Volatility Jumps

Viktor Todorov * and George Tauchen †

December 8, 2008

Abstract

The paper undertakes a non-parametric analysis of the high frequency movements in stock market volatility using very finely sampled data on the VIX index compiled by the CBOE. The data suggest that stock market volatility is best described as a pure jump process without a continuous component. The finding stands in contrast to nonparametric results, reported here and elsewhere, that the stock price itself is not a pure jump process but rather contains a continuous martingale component. The jumps in stock market volatility are found to be so active that this discredits many recently proposed stochastic volatility models, including the classic affine model with compound Poisson jumps that is widely used in financial modeling and practice. Additional empirical work shows that jumps in volatility and price level in most cases occur together, are strongly dependent, and have opposite sign. The latter suggests that jumps are an important channel for generating leverage effect.

Keywords: Stochastic volatility, activity index, jumps, jump risk premium, leverage effect, VIX index.

*Department of Finance, Kellogg School of Management, Northwestern University, Evanston, IL 60208; e-mail: v-todorov@kellogg.northwestern.edu
†Department of Economics, Duke University, Durham, NC 27708; e-mail: george.tauchen@duke.edu.
1 Introduction

Jumps are intrinsically a continuous time concept that can be defined only relative to a theoretical stochastic process satisfying mild regularity conditions. Models for such processes are convenient paradigms that should, of course, provide close approximations to the dynamics of discretely observed data. Models without jumps, i.e., models with continuous sample paths are especially convenient because then asset prices respond in a locally linear manner, hedging arguments work, and convenient, easy to manipulate closed-form expressions for the reduced forms of economic models are available. In the presence of jumps, however, markets are fundamentally incomplete and the analysis far less tractable. A fairly complete discussions of the complications induced by jumps is (Cont and Tankov, 2004, Chapter 10, pp. 319–351). Technical issues aside, jumps are important because they represent a significant source of non-diversifiable risk as discussed at length in (Bollerslev et al., 2008) and the references therein. Policy makers must make decisions in real time during times of jump-inducing chaotic conditions in financial markets, and it is thereby economically important to develop a statistical understanding of the time series behavior of jumps.

There is currently fairly compelling empirical evidence for jumps in the level of financial prices. The most convincing evidence comes from recent nonparametric work using high-frequency data as in Barndorff-Nielsen and Shephard (2007) and Ait-Sahalia and Jacod (2008a) among others. Preceding that evidence are the findings from parametric studies on daily data such as Chernov et al. (2003), Andersen et al. (2002), Eraker et al. (2003), which are strongly suggestive but arguably not overwhelming evidence for price jumps in the daily record.

A very prominent model that underlies much empirical work for continuous time processes with jumps is the setup of Duffie et al. (2000), which we call here the affine double-jump model. Since the double-jump model is in the affine class, just as in Heston (1993), it admits reduced form solutions for asset prices and derivatives that are closed form in the sense that they can be
readily computed on modern computing equipment using straightforward numerical techniques for
Fourier series and ordinary differential equations. The double-jump model presumes rare jumps,
e.g., compound Poisson process, for both asset prices and their variances. It has been applied
empirically by Eraker et al. (2003), Eraker (2004), Chernov et al. (2003), among others. It is especially useful for specification and estimation of continuous time models that use data on both
the underlying security and derivatives written on it. These studies generally find evidence for
both jumps in the price level and its volatility.

In this paper we aim to understand better the nature of changes, both small and big, in the
market volatility, which have important implications for volatility modeling, developing hedging
strategies and specification of market risk premia. In particular we answer the following questions.
Is the market volatility moving through occasional and relatively infrequent changes like in a
model driven by a compound Poisson process, or it involves a lot of small moves, which over short
intervals look like Gaussian as in the Heston (1993) model? Are there “sufficiently big” moves in
the volatility to justify inclusion of jumps in its modeling? Are volatility and price jumps related?

To date, the answers to these questions come through estimation of parametric models built
around the affine double-jump model. However, these questions are intrinsically nonparametric
and importantly they are related with the properties of the observed paths of the volatility for
which we do not need long-span asymptotics. Therefore, the persistence in volatility, e.g. how
many autoregressive factors are needed for its modeling, is a completely separate issue from the
type of changes through which the volatility evolves over time. Here we are interested in the
latter. Goodness-of-fit type tests for parametric volatility models would inevitably be joint type
hypothesis and therefore they should always be interpreted with caution when making conclusions
about pathwise properties of volatility. Here, we separate the pathwise properties of volatility
from its long-span ones (like persistence) by using high-frequency data and resorting to fill-in
asymptptotics. The analysis is fully nonparametric and thus the evidence we provide here is
robust.
Our estimation is based on inferring from the data the value of a generalized activity index (Ait-Sahalia and Jacod (2008a) and Todorov and Tauchen (2008)), which is a generalization of the classical Blumenthal-Getoor index of Blumenthal and Getoor (1961). The generalized activity index is defined for an arbitrary stochastic process unlike the Blumenthal-Getoor index which is defined for jump processes only. It lies in the interval $[0, 2]$ and measures the vibrancy of the process. The index divides the stochastic processes used in the volatility modeling into equivalent classes. For example the compound Poisson jump process, which is a building block in the affine double-jump model has an activity level of 0. On the other extreme is the Brownian motion (and any diffusion process), whose activity is 2. Values of the index in $(0, 1)$ correspond to jump processes of finite variation, i.e. processes whose trajectories over finite intervals are finite. Values of the index in $(1, 2)$ correspond to jump processes of infinite variation, i.e. their trajectories over finite intervals have infinite length.

We estimate the activity index of high-frequency data on the VIX volatility index computed by the Chicago Board of Options Exchange (CBOE), which is based on close-to-maturity S&P 500 index options, and then make inferences about activity level of the unobserved market volatility. Our estimation of the activity is based on constructing from the high-frequency data an activity signature function, a diagnostic tool proposed in Todorov and Tauchen (2008). The latter provides also evidence whether the “big” moves in volatility should be modeled as jumps. Finally, to explore the link between the discontinuities on market level and market volatility we use co-jumping statistics proposed in Jacod and Todorov (2008).

The nonparametric evidence regarding the types of moves in the market volatility provides empirical information on the plausibility of the various parametric volatility models that have been proposed in the literature. The set of parametric models includes the double-jump model discussed above along with many others reviewed in Section 3 below. In some of these models volatility is continuous, and in others it is a pure-jump process. Of course there are also models with both continuous and jump components. The various parametric models have different implications for
the activity level of the VIX index, the presence of jumps in it and their relationship with the ones in the price level. We find that our nonparametric evidence identifies with reasonable accuracy the most plausible class of parametric models and rules out many others.

There are certain advantages and also some notable pitfalls entailed with using the VIX data. High-frequency data, of course, provide far more information about jumps, both large and small, than do daily data, which is a major plus. Furthermore, since the VIX index is computed from quoted options prices, whose prices are highly sensitive to volatility, it provides far more information on volatility than does the financial price series itself. Some care is needed, however, because the VIX index is not a direct measure of volatility, but rather it is actually the forward price, and thus a risk-neutral expectation, of future variance. The issues are discussed in more detail below. Finally, volatility is known to be a long memory process, and this interacts with the VIX index in some subtle ways regarding traded securities, semimartingales, and lack of arbitrage. As discussed below, it turns out that use of the general activity index permits us to separate jumps from long memory, and therefore we can make statements about the characteristics of volatility jumps without having to account for the long memory.

Turning to our main empirical findings, we can summarize them as follows. First, we find that market volatility is a very vibrant process - it involves many small changes as well as occasional big moves. The presence of big moves justifies the use of jumps in volatility modeling. In terms of modeling the small moves in volatility we find evidence against using Brownian motion because it is somewhat more “active” than what the data implies for the volatility. On the other hand, the “activity” of the small volatility changes cannot be captured by a compound Poisson process or even a process of finite variation like a Lévy subordinator, (i.e. a jump process with non-negative jumps as in the non-Gaussian OU model of Barndorff-Nielsen and Shephard (2001)). The reason for this is that a finite variation jump process would imply too “little” activity in volatility than what is observed. We conclude that an appropriate model for the volatility that can reconcile the empirical evidence is a pure-jump model, where the driving jump process is far more active
than a process of finite variation, but on the other hand not as active as a continuous martingale. This is to be contrasted with our findings about the market level where we find that we need a continuous martingale to capture the small changes and jumps to capture the big ones.

