}(RRV_m^{\Delta} - IV)$ has a mixed Gaussian limit with a variance that can be much smaller relative to RV.

The paper is structured as follows. In the next section, we unfold the necessary diffusion theory, present various ways of measuring volatility and advance our methodological contribution by suggesting RRV and a version thereof that handles non-trading effects. Under mild conditions, we prove consistency for the estimation method and derive a mixed Gaussian central limit theorem (CLT). Section 3 illustrates the approach through Monte Carlo analysis to uncover the finite sample properties, and we present some empirical results in Section 4. Rounding up, Section 5 offers conclusions and sketches several directions for future research.

2. A semimartingale framework

In this section, we propose a new method for consistently estimating QV based on the price range. The theory is developed for the log-price of a univariate asset evolving in continuous time over some interval, say $p = (p_t)_{t\geqslant 0}$. p is defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$ and adapted to the filtration $(\mathscr{F}_t)_{t\geqslant 0}$, i.e. a collection of σ-fields with $\mathscr{F}_u \subseteq \mathscr{F}_t \subseteq \mathscr{F}$ for all $u \leqslant t < \infty$.

The basic building block is that p constitutes a continuous sample path semimartingale.[1] Hence, we write the time t log-price in the generic form:

$$p_t = p_0 + \int_0^t \mu_u \, du + \int_0^t \sigma_u \, dW_u \quad \text{for } t \geqslant 0, \tag{2.1}$$

where $\mu = (\mu_t)_{t\geqslant 0}$ (the drift) is locally bounded and predictable, $\sigma = (\sigma_t)_{t\geqslant 0}$ (the volatility) is càdlàg, and $W = (W_t)_{t\geqslant 0}$ is a standard Brownian motion.

Much work in financial econometrics is cast within this setting (see, e.g., Andersen et al., 2002b or BN–S, 2007 for reviews and references). Except for the continuity of the local martingale, we impose little structure on the model. In fact, for semimartingales with a continuous martingale component as above, the form $\left(\int_0^t \mu_u \, du\right)_{t\geqslant 0}$ is implicit, when the drift term is predictable (in the absence of arbitrage).[2]

The objective is to estimate a suitable measure of the return variation over a subinterval $[a, b] \subseteq [0, \infty)$, labeled the sampling period or measurement horizon. We assume $[a, b] = [0, 1]$; this will be thought of as representing a trading day, but the choice is arbitrary and can serve as a normalization. At any two sampling times t_{i-1} and t_i, with

[1] We adopt the continuity assumption as a starting point only. In subsequent work, we have been analyzing the properties of our estimator, when p exhibits jumps (see Christensen and Podolskij, 2006).

[2] Moreover, all continuous local martingales, whose QV (to be defined in a moment) is absolutely continuous, has the martingale representation of the second term in Eq. (2.1), e.g., Doob (1953). We refer to BN–S (2004, footnote 6) for further details.

$0 \leqslant t_{i-1} \leqslant t_i \leqslant 1$, the intraday return over $[t_{i-1}, t_i]$ is denoted by

$$r_{t_i,\Delta_i} = p_{t_i} - p_{t_{i-1}}, \tag{2.2}$$

where $\Delta_i = t_i - t_{i-1}$.

From the theory of stochastic integration, it is well-known that QV is a natural measure of sample path variability for the class of semimartingales. QV is defined by

$$\langle p \rangle = \text{p-}\lim_{n\to\infty} \sum_{i=1}^{n} r_{t_i,\Delta_i}^2, \tag{2.3}$$

for any sequence of partitions, $0 = t_0 < t_1 < \cdots < t_n = 1$, such that $\max_{1 \leqslant i \leqslant n}\{\Delta_i\} \to 0$ as $n \to \infty$ (e.g., Protter, 2004).

In our framework, QV is entirely induced by innovations to the local martingale and coincides with the IV, which is the object of interest:

$$IV = \int_0^1 \sigma_u^2 \, \mathrm{d}u. \tag{2.4}$$

IV is central to financial economics, whether in asset and derivatives pricing, portfolio selection or risk management (e.g., Andersen et al., 2002b). The econometric problem is that IV is latent, which complicates the empirical estimation of this quantity. We briefly review the literature on existing methods for measuring IV, before suggesting a new approach.

2.1. Return-based estimation of IV

Not long ago, the daily squared return was employed as a non-parametric estimator of IV. With the advent of high-frequency data, however, more recent work has computed RV, which is the sum of squared intraday returns sampled over non-overlapping intervals (see, e.g., ABDL, 2001 or BN–S, 2002). More formally, consider an equidistant partition $0 = t_0 < t_1 < \ldots < t_n = 1$, where $t_i = i/n$. Then, adopting the notation of Hansen and Lunde (2005), we define RV at sampling frequency n by setting

$$RV^{\Delta} = \sum_{i=1}^{n} r_{i\Delta,\Delta}^2. \tag{2.5}$$

RV builds directly on the theory of QV. From Eqs. (2.3) and (2.4), it follows that

$$RV^{\Delta} \xrightarrow{\mathrm{p}} IV, \tag{2.6}$$

as $n \to \infty$.[3]

BN–S (2002) derived a distribution theory for RV^{Δ} in relation to IV. The law of the scaled difference between RV^{Δ} and IV has a mixed Gaussian limit,

$$\sqrt{n}(RV^{\Delta} - IV) \xrightarrow{\mathrm{d}} MN(0, 2IQ), \tag{2.7}$$

[3]Though an irregular partition of the sampling period suffices for consistency, it is standard to compute an equidistant time series of intraday returns by various approaches, such as linear interpolation in, e.g., Andersen and Bollerslev (1997) or the previous-tick method suggested in Wasserfallen and Zimmermann (1985). A side-effect of linear interpolation is that $RV^{\Delta} \xrightarrow{\mathrm{p}} 0$ as $n \to \infty$, because the interpolated process is of continuous bounded variation, see Hansen and Lunde (2006, Lemma 1). Intuitively, a straight line is the minimum variance path between two points. Oomen (2005) characterizes RV under alternative sampling schemes.

where

$$IQ = \int_0^1 \sigma_u^4 \, du, \tag{2.8}$$

is the *integrated quarticity* (IQ). Thus, the size of the error bounds for RV^{Δ} is positively related to σ, so RV is a less precise estimator of IV when σ is high. BN–S (2002) also derived a feasible CLT, where all quantities except IV can be computed directly from the data. This was done by simply replacing IQ by a consistent estimator, such as *realized quarticity* (RQ):

$$RQ^{\Delta} = \frac{n}{3} \sum_{i=1}^{n} r_{i\Delta,\Delta}^4, \tag{2.9}$$

making it possible to construct confidence bands for RV^{Δ} to measure the size of the estimation error involved with finite sampling.

2.2. Range-based estimation of IV

The choice of volatility proxy is not obvious in practice, since microstructure bias affects RV if n is too large. With noisy prices, RV is both biased and inconsistent, see, e.g., Zhou (1996), Bandi and Russell (2005, 2006), or Hansen and Lunde (2006).[4] Previous studies have recognized this by developing bias reducing techniques (e.g., pre-whitening of the high-frequency return series with moving average or autoregressive filters as in Andersen et al., 2001a and Bollen and Inder, 2002, or kernel-based estimation as in Zhou, 1996 and Hansen and Lunde, 2006). Zhang et al. (2004) also suggest a subsample estimator that is robust to the noise in some situations. In empirical work, the benefits of more frequent sampling is traded off against the damage caused by cumulating noise, and—using various criteria to pick the optimal sampling frequency—the result is often sampling at a moderate frequency, e.g., every 5-, 10-, or 30-min, whereby data are discarded.

This pitfall of RV motivates our choice of another proxy with a long history in finance: the price range or high–low. Using the terminology from above, we define the intraday range at sampling times t_{i-1} and t_i as

$$s_{p_{t_i,\Delta_i}} = \sup_{t_{i-1} \leqslant s,t \leqslant t_i} \{p_t - p_s\}. \tag{2.10}$$

The subscript p indicates that we use the range of the price process. Below, we also need the range of a standard Brownian motion over $[t_{i-1}, t_i]$, which is denoted by

$$s_{W_{t_i,\Delta_i}} = \sup_{t_{i-1} \leqslant s,t \leqslant t_i} \{W_t - W_s\}. \tag{2.11}$$

2.2.1. The distribution of the range

The foundations of the range go back to Feller (1951), who found its distribution by using the theory of Brownian motion.[5] According to his work, the density of $s_{W_{t_i,\Delta_i}}$ is

[4]With IID noise, for instance, RV diverges to infinity, i.e. $RV^{\Delta} \xrightarrow{p} \infty$ as $n \to \infty$.

[5]There are two types of range-based volatility estimators: the first relies purely on the high–low, while the second combines the high–low with the open–close, e.g., Garman and Klass (1980) or Rogers and Satchell (1991). Throughout, we only consider the high–low estimator.

