The Econometrics of Financial Market Volatility: Past Developments and New Directions

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Economics 201 FS
January 2009

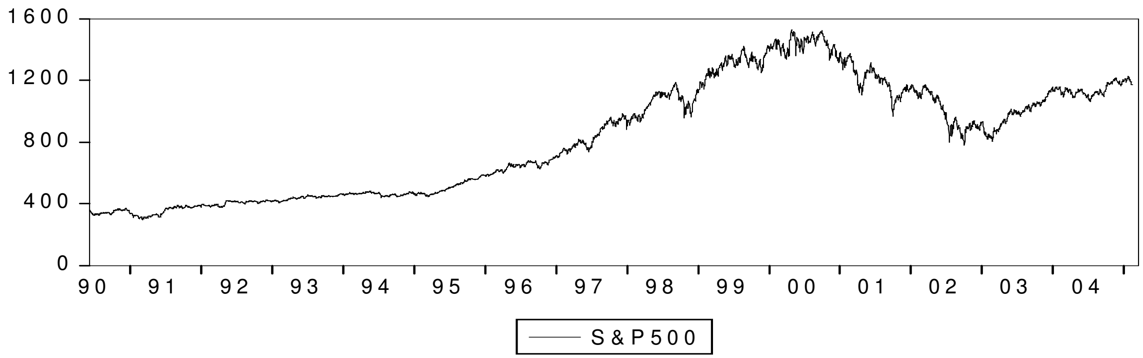
Brief Historical Perspective

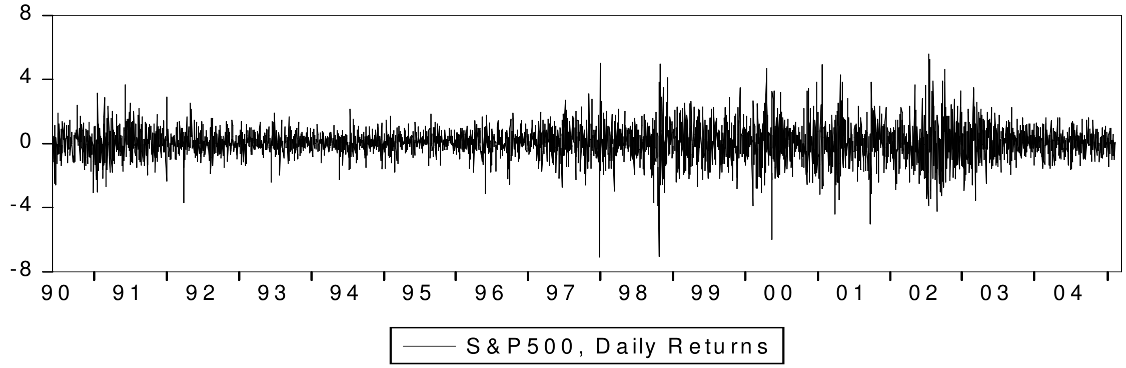
- Volatility Clustering and Why We Care
- ARCH, GARCH, and Stochastic Volatility Models
- Realized Volatility (RV)

Current Themes

- RV and Market Microstructure “Noise”
- RV, Jumps, and Distributional Implications
- Multivariate RV Measures
- RV, ImpV, and Volatility Risk Premia