Second, using both high-frequency data on the VIX index as well as the S&P 500 index, we find strong evidence that the jumps in the volatility and the price level occur at the same time. We also find that these jumps exhibit strong negative dependence. These findings suggest that the underlying risks behind the occurrence of stock market discontinuities and the spikes in market volatility (and the corresponding risk premia) are similar if not the same. Therefore plausible equilibrium-based models for the market risk premia should be able to generate endogenously such links between volatility and jump risk (and their compensation).

The paper is organized as follows. In Section 2 we define the measures of stochastic volatility and in particular the VIX index, data on which is used in the empirical part of the paper. In Section 3 we also present some popular stochastic volatility models and analyze their implications for the VIX index. Section 4 introduces our measure of activity associated with a continuous-time process and proposes methods for its inference from discrete observations of the process. Section 5 applies the estimation technique of Section 4 to simulated data from the models in Section 3. Section 6 contains the empirical part of the paper. Finally Section 7 concludes.

2 The VIX Index

Let \( \{S_t\}_{t \geq 0} \) denote the log of a financial price evolving in continuous time. We are interested in the high frequency dynamics of the so-called volatility index (VIX) pertaining to \( S_t \). The VIX index is computed by the CBOE for the S&P 500 index using written options on it, but the methodology for its computation can be applied to other assets as well. Theoretically, the VIX index is based on a portfolio of out-of-the-money options written on \( S_t \) over a continuum of strike prices whose value equals that of a variance swap, see e.g. Britten-Jones and Neuberger (2000), Jiang and Tian (2005) and Carr and Wu (2008). The latter is defined as a forward contract on
the total quadratic variation of the log-price of the underlying asset over a fixed interval into the future. Following (Protter, 2004, pp. 66–76), let \([S, S]\) denote the quadratic variation process associated with \(S_t\). Hence the VIX index is given by\(^1\)

\[
v_t \equiv \mathbb{E}^Q ([S, S]_{t+N} - [S, S]_t | \mathcal{F}_t),
\]

(2.1)

where \(N > 0\) is fixed, \(\{\mathcal{F}_t\}\) is the filtration on the probability space on which \(\{S_t\}_{t \geq 0}\) is defined, and the expectation is taken under the risk-neutral distribution \(Q\). The quadratic variation process \([S, S]\) is adapted, increasing, càdlàg (i.e. with paths that are a.s. right continuous with left limits), and it can be split into continuous and discontinuous components

\[
[S, S]_t = [S, S]^c_t + [S, S]^d_t,
\]

(2.2)

corresponding respectively to the quadratic variation of the continuous and discontinuous parts of the price process \(S_t\). We make a standard assumption in finance and impose absolute continuity of \([S, S]^c_t\) i.e.

\[
[S, S]^c_t = \int_0^t \sigma^2_s \, ds,
\]

(2.3)

where \(\sigma^2_t\) is the *spot variance* of \(S_t\), also referred to as the instantaneous variance by Andersen et al. (2008). The spot variance \(\sigma^2_t\) is the instantaneous increment to the quadratic variation of the continuous martingale component of \(S_t\). Thus the VIX can be written as

\[
v_t = \mathbb{E}^Q \left( \int_t^{t+N} \sigma^2_s \, ds \bigg| \mathcal{F}_t \right) + \mathbb{E}^Q \left( [S, S]_{t+N}^d - [S, S]_t^d \bigg| \mathcal{F}_t \right).
\]

(2.4)

The first term is the familiar risk-neutral expectation of the forward integrated variance while the second is the risk-neutral expected contribution of the price jumps.

In practice, to generate an empirical measure of \(v_t\) in (2.4) the CBOE uses a portfolio of short-maturity out-of-the-money options on the S&P 500 Index over a discrete grid of strike prices. The

\(^1\)The volatility index is typically quoted in terms of annualized volatility, which is easier to interpret, but the form (2.1) is much simpler to work with theoretically.
details of the computation are available at http://www.cboe.com/micro/vix/vixwhite.pdf.\(^2\) The measurement is considered to be very accurate, as documented by extensive theoretical and Monte Carlo analysis in Jiang and Tian (2005) and Carr and Wu (2008). Hence in what follows, we treat the CBOE measured VIX as coinciding directly to \(v_t\).

It is always important to keep in mind the distinction between the observed VIX and the unobserved spot variance. The observed VIX is the CBOE measurement of \(v_t\) in (2.4). We use these observations to make inferences about important characteristics of the random process \(\{\sigma_t^2\}_{t \geq 0}\) for the spot variance. The inference is complicated because the VIX is forward looking, and its increments are generated by movements in variables that influence the conditional expectations on the right hand side of (2.4). Furthermore, we only observe discretely-sampled observations on the VIX index which also complicates estimation and inference.

To the extent possible, we follow the convention of using the term “variance” for quantities that are squares and measures of variance and the term “volatility” to refer to measures of standard deviation. Variance measures are easier to work with mathematically because they add, while volatility measures are easier to interpret because they are expressed in the same units as the data itself.

As indicated by the many papers reprinted in Shephard (2005b) and the references therein, the dynamics of the spot variance \(\sigma_t^2\) are extremely important for modeling financial series. However, the spot variance itself is not directly observed. Our plan here is to adduce nonparametric evidence from high-frequency VIX data on the empirical plausibility of various models for the spot variance. The spot variance itself can also be split into continuous and discontinuous parts

\[
\sigma_t^2 = \sigma^2_{c,t} + \sigma^2_{d,t}. \tag{2.5}
\]

The jump component \(\sigma^2_{d,t}\) can be important even though it enters \(v_t\) in (2.4) through the integra-

\(^2\)There are two errors in replicating the price of a variance swap, which is the value on the right hand side of (2.4). The first comes from the fact that we use finite number of options in the calculation of the VIX index, while the theoretical variance swap rate is equal to the price of continuum portfolio of options. The second error arises when there are jumps in \(S_t\). It is equal to \(-2 \int_t^{t+N} \int_{R \cup 0} (e^x - 1 - x - x^2/2) dt \nu_0^2(dx)\), where \(dt \nu_0^2(dx)\) is the risk-neutral measure of the jumps. This error does not influence any of our subsequent results, see Theorem 1 and its proof.
tions involved in (2.3) and forming the conditional expectation in (2.4). A jump discontinuity $\sigma^2_{d,t} - \sigma^2_{d,t-}$ influences the entire trajectory $E(\sigma^2_{t+s} | \mathcal{F}_t)$, $s \geq 0$, and thereby induces a jump discontinuity in $v_t$.

Historically, stochastic volatility models have assumed that the spot variance is continuous, i.e., $\sigma^2_t \equiv \sigma^2_{c,t}$. However, more recently there has been interest in pure-jump stochastic volatility models, $\sigma^2_t \equiv \sigma^2_{d,t}$; see, e.g., Barndorff-Nielsen and Shephard (2001). Of course, the models can be combined, as in the double-jump model of Duffie et al. (2000). Two recent comprehensive reviews of stochastic volatility are Shephard (2005a) and Andersen and Benzoni (2007). In the subsequent section we highlight the more relevant models and their implications for the VIX index.

## 3 Parametric Models for the Spot Variance

Our objective in this paper is to use to nonparametric-type evidence to cast light on the empirical plausibility of the parametric volatility models that have been proposed in the literature. Thus we review the extant parametric models in this section, and we then proceed to the nonparametric analysis in the application section farther below. In the review, we leave unspecified whether the model pertains to the risk neutral distribution or the objective distribution, because common practice is to assume a risk premium structure that preserves the basic form of the model across the two distributions. We suppress here the presence of price jumps, since we are considering parametric spot volatility models (allowing for price jumps with intensity that is linear in the spot volatility factors will lead to affine transformations of the expressions for the VIX index below).