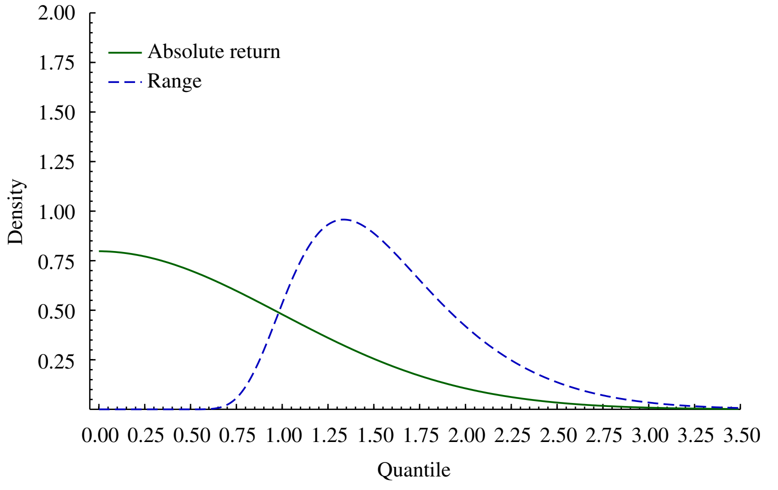


Fig. 1. The distribution of the absolute return and range of a standard Brownian motion over an interval of unit length.

given by

$$f(x) = 8\sum_{j=1}^{\infty}(-1)^{j-1}\frac{j^2}{\sqrt{\Delta_i}}\phi\left(\frac{jx}{\sqrt{\Delta_i}}\right) \quad \text{for } x>0, \tag{2.12}$$

with $\phi(y) = \exp(-y^2/2)/\sqrt{2\pi}$. The infinite series is evaluated by a suitable truncation. In Fig. 1, we plot the density function of $s_{W_{t_i,\Delta_i}}$ by taking $t_i = \Delta_i = 1$ (we use the shorthand notation s_W for this random variable in the rest of the paper).

The figure also displays the distribution of the absolute return. By comparing these proxies, it is suggestive that the efficiency of the range is higher, or in other words that its variance vis-à-vis the return is lower.

Parkinson (1980) used Feller's insights to derive the moment generating function of the range of a scaled Brownian motion, $p_t = \sigma W_t$.[6] For the rth moment:

$$\mathbb{E}[s^r_{p_{t_i,\Delta_i}}] = \lambda_r \Delta_i^{r/2}\sigma^r \quad \text{for } r \geqslant 1, \tag{2.13}$$

where $\lambda_r = \mathbb{E}[s^r_W]$.[7]

Arguably, a process without drift and constant σ is irrelevant from an empirical point of view. An overwhelming amount of research indicates that σ is time-varying, see, e.g., Ghysels et al. (1996). Nonetheless, to our knowledge there exists little theory about range-based estimation of IV in the presence of a continually evolving diffusion parameter.[8] Previous work accounts for (randomly) changing volatility by holding σ_t fixed within the

[6]Note, σ does double-duty; representing either the process $\sigma = (\sigma_t)_{t\geqslant 0}$ or a constant diffusion parameter $\sigma_t = \sigma$. The meaning is clear from the context.

[7]The explicit formula for λ_r is $\lambda_r = 4/\sqrt{\pi}(1 - 4/2^r)2^{r/2}\Gamma((r+1)/2)\zeta(r-1)$, for $r \geqslant 1$; where $\Gamma(x)$ and $\zeta(x)$ denote the Gamma and Riemann's zeta function, respectively.

[8]A notable exception is Gallant et al. (1999), who estimate two-factor stochastic volatility models in a general continuous time framework. They derive the density function of the range in this setting, but do not otherwise explore its theoretical properties.

trading day, while allowing for (stochastic) shifts between them (e.g., Alizadeh et al., 2002). Still, there are strong intraday movements in σ_t (e.g., Andersen and Bollerslev, 1997).

A major objective of this paper is, therefore, to extend the theoretical domain of the extreme value method to a more general class of stochastic processes. Contrary to extant research, we develop a statistical framework for the Brownian semimartingale in Eq. (2.1), featuring less restrictive dynamics for μ and σ.

2.2.2. A realized range-based estimator

As stated earlier, the (transformed) daily range is less efficient than RV for moderate values of n; 2- or 3-h returns suffice. But with tick-by-tick data at hand, we can construct more precise range-based estimates of IV by sampling high–lows within the trading day. Curiously, a rigorous analysis of intraday ranges has been missing in the volatility literature.[9]

Accordingly, consider again the equidistant partition with $t_i = i/n$, for $i = 1, \ldots, n$.[10] We then propose a RRV estimator of IV, which—at sampling frequency n—is defined as

$$RRV^{\Delta} = \frac{1}{\lambda_2} \sum_{i=1}^{n} s_{p_{i\Delta,\Delta}}^2. \tag{2.14}$$

RRV^{Δ} has two advantages over the previous return- and range-based methods suggested in the literature on volatility estimation. First, RRV^{Δ} inspects all data points (regardless of n), whereby we avoid neglecting information about IV. Second, the efficiency of RRV^{Δ} is several times that of RV^{Δ}, leading to narrower confidence intervals for IV (see below).

2.2.3. Properties of RRV

The properties of RRV^{Δ} are trivial for the scaled Brownian motion, $p_t = \sigma W_t$. As the infill asymptotics start operating by letting $n \to \infty$, we achieve an increasing sequence of IID random variables, $\{s_{p_{i\Delta,\Delta}}\}_{i=1,\ldots,n}$. Suitably transformed to unbiased measures of σ^2 using (2.13), the consistency of RRV^{Δ} follows from a standard law of large numbers by averaging. To see this, note that $\mathbb{E}[RRV^{\Delta}] = \sigma^2$ and $\mathrm{var}[RRV^{\Delta}] = \Lambda n^{-1}\sigma^4$ with

$$\Lambda = \frac{\lambda_4 - \lambda_2^2}{\lambda_2^2} \simeq 0.4073. \tag{2.15}$$

Hence, $RRV^{\Delta} \xrightarrow{\mathrm{p}} \sigma^2$ as $n \to \infty$. Also, for this process a standard CLT implies that

$$\sqrt{n}(RRV^{\Delta} - \sigma^2) \xrightarrow{\mathrm{d}} N(0, \Lambda\sigma^4). \tag{2.16}$$

If μ and σ are stochastic, establishing the large sample properties of RRV^{Δ} is more involved, but nonetheless possible. Overall, the basic idea extends to general Brownian semimartingales, given some regularity on μ and σ, as we next show.[11]

[9] In an independent, concurrent paper, Dijk and Martens (2006) have studied RRV for homoscedastic diffusions, but they do not derive a general asymptotic theory.

[10] We use equidistant estimation to ease notation. All our results generalize to an irregular subdivision of the sampling period, so long as $\max_{1\leqslant i\leqslant n}\{\Delta_i\} \to 0$ as $n \to \infty$, although the conditional variance in the CLT is modified slightly, as spelled out below.

[11] Throughout the paper, proofs of the theorems are presented in the Appendix.

Theorem 1. *Assume p satisfies the continuous time stochastic volatility model in Eq.* (2.1), *where μ is locally bounded and predictable, and σ is càdlàg. Then, as $n \to \infty$,*

$$RRV^{\Delta} \xrightarrow{\mathrm{p}} IV. \tag{2.17}$$

No knowledge about the dynamics of σ is needed for Theorem 1 to hold, except for weak technical conditions, so it considerably extends the theory of range-based volatility estimation. We allow for very general continuous time processes, including, but not limited to, models with leverage, long-memory, diurnal effects or jumps (in σ). This is certainly not true in the previous range-based literature. Moreover, the theorem allows for drift due to the fact that the variation induced by the expected move in p, $(\int_0^t \mu_u \,\mathrm{d}u)_{t\geqslant 0}$, is an order of magnitude lower than the variation induced by the continuous local martingale; comprised by $(\int_0^t \sigma_u \,\mathrm{d}W_u)_{t\geqslant 0}$.

2.2.4. Asymptotic distribution theory

In empirical work, the consistency of RRV^{Δ} becomes unreliable due to microstructure noise, if n is too large. Theorem 1 does not indicate the precision of RRV^{Δ} if n is fixed at a moderate level, and econometricians often compute confidence bands as a guide to the error made from estimation in finite samples. To strengthen the convergence in probability, we next develop a distribution theory for RRV^{Δ}.

The above weak assumptions on σ are too general to prove a CLT, and we need slightly stronger conditions:

Assumption (V). σ does not vanish (V_1) and satisfies

$$\sigma_t = \sigma_0 + \int_0^t \mu'_u \,\mathrm{d}u + \int_0^t \sigma'_u \,\mathrm{d}W_u + \int_0^t v'_u \,\mathrm{d}B'_u \quad \text{for } t\geqslant 0, \tag{V_2}$$

where $\mu' = (\mu'_t)_{t\geqslant 0}$, $\sigma' = (\sigma'_t)_{t\geqslant 0}$ and $v' = (v'_t)_{t\geqslant 0}$ are càdlàg, with μ' also being locally bounded and predictable, and $B' = (B'_t)_{t\geqslant 0}$ is a Brownian motion independent of W.

We prove our result by invoking stable convergence in law. This is standard in the RV literature. But to avoid any confusion about our terminology, we present the definition.

Definition 1. A sequence of random variables, $(X_n)_{n\in\mathbb{N}}$, converges stably in law with limit X, defined on an appropriate extension of $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$, if and only if for every $\mathscr{F}$-measurable, bounded random variable Y and any bounded, continuous function g, the convergence $\lim_{n\to\infty} \mathbb{E}[Yg(X_n)] = \mathbb{E}[Yg(X)]$ holds.

We use the symbol $X_n \xrightarrow{\mathrm{d_s}} X$ to denote stable convergence. Note that this implies weak convergence, which may be equivalently defined by taking $Y = 1$ (see, e.g., Rényi, 1963 or Aldous and Eagleson, 1978 for more details).

We now state the main result, which is a (non-standard) CLT.