Volatility Clustering



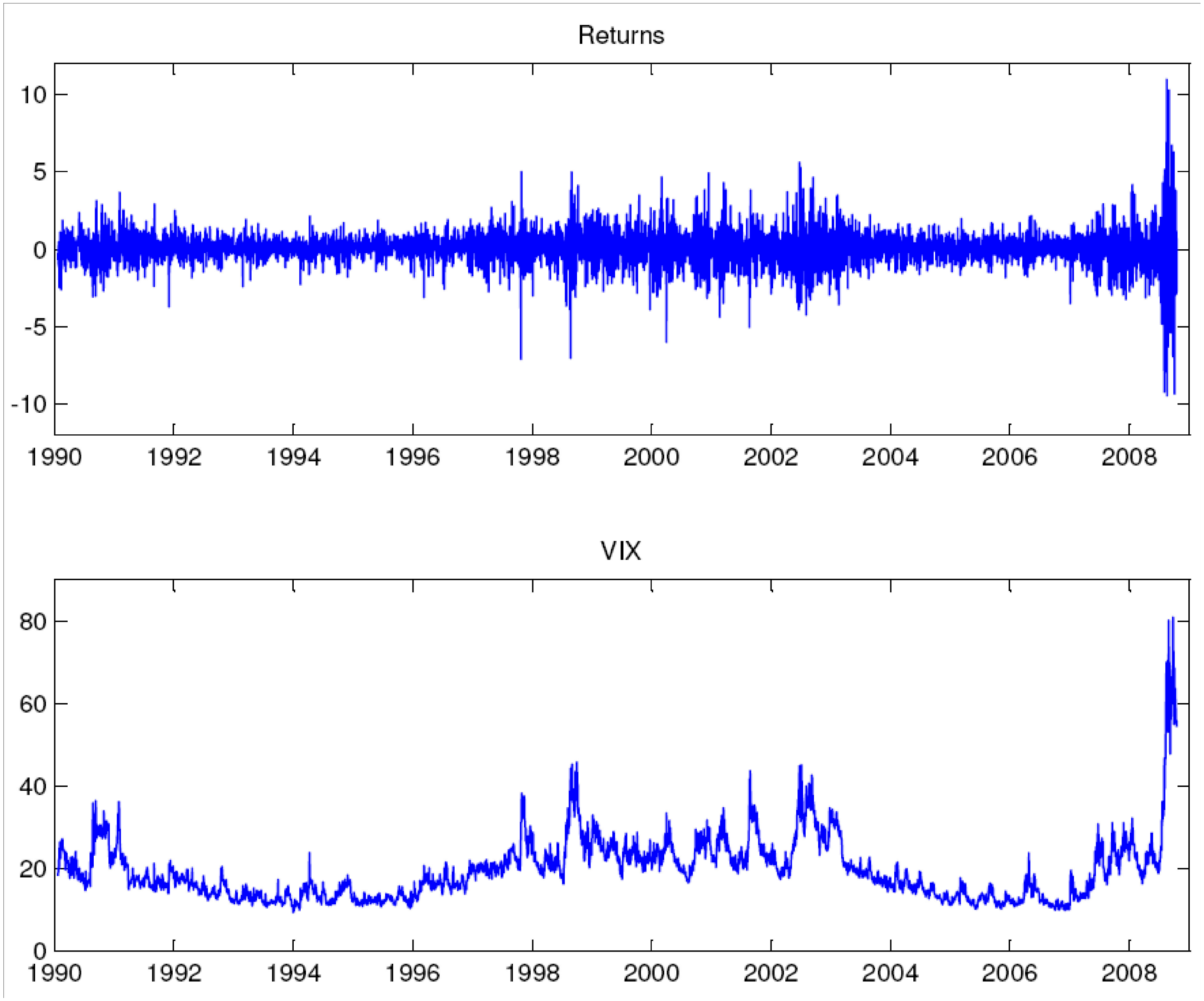


- Volatility clusters in time

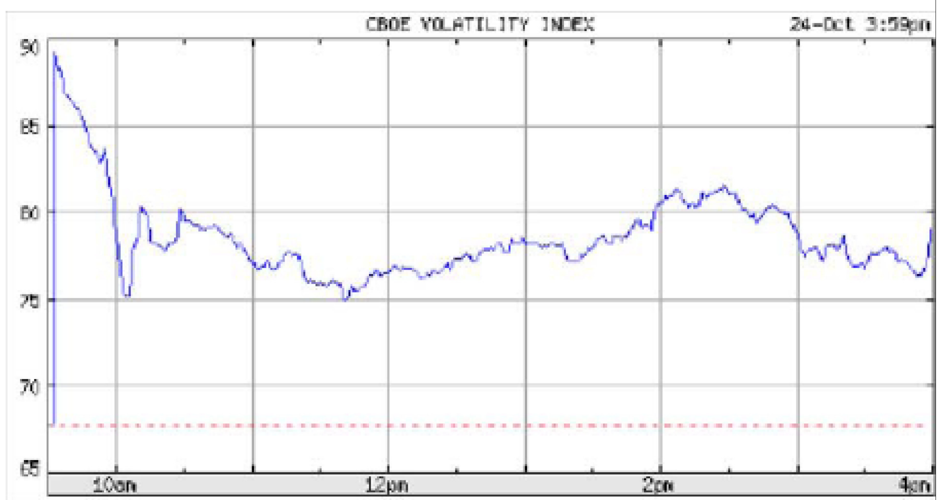
  "... *large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes ...* "

  Mandelbrot (1963, *J. Business*)