### 3.1 Continuous Path Models ($\sigma^2_t \equiv \sigma^2_{c,t}$)

A model of central importance is the square-root specification of Heston (1993):

---

3It is important to keep in mind the distinction that $\sigma^2_{d,t}$ in (2.5) pertains to the jump component of the spot variance while $[S, S]^d$ in (2.2) pertains to the jumps in the price.
**Affine Diffusion**

\[ d\sigma_t^2 = \rho(\sigma_t^2 - \psi_0)dt + \psi_1 \sigma_t dB_t, \quad \rho < 0, \psi > 0, \quad (3.1) \]

where \( B_t \) is standard Brownian motion. It is easy to show that with (3.1) the VIX index \( v_t \) becomes an affine function of the spot variance

\[ v_t = C_0 + C_1 \sigma_t^2, \quad (3.2) \]

where \( C_0 \) and \( C_1 \) are easy-to-derive constants depending on the parameters determining the motion of \( \sigma_t^2 \) in (3.1) and \( N \). The affine diffusion model is particularly convenient for derivatives pricing because of the exponentially-affine conditional characteristic function of the spot variance. This model can be extended by specifying \( \sigma_t^2 \) as a superposition of square-root processes, as in Duffie et al. (2000). In this case \( v_t \) will be simply an affine function of the volatility factors. This generalization allows to generate richer persistence patterns in the spot variance.

Another widely used model, particularly in econometric volatility modeling, is based on the exponential diffusion specification of Hull and White (1987):

**EXP-Gaussian-OU**

\[ \sigma_t^2 = \exp(\alpha_0 + \alpha_1 f_t), \quad (3.3) \]

\[ df_t = \rho f_t dt + dB_t, \quad \rho < 0. \quad (3.4) \]

In this model the spot variance is the exponential of a linear Gaussian process, \( f_t \), also called a Gaussian Ornstein-Uhlenbeck (OU) process; the model’s properties and related papers are thoroughly reviewed in Shephard (2005a). Standard calculations imply that the VIX index is

\[ v_t = \int_0^N \exp \left( \alpha_0 + \alpha_1 e^\rho u f_t + \alpha_1^2 \frac{1 - e^{2\rho u}}{4\rho} \right) du. \quad (3.5) \]

Similar to the affine diffusion model, the EXP-Gaussian-OU model can be extended to a multi-factor model \( \sigma_t^2 = e^{\alpha' f_t} \) where \( \alpha = (\alpha_1, ..., \alpha_k) \), \( f_t = (f_{1t}, ..., f_{kt}) \) and \( f_{i,t} \) are independent Gaussian OU processes. The formula for VIX in (3.5) is easy to extend to the multi-factor setting as well.
More generally, when $\sigma^2_t = G(f_t)$ for some measurable function $G : \mathbb{R}^k \to \mathbb{R}^+$ and $f$ is $k$-dimensional continuous-path Markov process, then from the properties of the Markov processes we have $v_t = F(f_t)$ for some function $F : \mathbb{R}^k \to \mathbb{R}^+$. Equations (3.2) and (3.5) provide examples of the function $F$ in the two popular continuous models for the volatility. In other words, when the stochastic volatility is modeled with Markov processes which is the case for most parametric models, the spot variance and the associated VIX index are both functions of the volatility factors. This has important implications for our analysis of the activity of volatility in the next section.

To capture the well-documented long range dependence in volatility (Baillie et al., 1996; Comte and Renault, 1998; Shephard, 2005a), we need to go out of the Markov setting for the stochastic volatility. Comte and Renault (1998) present a fractionally-integrated long memory model which can be written in differential form as:

**EXP-OU-FI**

\[
\sigma^2_t = \exp(\alpha_0 + \alpha_1 f_t), \quad (3.6)
\]

\[
df_t = \rho f_t dt + dB_{\delta,t}, \quad \rho < 0, \quad (3.7)
\]

where $B_{\delta,t}$ is fractionally integrated Brownian motion with fractional integration parameter $0 < \delta < \frac{1}{2}$. This model builds on the EXP-Gaussian-OU model by substituting the Brownian increments with those of a fractionally-integrated Brownian motion. In the EXP-OU-FI model the factor $f_t$ follows the continuous time analogue of an ARFIMA$(1, \delta, 0)$ discrete-time model and is no longer Markovian. It has the following stationary representation

\[
f_t = \int_{-\infty}^t a(t-s) dB_s, \quad (3.8)
\]

where $B_t$ is standard Brownian motion, and the function $a(\cdot)$ is given by

\[
a(u) = \frac{1}{\Gamma(1 + \delta)} (u^\delta + \rho e^{\rho u} \int_0^u e^{-\rho x} x^\delta dx), \quad (3.9)
\]

with $\Gamma(\cdot)$ being the gamma function. Therefore, the VIX index takes the following form for the
EXP-OU-FI model

\[ v_t = \int_0^N \exp \left( \alpha_0 + \alpha_1 \int_{t-u}^t a(t+u-s) dB_u + \frac{\alpha_1^2}{2} \int_0^u a^2(z) dz \right) du. \] (3.10)

Because of the non-Markovian structure, we can no longer express \( v_t \) as a function of the volatility factor \( f_t \). In this framework, long range dependence and arbitrage interact in an economically important manner. As noted in Comte and Renault (1998), the spot variance \( \sigma_t^2 \) in (3.6) is not a semimartingale. Nevertheless there are no arbitrage opportunities of the type discussed in Rogers (1997) because the spot variance is not a traded security. The observed VIX index, however, is a portfolio of traded securities and it should be a semimartingale to rule out arbitrage, and this indeed is the case for the model-implied VIX index in (3.10). We will illustrate this point in the next section.

3.2 Lévy-Driven Discontinuous Models \( (\sigma_t^2 \equiv \sigma_{d,t}^2) \)

Similar to the continuous path models we start with the most tractable Lévy-driven discontinuous model. This is the non-Gaussian OU model of Barndorff-Nielsen and Shephard (2001):

**Non-Gaussian OU**

\[ d\sigma_t^2 = \rho \sigma_t^2 dt + dL_t, \quad \rho < 0, \] (3.11)

where \( L_t \) is a pure-jump Lévy process with non-negative increments, also called a subordinator such that \( \int_{x>1} x\nu(dx) < \infty \) where \( \nu(dx) \) is the Lévy measure of \( L_t \). This model is a special case of the general affine jump-diffusion models of Duffie et al. (2003) and thus it enjoys the tractability offered by the exponentially-affine form of the conditional characteristic function of the spot variance (similar to the affine diffusion model). The VIX index in this case is an affine transformation of the spot variance exactly as in equation (3.2). The model can be extended by specifying the spot variance as a superposition of non-Gaussian OU factors. As for the affine diffusion model, the corresponding VIX index in such multivariate extension will be simply another affine function of the pure-jump Lévy-driven factors.
Non-negativity of $\sigma_i^2$ in (3.11) is assured by the fact that $L_t$ is a subordinator and hence of finite variation. In order to allow for driving processes of infinite variation and at the same time to ensure nonnegativity, a functional form transformation is needed. The analogue of the Exp-Gaussian-OU above is

**EXP-Lévy-OU**

\[
\sigma_i^2 = \exp(\alpha_0 + \alpha_1 f_i), \\
df_i = \rho f_i dt + dL_t, \quad \rho < 0,
\]

where $L_t$ is a pure-jump Lévy process of potential infinite variation such that $\int_{|x|>1} e^{\alpha_1 x} \nu(dx) < \infty$ where $\nu(dx)$ is the Lévy measure of $L_t$. This model can be viewed as a continuous-time limit of the discrete EGARCH model of Nelson (1991), see Haug and Czado (2007). Using (Sato, 1999, Theorem 25.17) the VIX index under the dynamics (3.12) process can be shown to take the form

\[
v_t = \int_0^N \exp [\alpha_0 + \rho u f_i + C(u)] \, du,
\]

where $C(u)$ is some function of $u$ determined by the characteristic exponent of the jump process $L_t$ (and the constants $\alpha_1$, $\rho$ and $N$).

The same comment as for the continuous path volatility models applies here. In general, if we model the spot variance via $\sigma_i^2 = G(f_i)$ for some $G: \mathbb{R}^k \to \mathbb{R}^+$ and $f$ is $k$-dimensional pure-jump Lévy-driven Markov process, then $v_t = F(f_i)$ for some function $F: \mathbb{R}^k \to \mathbb{R}^+$.