Theorem 2. *Assume that the conditions of Theorem* 1 *hold and Assumption* (V) *is satisfied. Then it holds that, as $n \to \infty$,*

$$\sqrt{n}(RRV^{\Delta} - IV) \xrightarrow{\mathrm{d_s}} \sqrt{\Lambda} \int_0^1 \sigma_u^2 \,\mathrm{d}B_u, \tag{2.18}$$

where $B = (B_t)_{t\geqslant 0}$ is a standard Brownian motion, independent from $\mathscr{F}$ (written $B \perp\!\!\!\perp \mathscr{F}$).

A critical feature of this theorem is that the left-hand side converges to a stochastic integral with respect to B, which is independent of the driving term σ. This implies $\sqrt{n}(RRV^{\Delta} - IV)$ has a mixed normal limit, with σ governing the mixture.[12] In general, this introduces heavier tails in the unconditional distribution of RRV^{Δ} than for Gaussian random variables. To summarize:

$$\sqrt{n}(RRV^{\Delta} - IV) \xrightarrow{\mathrm{d}} MN(0, \Lambda IQ). \tag{2.19}$$

Remark 1. The Λ scalar in front of IQ in Eq. (2.19) is roughly 0.4. In contrast, the number appearing in the CLT for RV^{Δ} is 2.

Hence, the sampling errors of RRV^{Δ} are about one-fifth of those based on RV^{Δ}. This is not surprising: RRV^{Δ} uses all the data, whereas RV^{Δ} is based on high-frequency returns sampled at fixed points in time. As, for the moment, p is assumed fully observed, RV^{Δ} is neglecting a lot of information.

IQ on the right-hand side in (2.19) is infeasible, i.e. it cannot be computed directly from the data. We can estimate it with the *realized range-based quarticity* (RRQ):

$$RRQ^{\Delta} = \frac{n}{\lambda_4} \sum_{i=1}^{n} s^4_{p_{i\Delta,\Delta}}. \tag{2.20}$$

With techniques similar to the proof of Theorem 1, we can show that $RRQ^{\Delta} \xrightarrow{\mathrm{p}} IQ$. Thus, by using the properties of stable convergence (e.g., Jacod, 1997), we get the next corollary.

Corollary 1. *Given the conditions of Theorem* 2, *it follows that*

$$\frac{\sqrt{n}(RRV^{\Delta} - IV)}{\sqrt{\Lambda RRQ^{\Delta}}} \xrightarrow{\mathrm{d}} N(0,1). \tag{2.21}$$

Remark 2. With irregular sampling schemes, the distributional result in (2.19)—and those in the next sections—changes slightly (the stochastic limit is unchanged). Set

$$RRV^{\Xi} = \frac{1}{\lambda_2} \sum_{i=1}^{n} s^2_{p_{t_i,\Delta_i}}, \tag{2.22}$$

$$H^{\Xi}_{n,u} = n \sum_{i=1}^{j:t_j \leqslant u} (t_i - t_{i-1})^2, \tag{2.23}$$

and assume that a pointwise limit H^{Ξ}_u of $H^{\Xi}_{n,u}$ exists and is continuously differentiable. Then, as $n \to \infty$ such that $\max_{1 \leqslant i \leqslant n}\{\Delta_i\} \to 0$:

$$\sqrt{n}(RRV^{\Xi} - IV) \xrightarrow{\mathrm{d}} MN\left(0, \Lambda \int_0^1 \frac{\partial H^{\Xi}_u}{\partial u} \sigma^4_u \,\mathrm{d}u\right). \tag{2.24}$$

The derivative $\partial H^{\Xi}_u / \partial u$ is small, when sampling runs quickly. Hence, there are potential gains in having more frequent observations when σ is high. Hansen and Lunde (2006) prove that such a sampling scheme minimizes the asymptotic variance of the RV.

[12]Earlier drafts of this paper had a non-mixed Gaussian CLT and the stronger conditions, $\mu = 0$ and σ is Hölder continuous of order $\gamma > \frac{1}{2}$, i.e. $\sigma_t - \sigma_s = \mathrm{O_p}(|\, t - s|^{\gamma})$ for $t \to s$. We have substantially weakened these restrictions and also proved the mixed Gaussian CLT. Svend E. Graversen was helpful in pointing our attention to a result that enabled us to remove these assumptions (see Lemma 1 in the Appendix).

Obviously, for equidistant subdivisions $H_u^{\Xi} = u$, so the extra term drops out. The theory is made feasible with

$$RRQ^{\Xi} = \frac{n}{\lambda_4} \sum_{i=1}^{n} s_{p_{t_i,\Delta_i}}^4 \xrightarrow{\text{p}} \int_0^1 \frac{\partial H_u^{\Xi}}{\partial u} \sigma_u^4 \, \mathrm{d}u. \tag{2.25}$$

2.2.5. Discretely sampled high-frequency data

In practice, we draw inference about IV from a finite sample and cannot extract the true range, so the intraday high–low statistic will be progressively more downward biased as n gets larger. Building on the simulation evidence of Garman and Klass (1980), Rogers and Satchell (1991) proposed a technique for bias correcting the range that largely removed the error from a numerical perspective.

Nonetheless, it is misleading to think about ranges as downward biased. The source of the bias is λ_2, which is constructed on the presumption that p is fully observed. Therefore, we will now develop an estimator that accounts for the number of high-frequency data used in forming the high–low, in order to scale properly. To formalize this idea, additional notation is required. Assume, without loss of generality, that $mn + 1$ equidistant observations of the price process are available, giving mn returns. These are split into n intervals each with m innovations. We denote the observed range over the ith interval by

$$s_{p_{i\Delta,\Delta},m} = \max_{0 \leqslant s,t \leqslant m} \{p_{(i-1)/n+t/mn} - p_{(i-1)/n+s/mn}\}. \tag{2.26}$$

Also, we let

$$s_{W,m} = \max_{0 \leqslant s,t \leqslant m} \{W_{t/m} - W_{s/m}\}, \tag{2.27}$$

and then define a new realized range-based estimator by setting

$$RRV_m^{\Delta} = \frac{1}{\lambda_{2,m}} \sum_{i=1}^{n} s_{p_{i\Delta,\Delta},m}^2, \tag{2.28}$$

where $\lambda_{r,m} = \mathbb{E}[s_{W,m}^r]$. $\lambda_{r,m}$ is the rth moment of the range of a standard Brownian motion over a unit interval, when we only observe m increments of the underlying continuous time process.

To our knowledge, there is no explicit formula for $\lambda_{r,m}$, but it is easily simulated to any degree of accuracy. Fig. 2 details this for $r = 2$ and all values of m that integer divide 23,400.

Of course, $\lambda_{2,m} \to \lambda_2$ as $m \to \infty$, but note also that $\lambda_{2,1} = 1$, which defines RV^{Δ}. The downward bias reported in simulation studies on the range-based estimator is a consequence of the fact that $1/\lambda_2$ was applied in place of $1/\lambda_{2,m}$, as the bias is in one-to-one correspondence with the difference.

Having completed these preliminaries, we prove consistency and asymptotic normality for the estimator in Eq. (2.28). Note that m is not required to approach infinity for the CLT to work; convergence to any natural number is sufficient.

Theorem 3. *Given the assumptions of Theorem* 1, *as* $n \to \infty$,

$$RRV_m^{\Delta} \xrightarrow{\text{p}} IV, \tag{2.29}$$

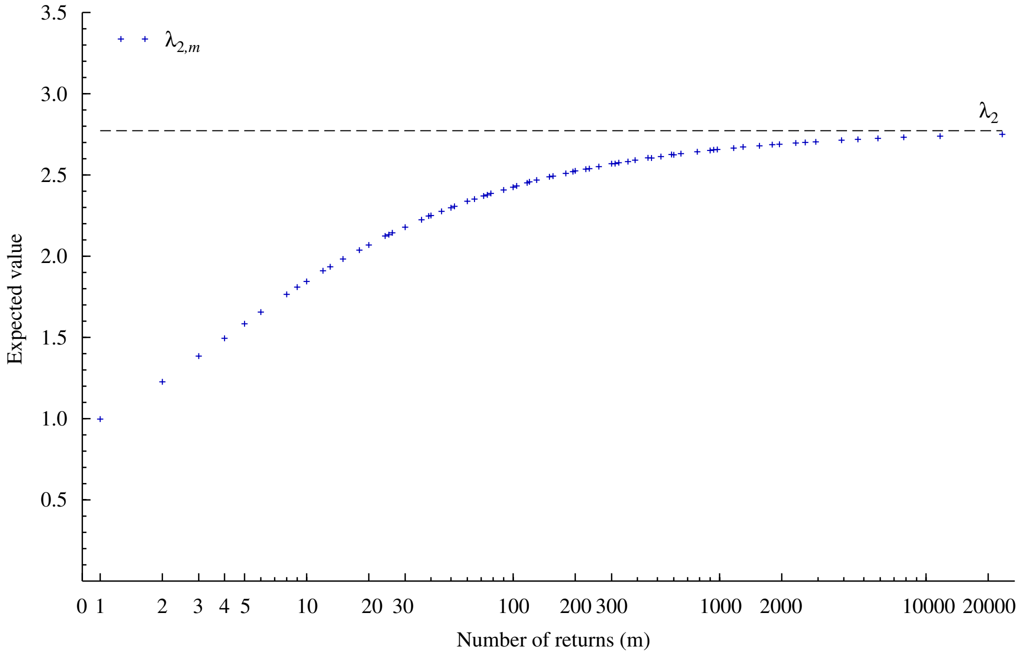


Fig. 2. $\lambda_{2,m}$ against m on a log scale. All estimates are from a simulation with 1,000,000 repetitions and the dashed line represents the asymptotic value.

where the convergence is uniform in m. Moreover, if assumption (**V**) *holds and* $m \to c \in \mathbb{N} \cup \{\infty\}$:

$$\sqrt{n}(RRV_m^\Delta - IV) \xrightarrow{\mathrm{d_s}} \sqrt{\Lambda_c} \int_0^1 \sigma_u^2 \, \mathrm{d}B_u, \tag{2.30}$$

where $\Lambda_c = \left(\lambda_{4,c} - \lambda_{2,c}^2\right)/\lambda_{2,c}^2$ *and* $B \perp\!\!\!\perp \mathscr{F}$. *Finally,*

$$\frac{\sqrt{n}(RRV_m^\Delta - IV)}{\sqrt{\Lambda_m RRQ_m^\Delta}} \xrightarrow{\mathrm{d}} N(0,1), \tag{2.31}$$

with $\Lambda_m = (\lambda_{4,m} - \lambda_{2,m}^2)/\lambda_{2,m}^2$ *and*

$$RRQ_m^\Delta = \frac{n}{\lambda_{4,m}} \sum_{i=1}^n s_{p_{i\Delta,\Delta},m}^4. \tag{2.32}$$

Remark 3. Theorem 3 provides a CLT for RV^Δ with $m = 1$, as also derived in, e.g., BN–S (2002) or Barndorff-Nielsen et al. (2006a).