S&P500 Daily Returns and Option Implied Volatilities, 1/2/1990 - 12/12/2008

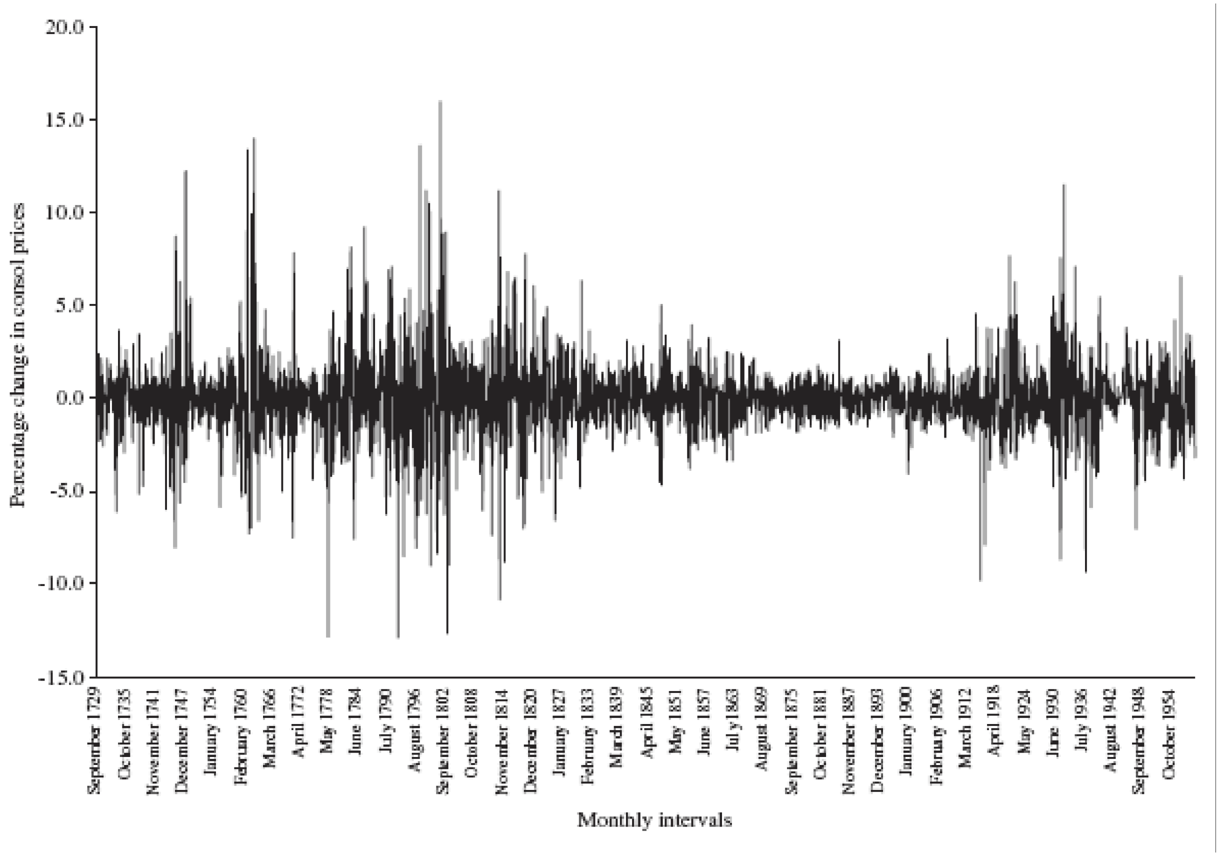


VIX, 10/20/2008 - 10/24/2008



– Intraday high of 89.53

U.K. consol prices, 1729-1957



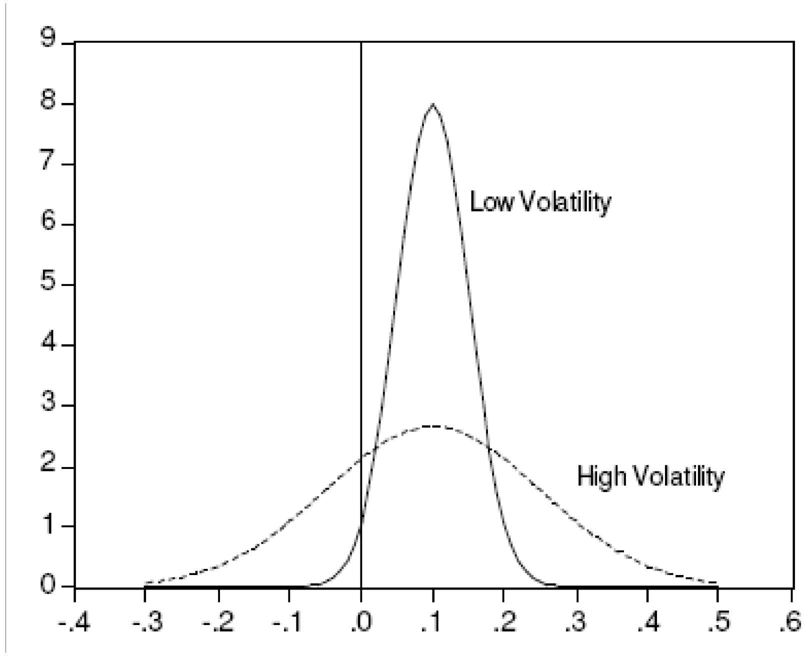
Brown, Burdekin and Weidenmeir (2006, *Journal of Financial Economics*)

- Why should we care about volatility clustering?

Why Do We Care?

- Volatility/risk is central to financial economics
  - Sign forecasting and market timing
  - Credit and default risk
  - Risk measurement and management
  - CAPM and other asset pricing models
  - Portfolio allocation and diversification
  - Option pricing and volatility trading
  - . . .