### 3.3 Combined Continuous/Jump Models

Evidently, there are all sorts of ways to combine the continuous and pure-jump models from Subsections 3.1–3.2. For example, the well-known affine jump-diffusion model for the spot variance of Duffie et al. (2000), estimated by Eraker et al. (2003), is of the form
Affine Jump-Diffusion

\[ d\sigma_t^2 = \rho (\sigma_t^2 - \psi_0) \, dt + \psi_1 \sigma_t \, dB_t + dL_t, \quad \rho < 0, \psi > 0, \]  

(3.15)

where \( L_t \) is typically a compound Poisson process with nonnegative jumps, and more generally a Lévy subordinator, satisfying \( \int_{x>1} x \nu(dx) < \infty \) where \( \nu(dx) \) is the Lévy measure of \( L_t \). The VIX index in this case is an affine function of the spot variance.

4 The Activity Level of Volatility

The models in the previous section have been all used in various applications for modeling stochastic volatility, and our aim is to provide nonparametric evidence on their empirical plausibility (and that of many others) using high-frequency observations on the VIX index. Towards this end, we now show in this section how to associate with each continuous-time process an index of its so-called activity. We further derive the activity index for the spot variance and the VIX index associated with the various models from the preceding section and show that the models differ in the implied volatility activity. We end the section with proposing an estimation strategy for the activity index from high-frequency observations of the process of interest (i.e. the VIX index here).

4.1 Activity Index

We start with consideration of a measure of activity for an arbitrary continuous-time process. Intuitively, by activity level we mean the “degree” of vibrancy of the process, i.e. the “roughness” of its trajectories. Formally, the statistical setup is as follows. We have \( n \) observations on a generic scalar process \( X \) over a long span \([0, T]\). During each subperiod \((t-1, t]\), where now \( t \) is an integer, we have high-frequency observations on \( X \) with a sampling interval of length \( \Delta_n \). That is, we observe \( X \) at times \( t-1, t-1+\Delta_n, \ldots, t-1+[1/\Delta_n]\Delta_n \) during the subperiod. Think of the subperiod as being either a day, week, or month. Following Ait-Sahalia and Jacod (2008a) and Todorov and Tauchen (2008), we can define the activity of \( X \) during an arbitrary interval \((t-1, t]\)
as

$$\beta_{X,t} := \inf \left\{ r > 0 : \lim_{\Delta_n \to 0} V_t(X, r, \Delta_n) < \infty \right\},$$

(4.1)

where $V_t(X, r, \Delta_n)$ is the power variation of $X$ over the interval $(t - 1, t]$.

$$V_t(X, r, \Delta_n) = \sum_{i=1}^{[1/\Delta_n]} \left| X_{t-1+i\Delta_n} - X_{t-1+(i-1)\Delta_n} \right|^r.$$  

(4.2)

The importance of the power variation (4.2) for financial econometrics was pointed out in Barndorff-Nielsen and Shephard (2003, 2004) and follow-up papers. Our interest is mainly in the case when $X$ is a semimartingale, because to avoid arbitrage, any traded security (and thus the VIX index as well) needs to be a semimartingale, see Delbaen and Schachermayer (1994). For a semimartingale the activity index takes values in the interval $[0, 2]$. Each semimartingale can be decomposed into drift term along with continuous and discontinuous (local) martingale parts. These components of the semimartingale process can be naturally ranked in terms of their activity in the following order from least to most active: finite activity (e.g. compound Poisson) jumps (activity of 0), infinite activity but finite variation jumps (activity in $[0, 1]$), drift (activity of 1), infinite variation jumps (activity of $(1, 2]$), continuous martingales (activity of 2). The activity of the semimartingale is determined by the activity of its most active component. Thus, for example, if $X$ is driven by both a Brownian motion and jumps, the continuous martingale dominates and the activity of $X$ is equal to that of its continuous martingale component, which is 2.

Apparently, the jumps are the most interesting component of a semimartingale in terms of their activity. Our measure of activity for pure-jump Lévy processes coincides with their so-called (generalized) Blumenthal-Getoor index (Blumenthal and Getoor (1961) and Aït-Sahalia and Jacod (2008a)). This index is entirely analogous to the parameter $\alpha$ of the $\alpha$-stable distribution and it measures the behavior of the very small jumps.

When $X$ is not a semimartingale things are different. For example, when $X$ is the OU process driven by fractional Brownian motion given in (3.8), then its activity index is determined by its

\footnote{Our focus in this paper is only on semimartingales with absolute continuous characteristics, known as Ito semimartingales, see Jacod and Shiryaev (2003) for a definition of the characteristic triplet of a semimartingale.}
degree of fractional integration, \( \delta \), and is equal to \( \frac{1}{\delta + 0.5} \), see Corcuera et al. (2006).

Finally, note that we defined the activity index over the subinterval \((t-1, t]\) instead of the whole sample. Thus, we implicitly allowed for the possibility that the process \( X \) can change its activity over time. While the activity of most parametric continuous-time models (including those for the spot variance of the previous section) is constant over time, we do not make this assumption apriori.

4.2 Linking the Activity Level of the VIX to that of the Spot Variance

Since we model stochastic volatility by specifying a process for the spot variance, naturally we are interested in the activity of the spot variance process. However, spot variance is unobservable and instead only high-frequency observations on the VIX index are available to us. Therefore, it is important to investigate the relationship between the activity level of the observed VIX index to that of the spot variance, and it turns out they are the same under very weak regularity conditions. Indeed, in a Markov setting, which is most often adopted in parametric volatility modeling, the agreement is established by the following theorem:

**Theorem 1** For the setting in (2.2)-(2.4) suppose in addition the following

(a) \( \sigma_t^2 = G^{(c)}(f_t) \) for some twice differentiable function \( G^{(c)} : \mathbb{R}^k \to \mathbb{R}^+ \) with non-vanishing first derivatives on the support of \( f_t \),

(b) the compensator of the jumps in \( \{S_t\} \) under the measure \( \mathbb{Q} \) is of the form \( G^{(d)}(f_t)dt \otimes \eta(dx) \)

for some twice differentiable function \( G^{(d)} : \mathbb{R}^k \to \mathbb{R}^+ \) and a measure \( \eta \) on \( \mathbb{R} \) satisfying

\[
\int_{\mathbb{R}}(|x|^2 \wedge 1)\eta(dx) < \infty,
\]

(c) \( f_t \) is a vector with independent elements each of which solves under \( \mathbb{Q} \)

\[
df^{(i)}_t = \sum_{j=1}^{d_i} g^{(i)}_j (j^{(i)}_{t-}) dZ^{(i)}_{tj}, \quad j = 1, \ldots, d_i, \quad i = 1, \ldots, k, \quad (4.3)
\]

where the functions \( g^{(i)}_j(\cdot) \) are twice differentiable and \( Z^{(i)}_{tj} \) are independent Lévy processes.
Assume further that the expectation \( \nu_t = \mathbb{E}^Q ( [S,S]_{t+N} - [S,S]_t | \mathcal{F}_t ) \) is well defined. Then, \( \nu_t = F(\mathbf{f}_t) \) for some continuously differentiable function \( F : \mathbb{R}^k \to \mathbb{R}^+ \). Furthermore, if \( \frac{\partial F}{\partial f_i} (\cdot) \neq 0 \) on the support of \( \mathbf{f}_t \) for \( i = 1, \ldots, k \), then we have for an arbitrary \( t > 0 \)

\[
\beta_{\sigma^2,t} \equiv \beta_{v,t} \quad \text{a.s.,} \tag{4.4}
\]

where \( \beta_{X,t} \) for an arbitrary semimartingale \( X_t \) is defined in (4.1).

The result of the theorem follows essentially from the fact that under its assumptions, both the spot variance and the associated VIX index are continuously differentiable functions of the volatility factors. Such transformations preserve the activity index, and therefore to determine the volatility activity we need only determine the activity of the most active volatility factor.

The theorem does not say that the VIX, which is a market-implied quantity, and the spot variance are the same. Indeed, their dynamics, including persistence and level, might be quite different as is often found in empirical work. The theorem does say, however, that a key feature of the stochastic processes, i.e., their activity levels, must agree.