To provide an impression of the efficiency of RRV_m^Δ, Fig. 3 depicts Λ_m on the y-axis, as a function of m along the x-axis. The steep initial decline in Λ_m renders the advantage of RRV_m^Δ large compared to RV^Δ even for moderate values of m. For $m = 10$, say, since the scalar appearing in front of IQ in the CLT for RRV_m^Δ is about 0.7, the confidence intervals

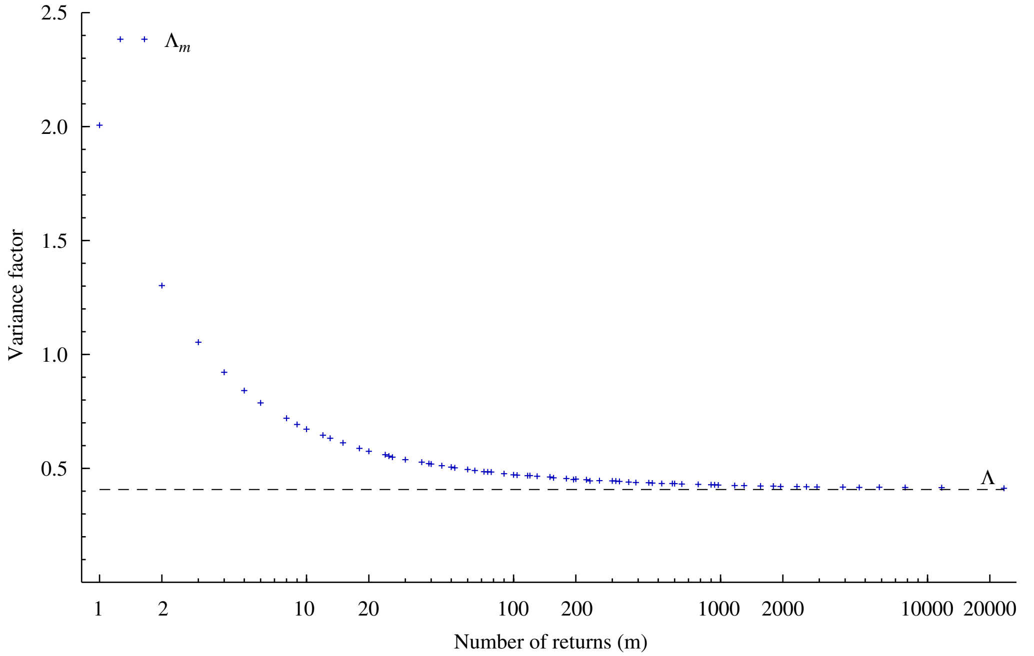


Fig. 3. We plot Λ_m, the variance factors appearing in the CLT of RRV_m^{Δ}, which are estimated from a simulation with 1,000,000 repetitions. The dashed line is the asymptotic value.

for IV are much narrower. In our experience $m = 10$, or higher values, is usually obtained for moderately liquid assets at empirically relevant frequencies, such as 5-min sampling.[13]

3. Monte Carlo experiment

To study the finite sample properties of RRV_m^{Δ}, this section uses repeated samples from a stochastic volatility model. We simulate the following system of stochastic differential equations:

$$\begin{aligned} \mathrm{d}p_t &= \sigma_t \, \mathrm{d}W_t, \\ \mathrm{d}\ln\sigma_t^2 &= \theta(\omega - \ln\sigma_t^2)\,\mathrm{d}t + \eta\,\mathrm{d}B_t, \end{aligned} \tag{3.1}$$

where W and B are independent Brownian motions, while (θ, ω, η) are parameters. Thus, the log-variance of spot prices evolves as a mean reverting Ornstein–Uhlenbeck process with mean ω, mean reversion parameter θ and volatility η (see, e.g., Gallant et al., 1999; Alizadeh et al., 2002; Andersen et al., 2002a). The vector $(\theta, \omega, \eta) = (0.032, -0.631, 0.115)$ is taken from Andersen et al. (2002a), who apply efficient method of moments (EMM) to calibrate numerous continuous time models.

[13]Under parametric assumptions and no microstructure noise, RV is the maximum likelihood estimator. Thus, our efficiency comparison should be viewed as the potential reduction in variance that can be achieved with RRV_m^{Δ} when microstructure noise is preventing RV^{Δ} from being sampled at the maximum frequency (mn) and n is set at a moderate level where the impact of noise is minimal.

Initial conditions are set at $p_0 = 0$ and $\ln \sigma_0^2 = \omega$, and we generate $T = 1,000,000$ daily replications from this model each with mn returns, where mn depends on the setting (see below). Throughout, we continue to ignore the irregular spacing of empirical high-frequency data and work with equidistant data.

3.1. Simulation results

The distributional result for RRV_m^{Δ} is detailed by setting $m = 10$. The reported results are not very sensitive to specific choices of m, but in general higher values improve the coverage rates of the asymptotic confidence bands. We simulate $n = 10$, 50, 100 for a total of $mn = 100$, 500, 1000 increments each day, allowing us to show the convergence in distribution to the standard normal for high-frequency sample sizes that resemble those of moderately liquid assets.

Fig. 4 (upper panel) graphs kernel densities for the standardized errors of RRV_m^{Δ}; cf. the ratio in Eq. (2.31). For $n = 10$, the distribution is left-skewed with a poor approximation in both the center and tail areas compared to the N(0,1) reference density. The distortions are diminished by progressively increasing the sample. With $n = 100$ the tails are tracked quite closely.

BN–S (2005) showed that log-based inference via standard linearization methods improved the raw distribution theory for RV^{Δ}. They reported a better finite sample behavior for the errors of the log-transform than those extracted with the feasible version of the CLT outlined in Eq. (2.7). The shape of the actual densities for RRV_m^{Δ} suggests that this also applies to our setting. By the delta method, the log-version of the CLT for RRV_m^{Δ} takes the form:

$$\sqrt{n}(\ln RRV_m^{\Delta} - \ln IV) \xrightarrow{\mathrm{d}} MN\left(0, \frac{\Lambda_c IQ}{IV^2}\right). \tag{3.2}$$

In the lower panel of Fig. 4, we plot the density functions of the feasible log-based t-statistics. The coverage probabilities of Eq. (3.2) are a much better guide for small values of n, with $n = 100$ providing a near perfect fit to the N(0,1) distribution. Hence, the results for RRV_m^{Δ} are consistent with the findings for RV^{Δ}.

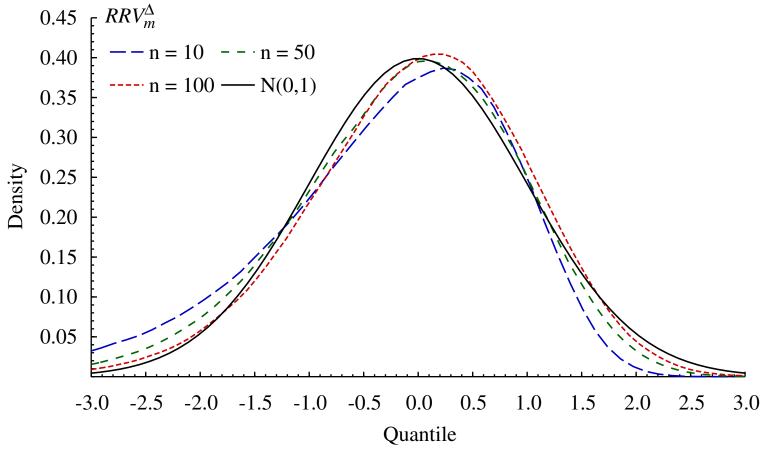
This technique is also applicable to study other (differentiable) functions of RRV_m^{Δ}. For convenience, we state the CLT of a particularly useful transformation, obtained by taking square roots:

$$\sqrt{n}\left(\sqrt{RRV_m^{\Delta}} - \sqrt{IV}\right) \xrightarrow{\mathrm{d}} MN\left(0, \frac{\Lambda_c IQ}{4IV}\right). \tag{3.3}$$

4. Empirical application: General Motors (GM)

We investigate the empirical properties of intraday ranges by analyzing a major stock from the Dow Jones Industrial Average, GM.