- Sign Forecasting and Market Timing



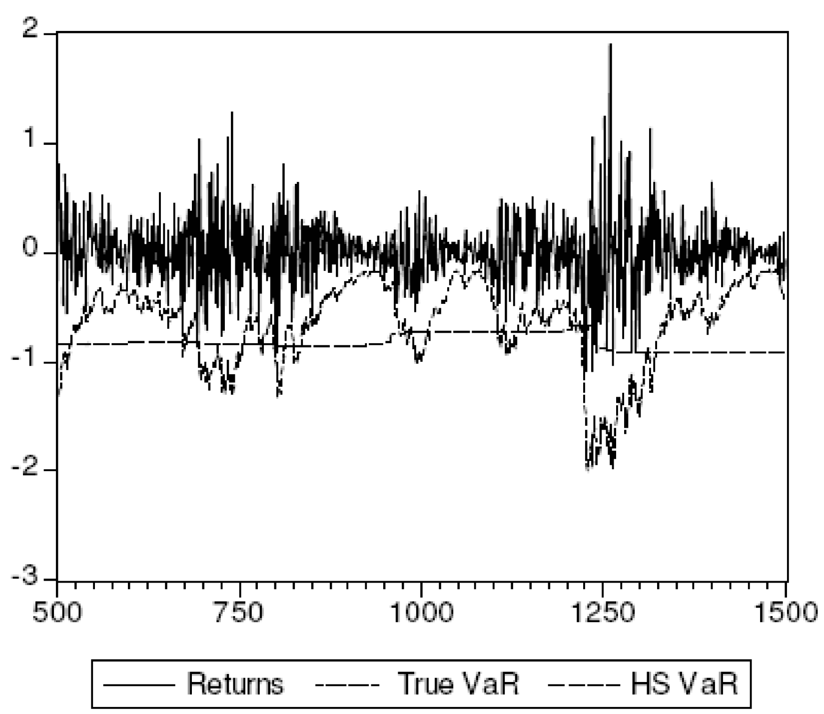
Andersen, Bollerslev, Christoffersen and Diebold (2006, *Handbook of Economic Forecasting*)

  - Same mean but $P(r_t) < 0$ differs
  - Bankruptcy (assets < liabilities)
  - Credit risk

- Risk Measurement and Management

  – Value-at-Risk (VaR)
    – Specific quantile in loss/return distribution
    – Basel Accord I and II

1% One-Day VaR's



Andersen, Bollerslev, Christoffersen and Diebold (2006, *Handbook of Economic Forecasting*)

  – Standard historical VaR correct on average, but ...

- Asset Pricing
  - CAPM

$$E(r_{i,t}) = r_f + \beta_i \lambda$$

$$\beta_i = \frac{Cov(r_{i,t}, r_{m,t})}{Var(r_{m,t})}$$ – Standard

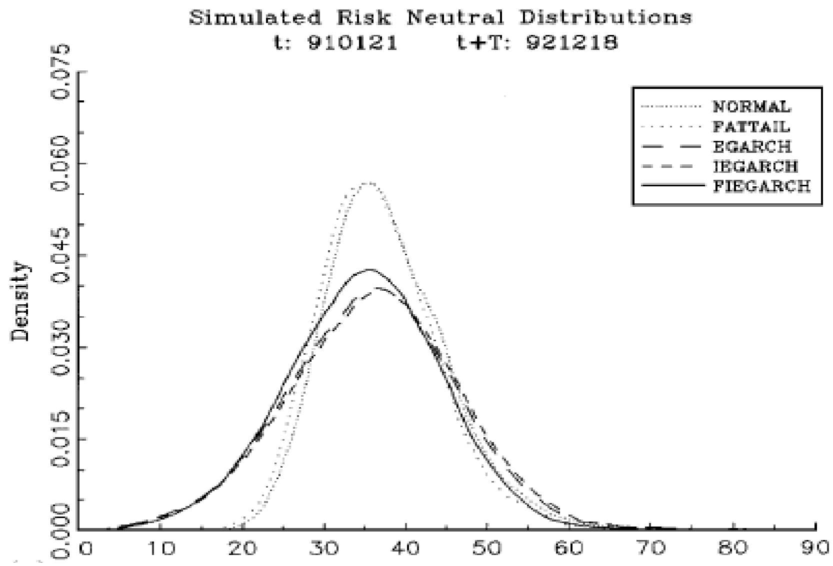
$$\beta_{i,t} = \frac{Cov_{t-1}(r_{i,t}, r_{m,t})}{Var_{t-1}(r_{m,t})}$$ – Conditional

  - Consumption-based CAPM
  - Multi-factor (APT) models

- Option Pricing and Volatility Trading

$$Call_t = e^{-r_f T} E_t^* (\max\{P_{t+T} - K, 0\})$$

$$Put_t = e^{-r_f T} E_t^* (\max\{K - P_{t+T}, 0\})$$



Bollerslev and Mikkelsen (1999, *Journal of Econometrics*)

– Black-Scholes assumes σ constant ...

- Volatility clusters in time

- Volatility is central to financial economics

- But, how to predict volatility?
  - Need a statistical/econometric model
  - ARCH/GARCH and stochastic volatility models

ARCH/GARCH Models

- ARCH (AutoRegressive Conditional Heteroskedasticity)

Engle (1982, *Econometrica*)

  - Discrete-time model
  - Explicitly parameterizes the time *t* conditional variance as a function of time *t-1* information

$$Var(r_t \mid \Omega_{t-1}) = \sigma_t^2$$

  - But how?