Based on Theorem 1 and the discussion in the previous section (see Todorov and Tauchen (2008) for formal results), we have the following levels of volatility activity for the Markov models of the previous section

(a) Affine Jump Diffusion and EXP-Gaussian-OU: \( \beta_{\sigma^2,t} = \beta_{v,t} = 2. \)

(b) Non-Gaussian OU: \( \beta_{\sigma^2,t} = \beta_{v,t} = 1. \)

(c) EXP-Lévy-OU: \( \beta_{\sigma^2,t} = \beta_{v,t} = \max\{\beta_L, 1\} \), where \( \beta_L \) is the Blumenthal-Getoor index of the driving Lévy process.

Note that for the non-Gaussian OU model the volatility activity is determined by the drift term in (3.11), since the driving jump process is a Lévy subordinator and thus of finite variation. On the other hand, for the affine jump-diffusion model (and all models in which Brownian motion is used in the specification of the stochastic volatility), the volatility activity is driven by the
continuous martingale part which “dominates” drift and arbitrary jump factors. Thus, the most tractable (and hence most used) stochastic volatility specifications from the different classes of models of the previous section, the affine jump-diffusion model and the non-Gaussian model, have very different implications for the activity level of the stochastic volatility.

Finally, the relationship between the activity of the spot variance and the associated VIX index is non-trivial outside of the Markov setting. For the EXP-OU-FI model, we have $\beta_{\sigma^2,t} = \frac{1}{\delta + 0.5}$ while $\beta_{v,t} = 2$. As already mentioned, the activity of the spot variance is driven by the degree of the fractional integration. The transformation implied by the VIX index restores the semimartingale property, and, since the model is driven by Brownian motion, we have that the activity of the VIX index is 2.0 as determined by this most active component. Therefore for the EXP-OU-FI model, the activity of the VIX index will not be informative about that of the spot variance. However, this is not a drawback of our analysis. For this model it is the activity of the VIX index that we are most interested in, since it tells us that the volatility process is modeled via (fractionally-integrated) Brownian motion and not jumps, and this is exactly what we are after.

4.3 Estimation of Activity Index

We finish this section with developing an estimation strategy for (1) detecting jumps and (2) estimating the activity index from discrete observations of the process of interest. We will use the setup of Section 4.1 and will work with the generic discretely-observed semimartingale process $X$. Following Todorov and Tauchen (2008), we introduce the so-called activity signature function, computed from the discrete observations of the process $X$ as follows:

$$b_{X,t}(p) = \frac{\ln(2)p}{\ln(k) + \ln[V_t(X,p,2\Delta_n)] - \ln[V_t(X,p,\Delta_n)]}, \quad p > 0.$$  \hspace{1cm} (4.5)

As shown in Todorov and Tauchen (2008), as we sample more frequently (i.e. $\Delta_n \to 0$) on any fixed interval $(t-1,t]$, the activity signature function $b_{X,t}(p)$ behaves as follows

A. $b_{X,t}(p) \xrightarrow{p} 2, \quad \forall p > 0$ if $X$ contains continuous martingale,
B. \( b_{X,t}(p) \xrightarrow{p} \max(p, 2), \quad \forall p > 0 \) if \( X \) contains continuous martingale plus jumps,

C. \( b_{X,t}(p) \xrightarrow{p} \max(p, \beta_{X,t}), \quad \forall p \neq \beta_{X,t} \) if \( X \) is driven by a pure jump process,

where the convergence is locally uniform in \( p \). Thus, \( b_{X,t}(p) \) converges to a flat line at 2 if the process is driven by Brownian motion and does not have jumps, whereas \( b_{X,t}(p) \) will converge to a flat line equal to \( \beta_{X,t} \) for \( p < \beta_{X,t} \) and then have a kink at \( \beta_{X,t} \), converging to \( p \) for \( p > \beta_{X,t} \).

The behavior of the activity signature function gives all the necessary information to determine the activity of the discretely-observed process \( X \). On the other hand, the behavior of the activity signature function for \( p > 2 \) can reveal us whether there are jumps, large or small, in the process \( X \) in the interval \((t-1, t]\) even in the case when they are dominated by a continuous semimartingale. Thus, that part of the activity signature function will be of use to us as a way of testing for jumps regardless of the presence of continuous component.\(^5\)

Todorov and Tauchen (2008) suggest graphical inspection as a way to make inference about \( \beta_{X,t} \) and the presence of jumps. Specifically, let

\[
B_q(p) = q^{th} \text{ quantile of } \{ b_{X,t}(p) \}_{t=1,2,\ldots,N}, \quad q \in (0, 1)
\]  

(4.6)

denote the \( q^{th} \) quantile of the \( b_{X,t} \) for each power \( p \). \( B_q(p) \) is called the quantile activity signature function. The most informative plots are obtained from the lower and upper quartiles, \( B_{0.25}(p), B_{0.75}(p) \), and the median \( B_{0.50}(p) \) over the range \( 0 \leq p \leq 4 \).

In addition to the quantile signature plots, here we also implement a direct estimation of the activity index over each subinterval \((t-1, t]\). Based, on the convergence results above, a very simple estimator for the activity can be formed by the value of the activity signature function evaluated at a power that is “sufficiently” small. Improvements of this estimator clearly can be derived, as there are benefits from integrating the activity signature function over different powers in the estimation. This is however beyond the scope of the present paper. Formally, we define

\(^5\)Formal jump test can be derived based on our activity signature function for some fixed value of \( p > 3 \) using the CLT result in Ait-Sahalia and Jacod (2008b).
our activity estimator as

\[ \hat{\beta}_{X,t} = b_{X,t}(p), \text{ for some fixed value of } p. \]  

(4.7)

This estimator will be consistent for the activity index, provided \( \beta_{X,t} > p \). The question is how to choose \( p \) in (4.7). There are several effects that need to be considered in this choice. First, obviously we need to pick \( p \) lower than the lowest possible activity that we assume the process \( X \) can possibly have. In our case, since we are interested in measuring activity of volatility, it is natural to assume that the lowest activity level is 1. This is because the volatility process is mean-reverting and this requires presence of a drift term in its dynamics, and the latter has an activity of 1. This was also illustrated with the different parametric models of Section 3. Second, it can be shown that for very small powers the estimator is relatively inefficient and thus higher values of \( p \) are preferred. Based on this discussion, an appropriate choice for \( p \) in (4.7) appears to be a value relatively close to, but lower than, 1.

Turning to the asymptotic behavior of the estimator (i.e. rate of convergence and asymptotic distribution), we mention that it is not trivial to determine it in a completely general setting. Therefore, for our analysis we will rely mostly on a Monte Carlo analysis to give us a guidance.

5 Monte Carlo

We now examine the behavior on simulated data of the quantile activity signature functions and the estimator \( \hat{\beta}_{X,t} \) defined by (4.7) for \( X \) corresponding to the VIX index and, in one case, the spot variance as well. The strategy is first to specify a parametric model for the spot variance, and then explore via Monte Carlo the relationship between the activity index of the model-implied VIX relative to the activity index of the spot variance in the underlying parametric model. From Theorem 1, the two indexes must agree asymptotically whenever the parametric model implies the spot variance is a semimartingale.
5.1 Setup

Following standard conventions annualized volatility is based on 252 trading days per year. We simulate different volatility models over a total of 4400 days. In each day we sample 78 times, which corresponds to a 5-minute sampling frequency in a standard 6.5 hours trading day, and this also is the frequency of our high-frequency data that we use in the empirical analysis of the next section. The interval \((t - 1, t]\) corresponds to 22 trading days, i.e., a calendar month, so the unit of time is thereby \(1 = \) one month in all calculations that follow. There are \(78 \times 22 = 1716\) high-frequency intervals per month and \(4400/22 = 200\) months worth of simulated data. The use of a month as the subinterval seems a reasonable compromise in the tradeoff between the presumption of constant activity, \(\beta_{X,t}\), over the subinterval and the associated reduction in sampling error inference with more data points per interval. Of course \(\beta_{X,t}\) is the same for all 200 simulated months, but that need not be the case with observed data.