High-frequency data were extracted from the TAQ database, which is a recording of trades and quotes from the securities listed on New York Stock Exchange (NYSE), American Stock Exchange (AMEX), and National Association of Securities Dealers Automated Quotation (NASDAQ). The sample period covers January 3, 2000 through December 31, 2004; a total of 1,255 trading days. We restrict attention to NYSE updates



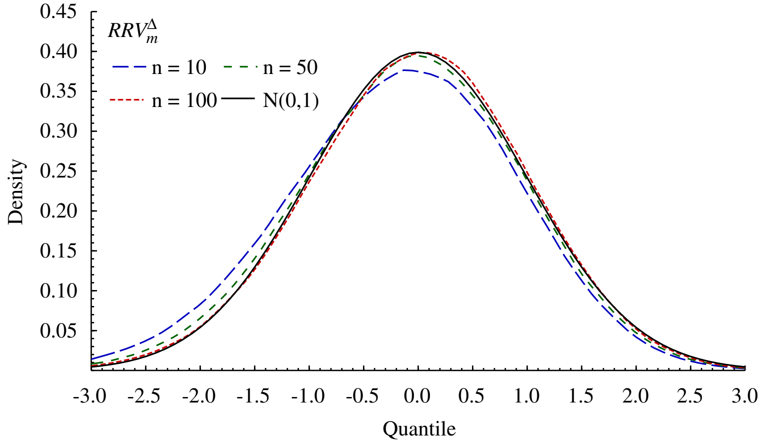


Fig. 4. Asymptotic normality for the standardized realized range-based statistic in estimating IV. The figure plots kernel densities of the sampling errors of RRV_m^{Δ} for the small sample settings $n = 10$, 50, 100 and $m = 10$. All plots are based on a simulation with 1,000,000 repetitions from a log-normal diffusion for σ, as explained in the main text. The upper panel depicts t-statistics of the feasible CLT for RRV_m^{Δ}, while the lower panel is the corresponding log-based version. The solid line is the N(0,1) density.

and only report the results of the quotation data, for which the midquote is used.[14] All raw data were filtered for irregularities (e.g., prices of zero, entries posted outside the NYSE opening hours, or quotes with negative spreads), and a second algorithm handled remaining outliers in the price series.

The average number of data points after filtering is given in Table 1. The column $\#r_{\tau_i} \neq 0$, where $r_{\tau_i} = p_{\tau_i} - p_{\tau_{i-1}}$ and τ_i is the arrival time of the ith tick, counts the number of price changes relative to the previous posting. $\#\Delta r_{\tau_i} \neq 0$ does the same for second differences, but after having removed updates with $r_{\tau_i} = 0$. These numbers are important to calculate $\lambda_{2,m}$ and $\lambda_{4,m}$ that are required to estimate RRV_m^{Δ} and construct confidence

[14]The analysis of transaction data is available upon request.

Table 1
Number of tick data pr. trading day

Ticker	Trades			Quotes		
	All	$\#r_{\tau_i} \neq 0$	$\#\Delta r_{\tau_i} \neq 0$	All	$\#r_{\tau_i} \neq 0$	$\#\Delta r_{\tau_i} \neq 0$
GM	2220	960	558	5144	1357	1017

The table contains information about the filtering of the General Motors high-frequency data. All numbers are averages across the 1,255 trading days in our sample from January 3, 2000 through December 31, 2004. $\#r_{\tau_i} \neq 0$ is the daily amount of tick data left after counting out price repetitions in consecutive ticks. $\#\Delta r_{\tau_i} \neq 0$ also removes price reversals.

Table 2
Sample statistics for RV^{Δ} and RRV_m^{Δ}

	Mean	Var.	Skew.	Kurt.	Min.	Max.	Correlation	
							RV^{Δ}	RRV_m^{Δ}
RV^{Δ}	7.276	47.740	3.624	25.085	0.472	79.332	1.000	
RRV_m^{Δ}	6.212	27.693	3.056	18.417	0.496	56.256	0.982	1.000

The table reports sample statistics of the annualized percentage RV^{Δ} and RRV_m^{Δ} for General Motors during January 3, 2000 up to December 31, 2004. We provide the mean, variance, skewness, kurtosis, minimum and maximum of the two time series, plus their correlation.

bands. Initially, we found mn on the basis of all non-zero returns; i.e. the $\#r_{\tau_i} \neq 0$ numbers. This meant mn was too high, because of instantaneous reversals (e.g., bid-ask bounce behavior). We assessed that a proper method to determine mn was to only count repeated reversals once. Thus, to compute mn we use the $\#\Delta r_{\tau_i} \neq 0$ numbers.

The estimation of RV^{Δ} and RRV_m^{Δ} proceeds with 5-min sampling through the trading session starting 9:30AM EST until 4:00PM EST; i.e. by setting $n = 78$ or $\Delta = 300\,\mathrm{s}$.[15] We use the previous-tick method to compute returns for RV^{Δ}. Note that since the empirical high-frequency data are irregularly distributed, there are, in general, different values of m in the 5-min intervals. This does not cause any problems, however, for the theory extends directly to this setting, provided we use the individual values of m in the estimation.

Sample statistics for the resulting time series are printed in Table 2. RV^{Δ} has a lower minimum and a higher maximum than RRV_m^{Δ}, while its overall mean is higher. Both kurtosis figures are consistent with a mixed Gaussian limit. The variance of RRV_m^{Δ} is only 58% that of RV^{Δ}. This is much lower, but as expected still somewhat higher than predicted by the theory (relative to RV^{Δ}). First off, here we are looking at a time series variance for the whole sample, so the CLT factors are not directly applicable. Second, the data from the empirical price process are, in all likelihood, not drawn from a Brownian semimartingale

[15]This choice was guided by signature plots, i.e. sample averages of the estimators across different sampling frequencies n. We found increasing signs of microstructure noise by moving below the 5-min frequency.

(e.g., there are jumps and microstructure frictions). RRV_m^{Δ}, in turn, behaves differently for other specifications, which we address elsewhere.

The correlation between RV^{Δ} and RRV_m^{Δ} is 0.982, pointing towards little gain—at relevant frequencies—from taking linear combinations of the estimators to further reduce sampling variation. From the joint asymptotic distribution of $(RV^{\Delta}, RRV_m^{\Delta})$, the conditional covariance matrix at time u is given by

$$\Sigma_u = \sigma_u^4 \begin{pmatrix} 2 & \\ \frac{\operatorname{cov}(W_1^2, s_{W,m}^2)}{\lambda_{2,m}} & \Lambda_m \end{pmatrix}. \tag{4.1}$$

The covariance term appearing in Σ_u is hard to tackle analytically. In unreported results, we used simulations to inspect the structure of the correlation coefficient around a grid of values for m that matches our sample. Based on this, we found that the estimated empirical correlation is slightly higher than the theoretical level.

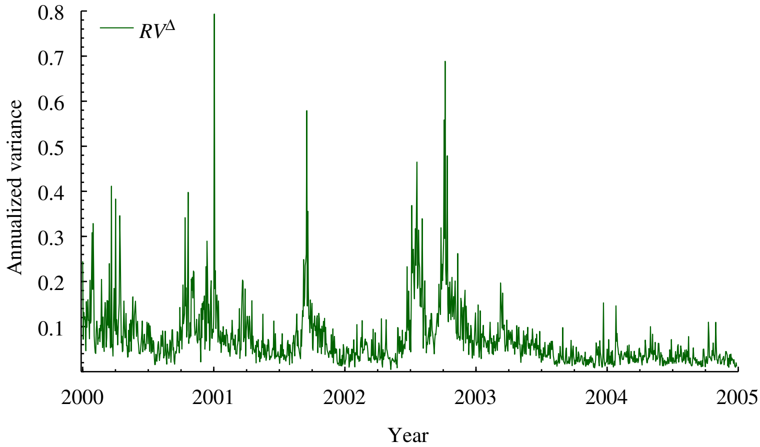
In Fig. 5, IV estimates are drawn for the two methods, RV^{Δ} and RRV_m^{Δ}. The time series agree on the level of IV. The key point is that the sample path of RRV_m^{Δ} is less volatile compared to RV^{Δ} (but still appears quite erratic). Again, this suggests that the sampling errors of RV^{Δ} are larger compared to RRV_m^{Δ}, and that the theoretical gains of the realized range-based estimator also hold for the empirical identification of the IV, at least for the 5-min frequency.

To underscore these insights, we extracted data from July 1, 2002 to December 31, 2002 to plot the IV estimates in Fig. 6 together with 95% confidence intervals, constructed from the log-based theory. The confidence bands widen as expected, when σ goes up. Nonetheless, the stability of RRV_m^{Δ} feeds into much smaller intervals, consistent with the theoretical relationship between the m and Λ_m scalars from Fig. 3. This implies that very few increments are required for RRV_m^{Δ} to gain a significant advantage in efficiency over RV^{Δ}.[16]

These empirical findings translate into a more persistent time series behavior for RRV_m^{Δ}, as shown by the autocorrelation functions in Fig. 7. We included the first 75 lags and report Bartlett two standard error bands for testing a white noise null hypothesis. All autocorrelations are positive, starting at about 0.60–0.70 and ending around 0.10–0.15. The decay pattern in the series is identical but it evolves more smoothly and at higher levels for RRV_m^{Δ}. Combined, these observations might be put to work in a forecasting exercise, although we do not pursue this idea here.

All told, realized range-based estimation of IV offers several advantages compared to RV, both from a theoretical and practical viewpoint. We acknowledge, however, that the probabilistic theory proposed in this paper needs further refinement at higher frequencies, where microstructure noise is more problematic. Statistical tools for controlling the impact of such noise is crucial for getting consistent estimates of IV. These techniques have already been developed for RV, see, e.g., Zhang et al., 2004 or Barndorff-Nielsen et al., 2006b. It presents a topic for future research to verify if our method extends along these lines, and we are currently undertaking a formal analysis of RRV and market microstructure noise.