- ARCH(q) model

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \ldots + \alpha_q \varepsilon_{t-q}^2$$

  - Like a rolling sample variance
  - In practice, large *q* and (too) many alpha's

- Generalized ARCH, or GARCH model

Bollerslev (1986, *J. of Econometrics*)

  - Like going from AR to ARMA
  - Simple GARCH(1,1) model often works well

$$\sigma_t^2 = \omega + \alpha\, \varepsilon_{t-1}^2 + \beta\, \sigma_{t-1}^2$$

- Numerous alternative GARCH type parameterizations and refinements ...

- An incomplete list of GARCH acronyms

Bollerslev (2008, *Engle Festschrift*)

ARCH	Engle (1982)
GARCH	Bollerslev (1986)
IGARCH	Bollerslev and Engle (1986)
Log-GARCH	Geweke (1986), Milhøj (1987), Pantula (1986)
TS-GARCH	Taylor (1986), Schwert (1989)
GARCH-t	Bollerslev (1987)
ARCH-M	Engle, Lilien and Robins (1987)
MGARCH	Bollerslev, Engle and Wooldridge (1998)
CCC GARCH	Bollerslev (1990)
AGARCH	Engle (1990)
CGARCH	Engle and Lee (1990)
EGARCH	Nelson (1991)
SPARCH	Engle and Gonzalez-Rivera (1991)
LARCH	Robinson (1991)
AARCH	Bera, Higgins and Lee (1992)
NGARCH	Higgins and Bera (1992)
QARCH	Sentana (1992)
STARCH	Harvey, Ruiz and Sentana (1992)
TGARCH	Zakoian (1994)
GJR-GARCH	Glosten, Jagannathan and Runkle (1993)
QTARCH	Gourieroux and Monfort (1992)
Weak GARCH	Drost and Nijman (1993)
VGARCH	Engle and Lee (1993)
APARCH	Ding, Granger and Engle (1993)
SWARCH	Hamilton and Susmel (1994)
GQARCH	Sentana (1995)
SGARCH	Liu and Brorsen (1995)
PGARCH	Bollerslev and Ghysels (1996)
HGARCH	Hentschel (1995)
FIGARCH	Baillie, Bollerslev and Mikkelsen (1996)
FIEGARCH	Bollerslev and Mikkelsen (1996)
ATGARCH	Crouchy and Rockinger (1997)
Aug-GARCH	Duan (1997)
STGARCH	Gonzalez-Rivera (1998)
OGARCH	Alexander (2001)
DCC GARCH	Engle (2002)
Flex-GARCH	Ledoit, Santa-Clara and Wolf (2003)
HYGARCH	Davidson (2004)
COGARCH	Klüppelberg, Lindner and Maller (2004)
. . . .	

Stochastic Volatility (SV) Models

- GARCH is a discrete-time model
  - Largely empirically motivated and ad hoc

- Finance theory, and derivatives pricing in particular, often cast in continuous-time
  - Continuous-time random walk

$$dp(t) = \mu \, dt + \sigma \, dW(t)$$

  - Black-Scholes and many other pricing formula
  - But, of course, σ is not constant

- Time-varying diffusive volatility

$$dp(t) = \mu \, dt + \sigma(t) \, dW(t)$$

- Time-varying diffusive volatility

$$dp(t) = \mu\, dt + \sigma(t)\, dW(t)$$

- First generation models
  - Cox-Ingersoll-Ross (CIR) model: $\sigma(t) = \eta\, p(t)^{1/2}$
  - Constant Elasticity of Variance (CEV): $\sigma(t) = \eta\, p(t)^{\gamma}$
  - Soundly rejected empirically

- Allow σ(t) to follow a separate stochastic process
  - But how?

- GARCH diffusion

Nelson (1990, *JoE*)

$$d\sigma^2(t) = (\alpha - \beta\sigma^2(t))\,dt + \eta\,\sigma^2(t)\,dB(t)$$

- Heston model

Heston (1993, *RFS*)

$$d\sigma^2(t) = (\alpha - \beta\sigma^2(t))\,dt + \eta\,\sigma(t)\,dB(t)$$

- Log-volatility model

$$d\log\sigma^2(t) = (\alpha - \beta\log\sigma^2(t))\,dt + \eta\,dB(t)$$

- Multi-factor models

$$\sigma^2(t) = \sigma_1^2(t) + \sigma_2^2(t)$$

- Estimation and inference for SV models generally much harder than for ARCH/GARCH models
  - $\sigma^2(t)$ is latent
  - $f(r_{t+1} \mid \Omega_t)$ not available in closed form

Realized Volatility

- Continuous-time stochastic volatility model

$$dp(t) = \mu \, dt + \sigma(t) \, dW(t)$$

  – What is the actual/true volatility from t to $t+1$?