The Monte Carlo investigation considers the following models for the spot variance from Section 3: (1) the affine diffusion/jump-diffusion models; (2) the long memory EXP-OU-FI model; (3) the non-Gaussian OU model; and (4) the EXP-Lévy-OU model. In each case, we plot the simulated VIX data, the activity signature quantile functions computed over the 200 months, and the distribution of the 200 monthly estimates \(\hat{\beta}_{X,t}\). For the estimator \(\hat{\beta}_{X,t}\) we set the power in (4.7) to \(p = 0.95\) (experiments with other powers lower than 1 produced similar results). In the EXP-OU-FI model we also consider the same computations for the spot variance, because it is not a semimartingale, and the contrast between it and the associated VIX is economically interesting. Nonetheless, since we observe only the VIX index and never the spot variance, we concentrate on inference using the simulated VIX index to guide the empirical work of the subsequent section.

For the models in the Monte Carlo with jumps the underlying Lévy processes are defined by the following Lévy densities:

- **Affine Jump-Diffusion**: compound Poisson with intensity \(\lambda\) and exponentially distributed
jump size, i.e. Lévy density of $\nu(x) = \frac{e^{-x/\mu}}{\mu}, x \geq 0$.

- **non-Gaussian OU**: tempered stable subordinator, i.e. Lévy density of $\nu(x) = c \frac{e^{-\lambda x}}{x^{1+\beta}} 1_{\{x>0\}}$.

- **Exp-Lévy-OU**: symmetric tempered stable process (also known as CGMY process), i.e. Lévy density of $\nu(x) = c \frac{e^{-\lambda|x|}}{|x|^{1+\beta}}$.

The parameter values for each of the simulation scenarios are given in Table 1. The parameters for the Affine Jump-Diffusion Model were taken from the estimation results of Eraker et al. (2003), while those of the EXP-OU-FI model are similar to the parameters used in the Monte Carlo study of Comte and Renault (1998).

## 5.2 Monte Carlo findings

The results from the estimation on the simulated data are summarized in Figures 1-3. We proceed with a short discussion on each of the cases.

### 5.2.1 Affine Diffusion and Jump-Diffusion Models

The outcomes reported in Figure 1 are in accordance with the theoretical predictions from Subsection 4.3. First, the distribution of the estimator of activity $\hat{\beta}_{X,t}$ is centered around 2, as predicted. Also, the addition of jumps to the spot variance produces almost no change to the empirical distribution of the activity estimator. Second, for $0 \leq p \leq 2$ the three quantile activity signature functions for both cases are all very close to 2, which again is in line with our theoretical results.

One important aspect of Figure 1 concerns the signature plots for the affine jump-diffusion, which is calibrated to point estimates from the published literature. For months in which there is one or more jumps, the asymptotic limit of the activity signature function is $\max(p, 2)$. The implied linear increase in the quantile plots for $p > 2$ is barely visible, because the frequency of the jumps is extremely low: on average, there is one jump in 180 days. Thus, on average less than 25 percent of the simulated months contain jumps, and their effects on the activity signature plots
is barely visible. Similar behavior for a jump-diffusion model for the stock price is reported in Todorov and Tauchen (2008).

5.2.2 The Long Memory EXP-OU-FI Model

The results for this case are reported in Figure 2. We report the results from estimation not only on the VIX index, but also on the spot variance. As already explained, the spot variance in this model is not a semimartingale and the activity estimator in this case converges to $\frac{1}{\delta+0.5} \in (1, 2)$ for values of $\delta \in (0, 0.5)$, which are the values of interest since only for them there is long-range dependence. The distribution of the estimator of the activity is nearly centered on its theoretical value although slightly upward biased. The quantile activity signature plots associated with the spot variance are also in line with theoretical predictions, since in this case the asymptotic signature plot is a straight line at $\frac{1}{\delta+0.5}$. This is to be contrasted with a jump martingale, whose activity signature quantiles will be flat at $\beta$ only for powers $p < \beta$ and after this will linearly increase.

Turning to the results for the VIX index in the simulated EXP-OU-FI model, we can see a vast difference. Both the activity estimator and the activity signature quantiles are centered around 2. This is predicted by theoretical results, since the integration and the expectation implicit in the computation of the VIX index restore the semimartingale property. In practice, of course we do not observe the spot variance and therefore activity index estimate of 2 for the VIX index can be consistent with a long-range dependence model.

5.2.3 Lévy-Driven Discontinuous Models

The results for this case are reported in Figure 3. First, in the non-Gaussian OU model, the drift term dominates the activity of the jumps (since they are of finite variation) and therefore the theoretical value of the activity for the VIX index is 1. Our estimator of the activity has excellent properties in this case. First, from the activity signature plots the kink is readily visible. We note

\footnote{Comte and Renault (1998) credit private communication with L. C. G. Rogers for pointing this out.}
Table 1: Parameter Setting for the Monte Carlo

<table>
<thead>
<tr>
<th>Case</th>
<th>Parameters</th>
<th>$\beta_{\sigma^2}$</th>
<th>$\beta_{\psi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A. Affine-Jump Diffusion Models</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>no jumps</td>
<td>$\sigma^2$</td>
<td>$-\rho$</td>
<td>$\psi$</td>
</tr>
<tr>
<td>with jumps</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.8136</td>
<td>0.0128</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.5585</td>
<td>0.0250</td>
</tr>
<tr>
<td><strong>Panel B. EXP-OU-FI</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\alpha_0$</td>
<td>$\alpha_1$</td>
<td>$\delta$</td>
</tr>
<tr>
<td></td>
<td>0.00</td>
<td>1.00</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Panel C. Non-Gaussian-OU</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-\rho$</td>
<td>$\beta$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td></td>
<td>0.03</td>
<td>0.50</td>
<td>5.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Panel D. EXP-Lévy-OU</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\alpha_0$</td>
<td>$\alpha_1$</td>
<td>$-\rho$</td>
</tr>
<tr>
<td></td>
<td>-0.70</td>
<td>1.00</td>
<td>0.07</td>
</tr>
</tbody>
</table>

that the three quantiles almost coincide illustrating the precision in our inference in this case.

This is further confirmed by the histogram of the activity estimator. Our estimator has very small variance and is slightly upward biased - it is centered around 1.05. Also, the activity signature plots clearly indicate the presence of jumps in volatility, since the three activity signature quantiles increase linearly after $p = 1$.

Turning to the simulation results for the EXP-Lévy-OU model, we can make similar conclusions. We can clearly “see” the presence of jumps because of the increase of the activity signature quantiles after the value of the Blumenthal-Getoor index of $\beta = 1.5$. Also, the activity estimate and the kink on the activity signature plot is around 1.7, which is slightly higher than the theoretical value of 1.5. This bias is due to sampling frequency, that is it disappears for more frequent sampling. Also this bias can be reduced if the estimator is based on averaging the activity signature function for lower values of $p$ as evident from the activity signature plots.
Table 2: Summary Statistics for the Data

<table>
<thead>
<tr>
<th>Statistics</th>
<th>VIX Index</th>
<th>S&amp;P 500 Index Daily Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>14.2476</td>
<td>9.7394</td>
</tr>
<tr>
<td>std</td>
<td>2.9991</td>
<td>11.5713</td>
</tr>
<tr>
<td>skewness</td>
<td>1.5989</td>
<td>−0.3043</td>
</tr>
<tr>
<td>kurtosis</td>
<td>7.6725</td>
<td>4.3764</td>
</tr>
</tbody>
</table>

Note: The first two summary statistics for the returns are annualized by multiplying by 252, respectively $\sqrt{252}$, and are reported in percentage terms.

6 Empirical Application

We turn now to the empirical study of the activity level of volatility using the estimation tools of Section 4, which we already tested on simulated data in Section 5. We use high-frequency data on the VIX index computed by the CBOE along with S&P 500 futures returns. The data set spans the period from September 22, 2004 till August 31, 2007, for a total of 942 trading days which corresponds to 48 calendar months. Within each day, we use 5-minute records of the VIX index and the S&P 500 futures contract from 9.35 till 16.00 (EST) corresponding to 78 price records per day. Table 2 shows simple summary statistics while the top two panels of Figure 4 show plots of the high-frequency series.