[16]With $\#\Delta r_{\tau_i} \neq 0$ equal to 1,017 on average for the midquote data, we have roughly $m = 13$ increments within each of the 78 5-min intervals during the trading day.



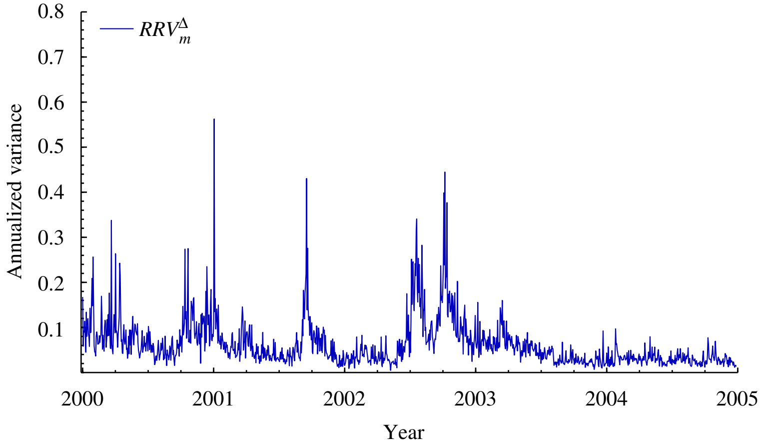


Fig. 5. The time series of RV^{Δ} and RRV_m^{Δ} for GM are shown through the sample period January 3, 2000 to December 31, 2004; or 1,255 trading days in total. These are constructed from 5-min midquote returns or ranges for data extracted from TAQ.

5. Conclusions and directions for future research

RRV is an approach based on intraday price ranges for non-parametric measurement of the IV of continuous semimartingales. Under weak regularity conditions, we have shown that it can extract IV more accurately than previous methods, when microstructure noise is preventing RV from being sampled at the maximum frequency. Another contribution of this paper, particularly useful in empirical analysis, is the solution to the downward bias problem that has haunted the range-based literature for decades.

The finite sample distributions of the estimator were inspected with Monte Carlo analysis. For moderate samples, the coverage probabilities of the confidence bands for the t-statistics correspond with the limit theory, in particular for log-based inference.

We highlighted the empirical potential of RRV vis-à-vis RV by applying our method to a set of high-frequency data for GM. Consistent with the theory, RRV has smaller

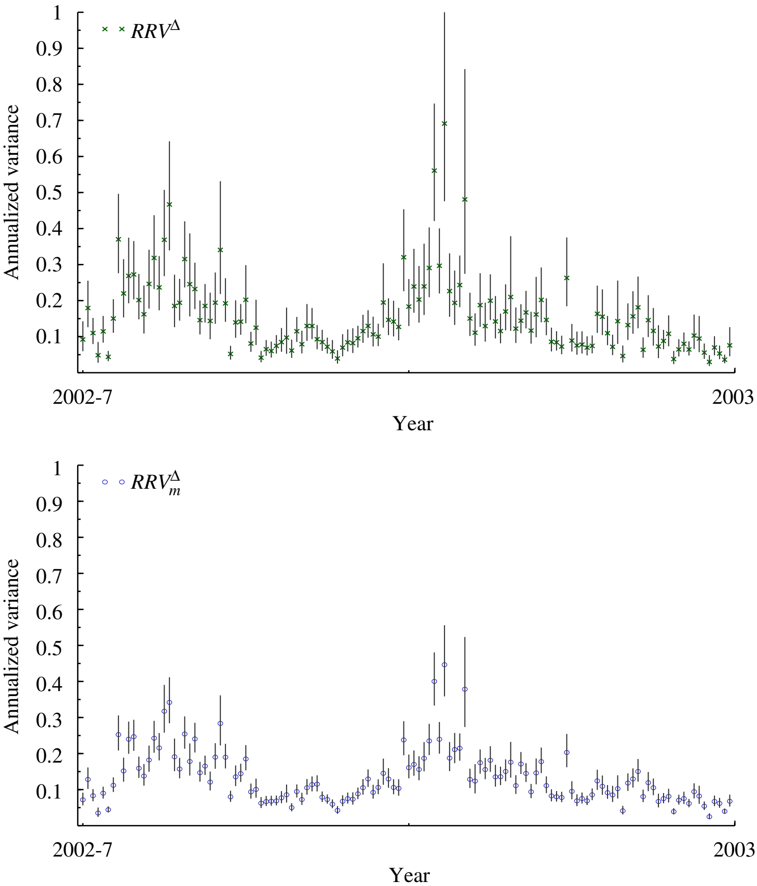


Fig. 6. RV^{Δ} and RRV_m^{Δ} are drawn with 95% confidence intervals (vertical lines) for the period July 1, 2002 through December 31, 2002. The bands are constructed from the log-based limit theory in Eq. (3.2).

confidence bands than RV. Although empirical price processes are very different from diffusion models and real data are noisy objects, we feel the results support our theory quite well and opens up alternative routes for estimating IV.

In future projects, we envision several extensions of the current framework. First, there is plenty of evidence against the continuous sample path diffusion. We are convinced that a range-based statistic can estimate QV, when the price also exhibits jumps. This theory is being developed in Christensen and Podolskij (2006), along with realized range-based bipower variation. Second, with microstructure noise in observed asset prices, further comparisons of RRV and RV are needed. Finally, we can handle the bivariate case with the polarization identities, so multivariate range-based analysis constitutes a promising future application.

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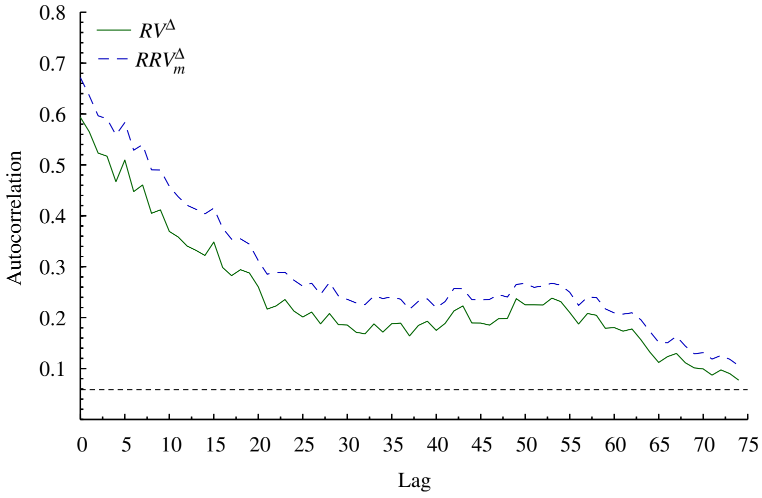

Fig. 7. Autocorrelation functions of RV^{Δ} and RRV_m^{Δ}. The first 75 lags are shown and the dashed horizontal line is Bartlett's standard error for testing a white noise hypothesis.

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Appendix A

Without loss of generality, in the following we restrict the functions μ and σ to be bounded (e.g., Barndorff-Nielsen et al., 2006a).

A.1. Proof of Theorem 1

First, define

$$\xi_i^n = \frac{1}{\lambda_2}\,\sigma^2_{(i-1)/n} s^2_{W_{i\Delta,\Delta}},$$

$$U_n = \sum_{i=1}^{n} \xi_i^n,$$

and note that

$$\mathbb{E}[\xi_i^n \mid \mathscr{F}_{(i-1)/n}] = \frac{1}{n}\,\sigma^2_{(i-1)/n}.$$

So

$$\sum_{i=1}^{n} \mathbb{E}[\xi_i^n \mid \mathscr{F}_{(i-1)/n}] \xrightarrow{\text{p}} IV. \tag{A.1}$$

Now, by setting

$$\eta_i^n = \xi_i^n - \mathbb{E}[\xi_i^n \mid \mathscr{F}_{(i-1)/n}],$$

we get

$$\mathbb{E}[(\eta_i^n)^2 \mid \mathscr{F}_{(i-1)/n}] = \Lambda\,\frac{1}{n^2}\,\sigma^4_{(i-1)/n}.$$

Therefore,

$$\sum_{i=1}^{n} \mathbb{E}[(\eta_i^n)^2 \mid \mathscr{F}_{(i-1)/n}] \xrightarrow{\text{p}} 0.$$

Hence, $U_n \xrightarrow{\text{p}} IV$ follows from (A.1). As a sufficient condition in the next step, we deduce that $RRV^{\Delta} - U_n \xrightarrow{\text{p}} 0$. Note the equality

$$\begin{aligned} RRV^{\Delta} - U_n &= \frac{1}{\lambda_2}\sum_{i=1}^{n}(s_{p_{i\Delta,\Delta}} - \sigma_{(i-1)/n}s_{W_{i\Delta,\Delta}})(s_{p_{i\Delta,\Delta}} + \sigma_{(i-1)/n}s_{W_{i\Delta,\Delta}}) \\ &\equiv R_n^1 + R_n^2, \end{aligned}$$

with R_n^1 and R_n^2 defined by

$$\begin{aligned} R_n^1 &= \frac{2}{\lambda_2}\sum_{i=1}^{n} \sigma_{(i-1)/n}s_{W_{i\Delta,\Delta}}(s_{p_{i\Delta,\Delta}} - \sigma_{(i-1)/n}s_{W_{i\Delta,\Delta}}), \\ R_n^2 &= \frac{1}{\lambda_2}\sum_{i=1}^{n}(s_{p_{i\Delta,\Delta}} - \sigma_{(i-1)/n}s_{W_{i\Delta,\Delta}})^2. \end{aligned}$$