- Integrated volatility/variation

$$IV_{t+1} \equiv \int_{t}^{t+1} \sigma^2(\tau) \, d\tau$$

  – Variance of $r_{t+1} = p(t+1) - p(t)$ given $\{ \sigma(\tau) \, , t \leq \tau \leq t+1 \}$

  – Option pricing

Hull and White (1987, *JF*)

  – But, how do you measure the integrated volatility in practice?

- Realized volatility from high(er)-frequency data

$$RV_{t+1}(\Delta) \equiv \sum_{j=1}^{1/\Delta} (p(t+j\Delta) - p(t+(j-1)\Delta))^2 \equiv \sum_{j=1}^{1/\Delta} r_{t+j\Delta,\Delta}^2$$

  – Earlier precedent in finance

French, Schwert and Stanbaugh (1987, *JFE*)
Hsieh (1998, *JF*), Schwert (1989, *JF*; 1990, *RFS*)

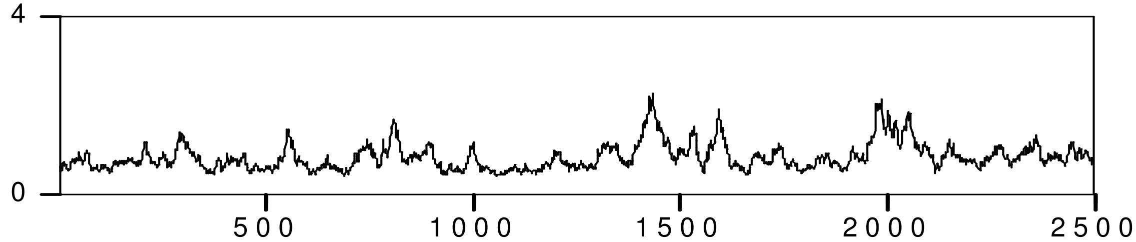
- Theory of quadratic variation

$$\lim_{\Delta\to 0} RV_{t+1}(\Delta) \to IV_{t+1}$$

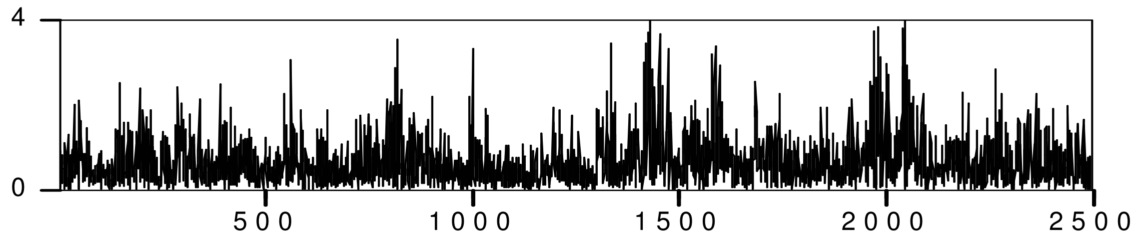
  – High-frequency data crucial

Andersen and Bollerslev (1998, *IER*)
Andersen, Bollerslev, Diebold and Labys (2011, *JASA*)
Barndorff-Nielsen and Shephard (2001, *JRSS*)