The empirical analysis is divided into two parts. We start with investigating the activity of market volatility and the stock market index, and we then proceed with analysis of the relationship between the discontinuities in the stock market index and market volatility. The analysis excludes the overnight period, as the estimation developed in Section 4.3 is based on high-frequency data. However, this does not affect any of our conclusions about the plausibility of various parametric models as long as we are willing to assume that the continuous-time process behind the trading part of a day and the overnight period stays the same. Note that this is different from the measurement of volatility through e.g. realized volatility where the overnight period have to be accounted for in some way if we are interested for example in forecasting future volatility.
6.1 How Active is Stock Market Volatility?

To answer this question we compute for each of the months in the sample the activity signature function. Using it we construct an activity signature plot, consisting of the 25-th, 50-th and 75-th quantiles of the activity signature functions in the sample at each value of the power $p$. We also report the activity index estimator in (4.7). The activity signature plot and the histogram of the activity index estimates are plotted on the right side of Figure 4. For comparison we did the same calculations for the S&P 500 index. The results are shown on the left side of Figure 4.

The median of the activity estimate over the sample is 2.0553 for the S&P 500 and 1.8551 for the VIX index. This suggests that the S&P 500 index needs a continuous martingale component to capture the many small moves observed in it. On the other hand, we find the activity of the VIX index, and therefore the stochastic volatility, to be slightly less than 2 but still relatively high. Thus, at least for some of the months in the sample, we can find support for pure-jump dynamics of the stochastic volatility. These observations are supported from the activity signature plots for the two indexes. The activity signature quantiles for the S&P 500 index start increasing around 2, while those for the VIX index start increasing slightly before 2.

What does this evidence imply in terms of modeling the stochastic variance? First, the empirical results suggest that both the market stochastic volatility and the market price level (i.e. the S&P 500 index) contain jumps during the sample. This can be clearly seen from the behavior of the activity quantiles for powers after 2 - they tend to increase, more prominently the ones associated with the VIX index. By contrast models with no jumps will imply activity quantiles that are flat at 2. Thus, based on our empirical findings, we can clearly reject pure-continuous models for volatility like those in Section 3.1. We also note that we can rule out as plausible volatility models jump-diffusion ones in which the jumps are relatively rare events, like the one simulated in the Monte Carlo. The data suggests that the jumps in the volatility happen far more frequently than just a handful of times in the year. Our second conclusion is that both
the stock market index and the market volatility are very “active” processes, i.e. they evolve over time with many small moves. By contrast, we noticed in Section 3.2 that by far the most popular pure-jump models for the stochastic variance, the non-Gaussian OU model and superpositions of it, are driven by jumps of finite variation. Therefore, their “activity” is bounded by that of the drift term, which is 1. Thus, our estimation results here can clearly reject such modeling of the volatility. The data suggests far more “active” movements in the volatility that a jump process of finite variation can create. Third, the activity estimates single out as good candidates for modeling the stochastic volatility the pure-jump models in the EXP-Lévy-OU class and/or for some of the months in the sample a mixture of continuous plus jump models. The nonnegativity transformation in the EXP-Lévy-OU models is important because it allows to use in the modeling very active jumps, with activity well above that of nonnegative jump processes which is 1.

To check formally our conclusions we conduct the following test which is based on the activity signature function evaluated at $p = 0.95$, i.e. our estimator of the activity $\hat{\beta}_{X,t}$. The null hypothesis of the test is that the underlying process contains continuous martingale plus possibly jumps with activity (i.e. Blumenthal-Getoor index) of at most 0.97. To conduct the test we construct a 95% critical region for $\ln(b_{X,t}(0.95))$ using its asymptotic distribution under the null (the asymptotics is fill-in, i.e. for $\Delta_n \to 0$). The latter is normal with estimated standard error of $\hat{\text{AsySE}}(\ln(b_{X,t}(0.95)))$ constructed from the high-frequency data within the month. Details on the asymptotic result and the calculation of the feasible standard error can be found in Todorov and Tauchen (2008). The critical region of the test is given by $\{\ln(b_{X,t}(0.95)) < \ln(2) - 1.6649 \times \hat{\text{AsySE}}(\ln(b_{X,t}(0.95)))\}$. The results from the testing for the S&P 500 index and the VIX index are summarized on Figure 5. We can see from the figure that our conclusion regarding the presence of continuous martingale component are confirmed by the test. There is overwhelming evidence for presence of continuous martingale in the stock market index. In only 12 out of the 48 months is the null hypothesis rejected and even for those months for which we reject the null, the values of $\ln(b_{v,t}(0.95))$ are relatively close to the critical values. By contrast, for the VIX
index we fail to reject the null in only 16 out of the 48 months in the sample. We note that the results from the test provide evidence not only against pure-continuous modeling of the volatility but also against continuous plus jump models like the jump-diffusion volatility specifications.

6.2 What is the Relationship between Volatility Jumps and Stock Market Jumps?

Having detected presence of jumps both in the S&P 500 index and the VIX index, a natural question arises about their dependence. We address this question in this section using the nonparametric tests developed in Jacod and Todorov (2008). Before presenting the tests and applying them to our data set, we briefly summarize previous findings based on parametric or semiparametric specifications. As mentioned in the introduction, the most commonly used model in finance which allows for jumps both in the price and the stochastic volatility is the double-jump model of Duffie et al. (2000). In their general specification, Duffie et al. (2000) allow for independent as well as dependent jumps in the index and its stochastic volatility. The studies that estimate double-jump models restrict them to arrive always together, see e.g. Chernov et al. (2003), Eraker et al. (2003). These papers, however find that the correlation between the jump sizes in the price and volatility is not statistically different from zero. On the other hand, using high-frequency data and in the context of a pure-jump model for the volatility, Todorov (2008) finds strong semiparametric evidence for dependent price and volatility jumps although perfect dependence is rejected.

Determining whether the jumps in the price and volatility arrive together and if so whether they are dependent is crucial from the perspective of successful risk management as well as determining the volatility and jump risk premia. Therefore, here we investigate this important question in a completely non-parametric framework using the high-frequency data on the S&P 500 and VIX indices. First, we investigate whether in our sample the jumps in the S&P 500 index and the VIX index arrived at the same time. For this, following Jacod and Todorov (2008), we use the following test statistic defined for two arbitrary processes $X$ and $Y$ observed over the
time interval \((t - 1, t)\) at frequency \(\Delta_n\)

\[
T_{cj}(t) = \frac{V_t(X, Y, 2, 2\Delta_n)}{V_t(X, Y, 2, \Delta_n)},
\]

(6.1)

where \(V_t(X, Y, r, \Delta_n)\) is the following analogue of the realized power variation in a two-dimensional context

\[
V_t(X, Y, r, \Delta_n) = \sum_{i=1}^{\lfloor 1/\Delta_n \rfloor} |X_{t-1+i\Delta_n} - X_{t-1+(i-1)\Delta_n}|^r |Y_{t-1+i\Delta_n} - Y_{t-1+(i-1)\Delta_n}|^r.
\]

(6.2)

If there is common arrival of jumps in \(X\) and \(Y\) over the interval \((t - 1, t]\), then this statistic converges to 1 (as \(\Delta_n \to 0\)), while if the jumps in the two series never arrive together the limiting value of \(T_{cj}(t)\) is “around” 2.\(^7\) For consistency with the literature on testing for jumps, we calculated \(T_{cj}\) for each day in our sample. The histogram of the estimated \(T_{cj}\) is plotted on the left panel of Figure 6. As seen from this plot, the histogram clearly has well distinguishable mode right around 1, which is very strong evidence for common arrival of jumps in the price and the stochastic volatility. We also conducted a formal test using \(T_{cj}\) and the testing procedure outlined in Jacod and Todorov (2008). For 5% significance we failed to reject the null of common arrival of jumps in 654 out of the 942 days in the sample.\(^8\)

Another useful statistic that allows us to analyze cojumping in market volatility and market price level is the “realized” correlation between the squared jumps in those two series. For two arbitrary processes \(X\) and \(Y\) observed over the time interval \((t - 1, t)\) at frequency \(\Delta_n\), the realized correlation is defined as

\[
R_{cj}(t) = \frac{V_t(X, Y, 2, \Delta_n)}{\sqrt{V_t(X, 4, \Delta_n)V_t(Y, 4, \Delta_n)}}.
\]

(6.3)

A value of zero of this statistic means disjoint arrival of jumps, while value close to 1 is evidence for a perfect dependence between the jumps in the two series over the given interval of time.