We decompose the second term further:

$$\begin{aligned} R_n^2 &\leqslant \frac{1}{\lambda_2}\sum_{i=1}^{n}\left(\sup_{(i-1)/n\leqslant s,t\leqslant i/n} \left| \int_s^t \mu_u\,\mathrm{d}u + \int_s^t (\sigma_u - \sigma_{(i-1)/n})\,\mathrm{d}W_u \right| \right)^2 \\ &\leqslant \frac{2}{\lambda_2}\sum_{i=1}^{n}\left(\sup_{(i-1)/n\leqslant s,t\leqslant i/n} \left| \int_s^t \mu_u\,\mathrm{d}u \right| \right)^2 + \frac{2}{\lambda_2}\sum_{i=1}^{n}\left(\sup_{(i-1)/n\leqslant s,t\leqslant i/n} \left| \int_s^t \left(\sigma_u - \sigma_{(i-1)/n}\right)\mathrm{d}W_u \right| \right)^2 \\ &\equiv R_n^{2.1} + R_n^{2.2}. \end{aligned}$$

It is straightforward to verify that $\mathbb{E}\left[R_n^{2.1}\right] = \mathrm{O}\left(n^{-1}\right)$. For the latter term, we exploit Burkholder's inequality (e.g., Revuz and Yor, 1998):

$$\begin{aligned}\mathbb{E}[R_n^{2.2}] &\leqslant \frac{2C}{\lambda_2}\sum_{i=1}^{n} \mathbb{E}\left[\int_{(i-1)/n}^{i/n} (\sigma_u - \sigma_{(i-1)/n})^2 \,\mathrm{d}u\right] \\ &= \frac{2C}{\lambda_2}\,\mathbb{E}\left[\int_0^1 (\sigma_u - \sigma_{[nu]/n})^2 \,\mathrm{d}u\right] \\ &= \mathrm{o}(1),\end{aligned}$$

for some constant $C>0$. Thus, $R_n^2 = \mathrm{o_p}(1)$. Using a decomposition as above and the Cauchy–Schwarz inequality, we have that $R_n^1 = \mathrm{o_p}(1)$. By collecting terms, $RRV^{\Delta} - U_n \xrightarrow{\mathrm{p}} 0$.

A.2. Proof of Theorem 2

We need the following lemma.

Lemma 1. *Given two continuous functions $f, g : I \to \mathbb{R}$ on compact $I \subseteq \mathbb{R}^n$, assume t^* is the only point in I where the maximum of f is achieved. Then it holds*:

$$M_\varepsilon(g) \equiv \frac{1}{\varepsilon}\left[\sup_{t\in I}\{f(t) + \varepsilon g(t)\} - \sup_{t\in I}\{f(t)\}\right] \to g(t^*) \quad \text{as } \varepsilon \downarrow 0.$$

Proof. Construct the set

$$\bar{G} = \{h \in C(I) \mid h \text{ is constant on } B_\delta(t^*) \cap I \text{ for some } \delta>0\}.$$

As usual, $C(I)$ is the set of continuous functions on I and $B_\delta(t^*)$ is an open ball of radius δ centered at t^*. Take $\bar{g} \in \bar{G}$ and recall $\bar{g}$ is bounded on I. Thus, for ε sufficiently small:

$$\begin{aligned}\sup_{t\in I}\{f(t) + \varepsilon\bar{g}(t)\} &= \max\left\{\sup_{t\in I\cap B_\delta(t^*)}\{f(t) + \varepsilon\bar{g}(t)\}, \sup_{t\in I\cap B_\delta^c(t^*)}\{f(t) + \varepsilon\bar{g}(t)\}\right\} \\ &= \sup_{t\in I\cap B_\delta(t^*)}\{f(t) + \varepsilon\bar{g}(t)\} \\ &= f(t^*) + \varepsilon\bar{g}(t^*).\end{aligned}$$

So,

$$M_\varepsilon(\bar{g}) \to \bar{g}(t^*),$$

$\forall \bar{g} \in \bar{G}$. Now, let $g \in C(I)$. As $\bar{G}$ is dense in $C(I)$, $\exists \bar{g} \in \bar{G} : \bar{g}(t^*) = g(t^*)$ and $|\bar{g} - g|_\infty < \varepsilon'$ ($|\cdot|_\infty$ is the sup-norm). We see that $|M_\varepsilon(\bar{g}) - M_\varepsilon(g)| < \varepsilon'$, and

$$|M_\varepsilon(g) - g(t^*)| \leqslant |M_\varepsilon(\bar{g}) - \bar{g}(t^*)| + |M_\varepsilon(g) - M_\varepsilon(\bar{g})| \to 0.$$

Thus, the assertion is established. $\square$

With this lemma at hand, we proceed with a three-stage proof of Theorem 2. In the first part, a CLT is proved for the quantity

$$\bar{U}_n = \sqrt{n}\sum_{i=1}^{n} \eta_i^n.$$

The second step is to define a new sequence:

$$U'_n = \sqrt{n}\frac{1}{\lambda_2}\sum_{i=1}^{n}(s^2_{p_{i\Delta,\Delta}} - \mathbb{E}[s^2_{p_{i\Delta,\Delta}} \mid \mathscr{F}_{(i-1)/n}]),$$

and show the result

$$U'_n - \bar{U}_n \xrightarrow{\mathrm{p}} 0.$$

The interested reader may note that assumption (V) is not needed for Part I and II. Finally, in Part III, the theorem follows from:

$$\sqrt{n}\sum_{i=1}^{n}\left(\frac{1}{\lambda_2}\mathbb{E}[s^2_{p_{i\Delta,\Delta}} \mid \mathscr{F}_{(i-1)/n}] - \mathbb{E}[\xi^n_i \mid \mathscr{F}_{(i-1)/n}]\right) \xrightarrow{\mathrm{p}} 0, \quad \text{and}$$

$$\sqrt{n}\left(\sum_{i=1}^{n}\mathbb{E}[\xi^n_i \mid \mathscr{F}_{(i-1)/n}] - IV\right) \xrightarrow{\mathrm{p}} 0.$$

Proof of Part I. Notice that:

$$n\sum_{i=1}^{n}\mathbb{E}[(\eta^n_i)^2 \mid \mathscr{F}_{(i-1)/n}] \xrightarrow{\mathrm{p}} \Lambda IQ,$$

and by the scaling property of Brownian motion,

$$\sqrt{n}\sum_{i=1}^{n}\mathbb{E}[\eta^n_i(W_{i/n} - W_{(i-1)/n}) \mid \mathscr{F}_{(i-1)/n}] \xrightarrow{\mathrm{p}} \frac{v}{\lambda_2} IV,$$

where $v = \mathbb{E}[W_1 s^2_W]$. As $W \stackrel{\mathrm{d}}{=} -W$, it follows that $v = -v$ and, hence, $v = 0$.

Next, let $N = (N_t)_{t\in[0,1]}$ be a bounded martingale on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,1]}, P)$, which is orthogonal to W (i.e. with quadratic covariation $\langle W, N\rangle_t = 0$, almost surely). Then

$$\sqrt{n}\sum_{i=1}^{n}\mathbb{E}[\eta^n_i(N_{i/n} - N_{(i-1)/n}) \mid \mathscr{F}_{(i-1)/n}] = 0. \tag{A.2}$$

For this result, we use Clark's Representation Theorem (see, e.g., Karatzas and Shreve, 1998, Appendix E):

$$s^2_{W_{i\Delta,\Delta}} - \frac{1}{n}\lambda_2 = \int_{(i-1)/n}^{i/n} H^n_u \,\mathrm{d}W_u, \tag{A.3}$$

for some predictable function H^n_u. Notice $\mathbb{E}[\int_a^b f_u \,\mathrm{d}W_u(N_b - N_a) \mid \mathscr{F}_a] = 0$, for any $[a,b]$ and predictable f. To prove this assertion, take a partition $a = t^*_0 < t^*_1 < \cdots < t^*_n = b$ and compute

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n} f_{t^*_{i-1}}(W_{t^*_i} - W_{t^*_{i-1}})(N_b - N_a)|\mathscr{F}_a\right]\\
&\quad= \mathbb{E}\left[\sum_{i=1}^{n} f_{t^*_{i-1}}(W_{t^*_i} - W_{t^*_{i-1}})N_b|\mathscr{F}_a\right]\\
&\quad= \mathbb{E}\left[\sum_{i=1}^{n}\mathbb{E}[\mathbb{E}[f_{t^*_{i-1}}(W_{t^*_i} - W_{t^*_{i-1}})N_b|\mathscr{F}_{t^*_i}]|\mathscr{F}_{t^*_{i-1}}] \mid \mathscr{F}_a\right]\\
&\quad= 0.
\end{aligned}$$

From Eq. (A.3), (A.2) is attained. Finally, stable convergence in law follows by Theorem IX 7.28 in Jacod and Shiryaev, 2003:

$$\bar{U}_n \xrightarrow{d_s} \sqrt{\Lambda} \int_0^1 \sigma_u^2 \mathrm{d}B_u.$$

Proof of Part II. We begin by setting

$$\zeta_i^n = \sqrt{n}\left(\frac{1}{\lambda_2} s^2_{p_{i\Delta,\Delta}} - \xi_i^n\right),$$

and obtain the identity:

$$U'_n - \bar{U}_n = \sum_{i=1}^n (\zeta_i^n - \mathbb{E}[(\zeta_i^n)^2 \mid \mathscr{F}_{(i-1)/n}]).$$

To complete the second step, it suffices that

$$\sum_{i=1}^n \mathbb{E}[(\zeta_i^n)^2] \to 0.$$

We can show this result with the same methods applied to the estimates of R_n^1 and R_n^2 in the proof of Theorem 1.