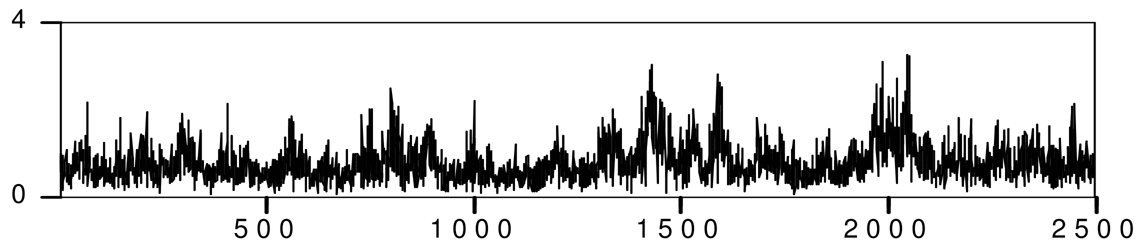
IV_t



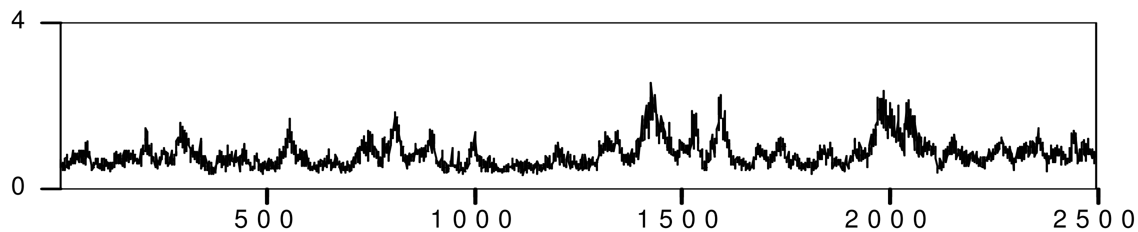
$RV_t(1)$



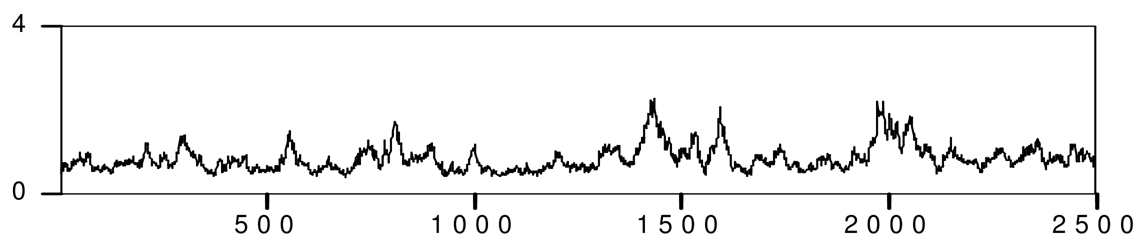
$RV_t(1/3)$



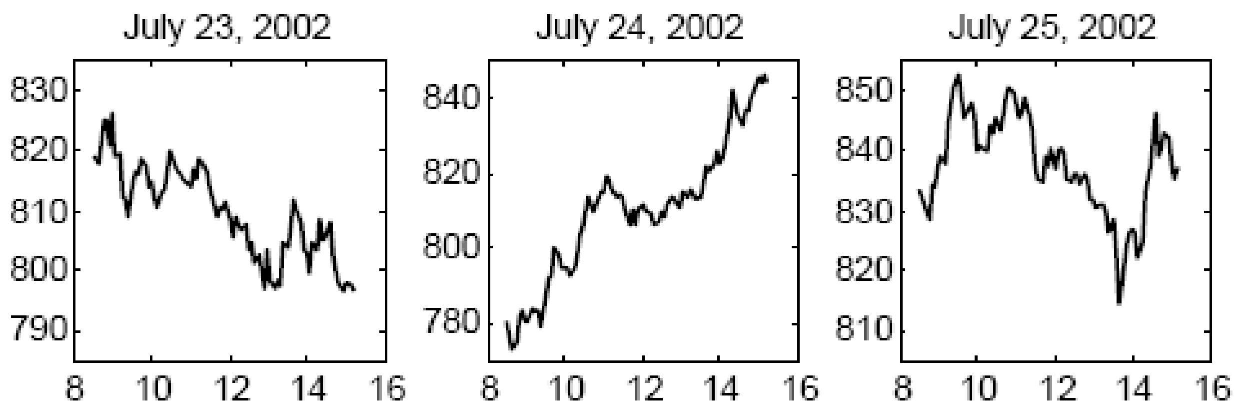
$RV_t(1/24)$



$RV_t(1/288)$



- S&P500 5-minute futures prices:



r_t = -2.75%	r_t = 7.88%	r_t = 0.40%
$RV_t(1)$ = 43.5%	$RV_t(1)$ = 124.6%	$RV_t(1)$ = 6.3%
$RV_t(1/81)$ = 53.3%	$RV_t(1/81)$ = 59.4%	$RV_t(1/81)$ = 55.4%

- Nice theory, but does it work in practice?
  - Yes!

- By measuring the variation over non-trivial daily (and longer) time-intervals the realized volatility avoids
  - Intraday volatility patterns, or circadian rhythms

    Andersen and Bollerslev (1997, *JoEF*)
    Andersen and Bollerslev (1998, *JF*)

  - Double asymptotic usually required for consistently estimating, or filtering $\sigma(t)$

    Merton (1980, *JFE*)
    Nelson (1992, *JoE*), Nelson and Foster (1994, *Ect.*)

- The realized volatility is completely model-free

Modeling and Forecasting Realized Volatility

- The theory of realized volatility (in theory) permits a direct look at the empirical distribution of (the true ex-post) volatility

- Remarkable similarities across different assets/asset classes

Andersen, Bollerslev, Diebold and Labys (2001, *JASA*)
Andersen, Bollerslev, Diebold and Ebens (2001, *J.Fin.Eco.*)

  - The unconditional distribution of realized volatility is approximately log-normal
  - The conditional dependencies in realized volatility are approximately long-memory
  - Standardized returns, $r_t / RV_t(\Delta)^{1/2}$, are approximately normal

- Reduced form time series models for $RV_{t+1}(\Delta)$ may be used in modeling and forecasting volatility

Andersen, Bollerslev, Diebold and Labys (2003, *Econometrica*)
Andersen, Bollerslev and Diebold (2008, *Handbook Ch.*)
Andersen, Bollerslev and Meddahi (2004, *IER*)

  - Avoids the problems associated with estimation and inference in GARCH and latent stochastic volatility models

  - Avoids having to directly model intraday returns, while incorporating (most of) the relevant longer-run information

  - But does it work in practice?

    - Yes!