\(^7\)It is exactly 2 if both series do not have jumps over \((t-1,t]\).

\(^8\)The critical region for the test of common jumps in Jacod and Todorov (2008) is developed under the assumption that both components contain continuous component. While this is the case for the price level, in the previous section we found evidence that such an assumption is questionable for the VIX index. Therefore, to check the robustness of the results we added to the VIX index series a simulated Brownian motion with standard deviation of its increments being 10% of the estimated standard deviation of the 5-minute increments of the VIX index (similar approach was taken in Cont and Mancini (2007) for testing the null of pure-jump models). The value of the statistics \(T_{cj}\) remained virtually unchanged and the reported number of days with evidence of cojumps remained the same.
The histogram of the realized correlation between the jumps in the S&P 500 index and the VIX index is plotted on the right panel of Figure 6. As seen from the histogram, there is not only overwhelming evidence for common arrival of jumps, but also for a strong dependence between the realized jumps in the two series. This suggests that the jumps in volatility and market level should be modeled jointly. This result casts also doubt on the plausibility of empirical findings, based on jump-diffusion models, for statistically insignificant dependence between the jump size of volatility and price jumps.

Given the strong dependence between price and volatility jumps, we next explore whether the common jumps in the two series happen in the same direction. We do this by splitting $V_t(X,Y,r,\Delta_n)$ into cojump variation due to jumps in the same direction and one due to jumps in the opposite direction:

$$V_t^+(X,Y,2,\Delta_n) = \sum_{i=1}^{[1/\Delta_n]} \left( |X_{t-1+i\Delta_n} - X_{t-1+(i-1)\Delta_n}|^2 |Y_{t-1+i\Delta_n} - Y_{t-1+(i-1)\Delta_n}|^2 \right) \cdot \mathbf{1}_{\{X_{t-1+i\Delta_n} - X_{t-1+(i-1)\Delta_n}(X_{t-1+i\Delta_n} - X_{t-1+(i-1)\Delta_n}) > 0\}}.$$  

$$V_t^-(X,Y,2,\Delta_n) = \sum_{i=1}^{[1/\Delta_n]} \left( |X_{t-1+i\Delta_n} - X_{t-1+(i-1)\Delta_n}|^2 |Y_{t-1+i\Delta_n} - Y_{t-1+(i-1)\Delta_n}|^2 \right) \cdot \mathbf{1}_{\{X_{t-1+i\Delta_n} - X_{t-1+(i-1)\Delta_n}(X_{t-1+i\Delta_n} - X_{t-1+(i-1)\Delta_n}) < 0\}}.$$  

The median value of the ratio $\frac{V_t^-(X,Y,2\Delta_n)}{V_t^+(X,Y,2\Delta_n)}$ in our sample is 0.9951. Thus, almost all of the common jump variation in price and volatility is due to jumps in opposite directions. This is consistent with models generating dynamic leverage effect through jumps, e.g. Barndorff-Nielsen and Shephard (2001) and Todorov and Tauchen (2006), in which a negative price jump leads to an increase in the future volatility.

### 7 Concluding Remarks

This paper conducts a non-parametric analysis of the market volatility dynamics using high-frequency data on the VIX index and the S&P 500 index. Our results imply that a plausible
model for stochastic volatility is a model of pure-jump type whose driving jumps come from a very active Lévy process. Also, the volatility jumps and market price jumps occur in most cases at the same time and exhibit high negative dependence.

The findings lead to several economically important conclusions. First, on an individual investor level, the pure-jump dynamics of stochastic volatility implies that hedging is quite complicated. This is in sharp contrast with diffusive volatility dynamics in which a derivative instrument sensitive to the volatility suffices, see e.g. Liu and Pan (2003). The very active pure-jump nature of volatility we find here means that the volatility risk cannot be spanned with a handful of derivatives instruments. Also, the finding of strong dependence between the price and volatility jumps additionally complicates hedging. If volatility and price jumps were independent, then the investor could use deep-out-of-the-money put options to hedge against the price jump risk and at-the-money options to hedge the volatility risk. Our findings suggest that volatility and jump risks share common origins and therefore such separate hedging cannot be expected to work well. Furthermore the two jump risks cannot be spanned with commonly traded derivative instruments, including variance swaps.

Second, on a macro level our findings have important implications for the risk premia associated with price jumps and volatility risk. Typically these risk premia are modeled separately, e.g. price jump risk is modeled as a compensation for jump size risk only which is independent from the stochastic volatility. However, our results suggest that (negative) jumps on the market are associated with increase in the stochastic variance $\sigma_t^2$ and therefore at least part of the volatility risk either coincides or is highly correlated with the price jump risk. Thus, volatility and price jump risk premia share compensations for similar risks, and therefore should be modeled jointly.

8 Proof of Theorem 1

First, using e.g. Theorem V.32 in Protter (2004), we have that the vector $f_t$ is a strong Markov process. Therefore, the probability of $f_s$ under $Q$ conditioned on the filtration $\mathcal{F}_t$ for $s > t$ is
a function only of $f_t$. Also, using the differentiability assumption on the functions $g_j^{(i)}(\cdot)$, we have that for $s \in [t, t + N]$, $f_s$ conditional on $\mathcal{F}_t$ is a random function of $f_t$ which by Theorem V.40 in Protter (2004) is continuously differentiable. Therefore, $\mathbb{E}^\mathbb{Q}(\sigma_s^2 | \mathcal{F}_t)$ is a continuously differentiable function of $f_t$ for $s \geq t$ and from here we also have the continuous differentiability of $\mathbb{E}^\mathbb{Q}([S, S]_{t+N}^c - [S, S]_{t}^c | \mathcal{F}_t)$ in $f_t$.

For the discontinuous part of the quadratic variation, using the definition of a jump compensator (see Jacod and Shiryaev (2003), Theorem II.1.8), we have that

$$\mathbb{E}^\mathbb{Q}([S, S]_{t+N}^d - [S, S]_{t}^d | \mathcal{F}_t) = \int_\mathbb{R} x^2 \eta(dx) \mathbb{E}^\mathbb{Q} \left( \int_t^{t+N} G^{(d)}(f_s) \bigg| \mathcal{F}_t \right),$$

and from here repeating the analysis for the continuous quadratic variation above, we have the continuous differentiability of $\mathbb{E}^\mathbb{Q}([S, S]_{t+N}^d - [S, S]_{t}^d | \mathcal{F}_t)$ in $f_t$ as well. Hence $\nu_t$ is continuously differentiable in $f_t$.

Given the continuous differentiability of $\nu_t$ (and the non-vanishing first derivatives of $F(\cdot)$) for an arbitrary $\omega$ in the probability space we have

$$k(\omega)V_t(\sigma^2, r, \Delta_n) \leq V_t(\nu, r, \Delta_n) \leq K(\omega)V_t(\sigma^2, r, \Delta_n), \quad t > 1, \quad r > 0,$$

where $0 < k(\omega) \leq K(\omega)$, where we made use of the fact that the first derivatives of $G^{(c)}(\cdot)$, $G^{(d)}(\cdot)$ and $F(\cdot)$ are continuous functions of càdlàg processes and hence are locally bounded. From here, using the definition (4.1), we have the result in (4.4).
Figure 1: Monte Carlo Results for Affine Diffusion and Jump-Diffusion Models. The left-side panels pertain to the continuous diffusion (3.1); the right-side panels pertain to the jump-diffusion (3.15).
Figure 2: Monte Carlo Results for the long memory model EXP-OU-FI model (3.6).
Figure 3: Monte Carlo Results for pure-jump models. The left-side panels pertain to non-Gaussian OU model (3.11); the right-side panels pertain to Exponential Lévy-OU model (3.12).
Figure 4: Estimation results for “activity” of S&P 500 index and VIX index.
Figure 5: Testing the null for presence of continuous component. Monthly log \( b_{X,t}(0.95) \) are marked with * and the 95% rejection region of the test is the area below the lines on the plots.

Figure 6: Testing for common arrival of jumps. The left panel plots the histogram of the non-standardized test for common arrival of jumps in S&P 500 index and VIX index: value of 1 is evidence for common jumps, while value “around” 2 is evidence for disjoint or no jumps in both series. The right panel plots the histogram of the daily “realized” correlation between price and volatility jumps.
References