Proof of Part III. It holds that

$$\sqrt{n}\left(\sum_{i=1}^n \mathbb{E}[\xi_i^n \mid \mathscr{F}_{(i-1)/n}] - IV\right) = \sqrt{n}\sum_{i=1}^n \int_{(i-1)/n}^{i/n} (\sigma^2_{(i-1)/n} - \sigma_u^2)\,\mathrm{d}u.$$

Exploiting the results of Barndorff-Nielsen et al. (2006a), we find that, under Assumption (V_2),

$$\sqrt{n}\left(\sum_{i=1}^n \mathbb{E}[\xi_i^n \mid \mathscr{F}_{(i-1)/n}] - IV\right) \xrightarrow{p} 0.$$

Now, we prove the first convergence of Part III stated above. After some computations—identical to the methods in Theorem 1—we get, using (V_2),

$$\begin{aligned}
&\sqrt{n}\sum_{i=1}^n \left(\frac{1}{\lambda_2} \mathbb{E}[s^2_{p_{i\Delta,\Delta}} \mid \mathscr{F}_{(i-1)/n}] - \mathbb{E}[\xi_i^n \mid \mathscr{F}_{(i-1)/n}]\right) \\
&= \sqrt{n}\,\frac{2}{\lambda_2}\sum_{i=1}^n \mathbb{E}[\sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}} (s_{p_{i\Delta,\Delta}} - \sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}}) \mid \mathscr{F}_{(i-1)/n}] + o_p(1) \\
&= \sqrt{n}\frac{2}{\lambda_2}\sum_{i=1}^n \mathbb{E}\left[\sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}} \left(\sup_{(i-1)/n \leqslant s,t \leqslant i/n}\left(\sigma_{(i-1)/n}(W_t - W_s) + \int_s^t \mu_u\,\mathrm{d}u\right.\right.\right. \\
&\quad \left.\left.\left. + \int_s^t (\sigma_u - \sigma_{(i-1)/n})\,\mathrm{d}W_u\right) - \sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}}\right) \mid \mathscr{F}_{(i-1)/n}\right] + o_p(1).
\end{aligned}$$

By appealing to assumption (V_2) again, we get the decomposition:

$$\sqrt{n}\sum_{i=1}^n \left(\frac{1}{\lambda_2} \mathbb{E}[s^2_{p_{i\Delta,\Delta}} \mid \mathscr{F}_{(i-1)/n}] - \mathbb{E}[\xi_i^n \mid \mathscr{F}_{(i-1)/n}]\right) = V_n^1 + V_n^2 + o_p(1),$$

with the random variables V_n^1 and V_n^2 defined by

$$V_n^1 = \frac{2}{\lambda_2}\sum_{i=1}^{n} \mathbb{E}\Bigg[\sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}} \Bigg\{ \sup_{(i-1)/n \leqslant s,t \leqslant i/n} (\sqrt{n}\sigma_{(i-1)/n}(W_t - W_s) + \sqrt{n}\int_s^t \mu_{(i-1)/n}\,\mathrm{d}u$$
$$+ \sqrt{n}\int_s^t \{\sigma'_{(i-1)/n}(W_u - W_{(i-1)/n}) + v_{(i-1)/n}(B'_u - B'_{(i-1)/n})\}\,\mathrm{d}W_u$$
$$- \sqrt{n}\sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}} \Bigg\} | \mathscr{F}_{(i-1)/n}\Bigg],$$

and

$$V_n^2 = \sqrt{n}\frac{2}{\lambda_2}\sum_{i=1}^{n} \mathbb{E}\Bigg[\sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}} \Bigg\{ \sup_{(i-1)/n \leqslant s,t \leqslant i/n} \Bigg(\sigma_{(i-1)/n}(W_t - W_s) + \int_s^t \mu_u\,\mathrm{d}u$$
$$+ \int_s^t (\sigma_u - \sigma_{(i-1)/n})\,\mathrm{d}W_u\Bigg) - \sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}} \Bigg\} \mid \mathscr{F}_{(i-1)/n}\Bigg] - V_n^1$$
$$\leqslant \sqrt{n}\,\frac{2}{\lambda_2}\sum_{i=1}^{n} \mathbb{E}\Bigg[\sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}} \Bigg\{ \sup_{(i-1)/n \leqslant s,t \leqslant i/n} \Bigg(\int_s^t (\mu_u - \mu_{(i-1)/n})\,\mathrm{d}u$$
$$+ \int_s^t \Bigg\{\int_{(i-1)/n}^u \mu'_r \mathrm{d}r + \int_{(i-1)/n}^u (\sigma'_r - \sigma'_{(i-1)/n})\,\mathrm{d}W_r$$
$$+ \int_{(i-1)/n}^u (v'_r - v'_{(i-1)/n})\,\mathrm{d}B'_r\Bigg\}\,\mathrm{d}W_u\Bigg)\Bigg\} | \mathscr{F}_{(i-1)/n}\Bigg].$$

From the Cauchy–Schwarz and Burkholder inequalities, we find that

$$V_n^2 = \mathrm{o_p}(1).$$

At this point, we invoke Lemma 1 by setting:

$$f_{in}(s,t) = \sqrt{n}\sigma_{(i-1)/n}(W_t - W_s),$$

$$g_{in}(s,t) = n\int_s^t \mu_{(i-1)/n}\,\mathrm{d}u + n\int_s^t \{\sigma'_{(i-1)/n}(W_u - W_{(i-1)/n}) + v'_{(i-1)/n}(B'_u - B'_{(i-1)/n})\}\,\mathrm{d}W_u$$
$$= \mu_{(i-1)/n} g_{in}^1(s,t) + \sigma'_{(i-1)/n} g_{in}^2(s,t) + v'_{(i-1)/n} g_{in}^3(s,t).$$

Note that $\varepsilon = 1/\sqrt{n}$. Through assumption (V_1), we get the following identity:

$$\begin{aligned}(t_{in}^*(W), s_{in}^*(W)) &= \underset{(i-1)/n \leqslant s,t \leqslant i/n}{\arg\sup} f_{in}(s,t)\\ &= \underset{(i-1)/n \leqslant s,t \leqslant i/n}{\arg\sup} \sqrt{n}(W_t - W_s)\\ &\overset{\mathrm{d}}{=} \underset{0 \leqslant s,t \leqslant 1}{\arg\sup}(W_t - W_s).\end{aligned}$$

A standard result then states that the points $t^*_{in}(W)$ and $s^*_{in}(W)$ are unique, almost surely, so the lemma applies. Hence, by repeating the proof of the lemma, we get the decomposition:

$$V^1_n = \frac{2}{\lambda_2}\sum_{i=1}^{n} \mathbb{E}\left[\sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}}\left(\frac{1}{\sqrt{n}}\, g_{in}(t^*_{in}(W), s^*_{in}(W)) + R_{in}\right)|\mathscr{F}_{(i-1)/n}\right],$$

where the term R_{in} satisfies:

$$\mathbb{E}[(R_{in})^2] = \mathrm{o}(n^{-1}),$$

(uniformly in i). By the Cauchy–Schwarz inequality, we have the estimation:

$$\frac{2}{\lambda_2}\sum_{i=1}^{n} \mathbb{E}[\sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}} R_{in}|\mathscr{F}_{(i-1)/n}] = \mathrm{o_p}(1).$$

As $g^1_{in}(s,t)$, $g^2_{in}(s,t)$ and $g^3_{in}(s,t)$ are independent of $\mathscr{F}_{(i-1)/n}$, we obtain:

$$\mathbb{E}\left[\sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}} \frac{1}{\sqrt{n}} g^1_{in}(t^*_{in}(W), s^*_{in}(W))|\mathscr{F}_{(i-1)/n}\right] \equiv \frac{1}{\sqrt{n}}\sigma_{(i-1)/n}\mu_{(i-1)/n}v_1$$
$$\mathbb{E}\left[\sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}} \frac{1}{\sqrt{n}} g^2_{in}(t^*_{in}(W), s^*_{in}(W))|\mathscr{F}_{(i-1)/n}\right] \equiv \frac{1}{\sqrt{n}}\sigma_{(i-1)/n}\sigma'_{(i-1)/n}v_2$$
$$\mathbb{E}\left[\sigma_{(i-1)/n} s_{W_{i\Delta,\Delta}} \frac{1}{\sqrt{n}} g^3_{in}(t^*_{in}(W), s^*_{in}(W))|\mathscr{F}_{(i-1)/n}\right] \equiv \frac{1}{\sqrt{n}}\sigma_{(i-1)/n}v_{(i-1)/n}v_3,$$

with

$$v_k = \mathbb{E}[s_{W_{i\Delta,\Delta}} g^k_{in}(t^*_{in}(W), s^*_{in}(W))] \quad \text{for } k = 1, 2, \text{ and } 3.$$

Note that,

$$(t^*_{in}(W), s^*_{in}(W)) = (s^*_{in}(-W), t^*_{in}(-W)). \tag{A.4}$$

Using (A.4) and the relationship $(W, B) \overset{\mathrm{d}}{=} (-W, -B)$, it follows that $v_k = -v_k$ and, hence, $v_k = 0$ for $k = 1$, 2, and 3. This yields the estimation:

$$V^1_n = \mathrm{o_p}(1),$$

and the proof is complete. □

A.3. Proof of Theorem 3

The result is shown in the same manner as the proofs of Theorem 1 and 2. □

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