– Reduced form AR-RV model

Andersen, Bollerslev, Diebold and Labys (2003, *Econometrica*)

$$A(L)\,(log\,(RV_t(\Delta))\,-\,\mu)\;=\;\varepsilon_t$$

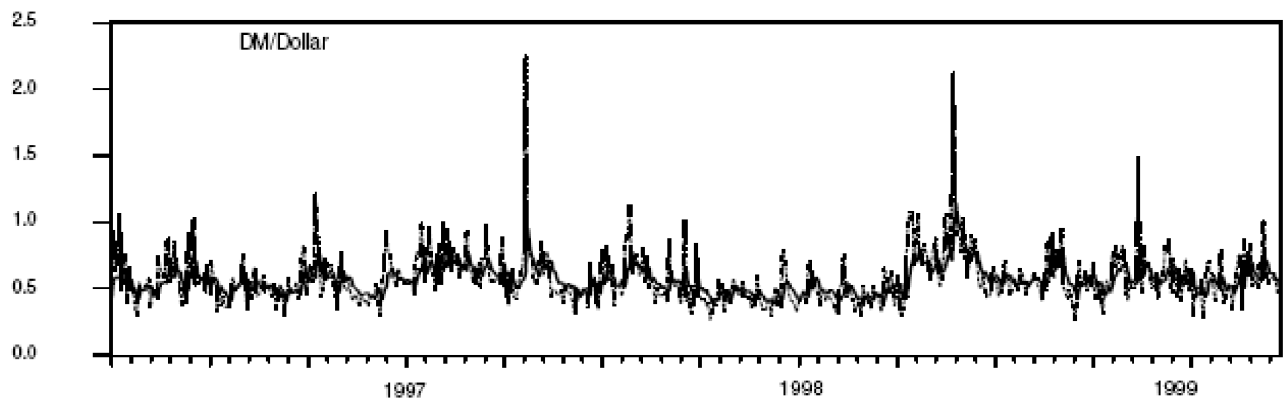
– Mincer-Zarnowitz style ex-post forecast evaluation regression

$$RV_{t+1}(\Delta)\;=\;b_0\;+\;b_1 AR{-}RV_{t+1|t}(\Delta)\;+\;b_2 Other_{t+1|t}\;+\;\xi_{t+1}$$

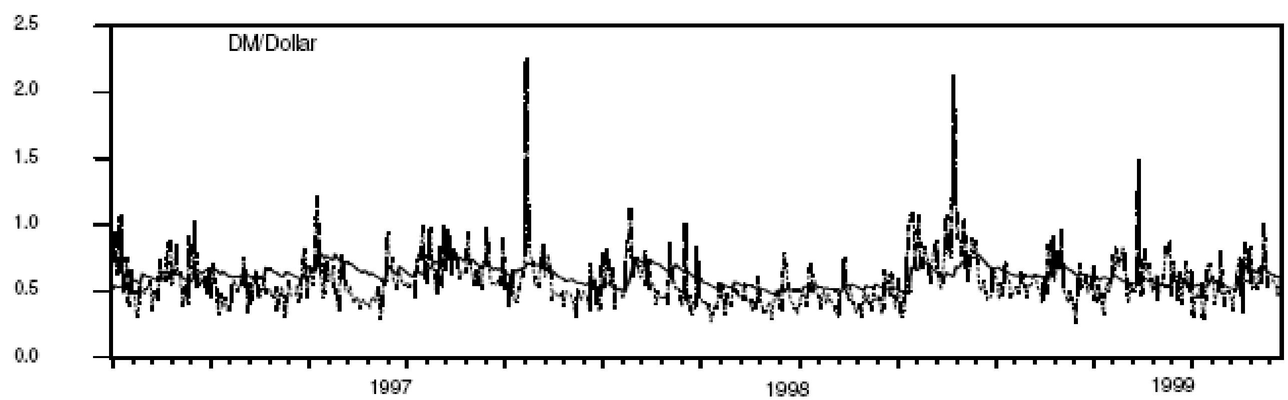
– Out-of-sample, DM/$, one-day ahead

	b_0	b_1	b_2	R^2
AR-RV	0.02 (.05)	0.99 (.09)	-	0.25
AR-ABS	0.44 (.03)	-	0.45 (.09)	0.03
RiskMetrics	0.22 (.04)	-	0.62 (.08)	0.10
GARCH	0.05 (.06)	-	0.85 (.11)	0.10
HF-FIEGARCH	-0.07 (.06)	-	1.01 (.10)	0.26
AR-RV+AR-ABS	0.04 (.05)	1.02 (.11)	-0.11 (.10)	0.25
AR-RV+RiskMetrics	0.02 (.05)	0.98 (.13)	0.01 (.11)	0.25
AR-RV+GARCH	0.02 (.06)	0.98 (.13)	0.02 (.16)	0.25
AR-RV+HF-FIEGARCH	-0.07 (.06)	0.40 (.19)	0.66 (.20)	0.27

– RV-AR based forecasts



– GARCH based forecasts



Andersen, Bollerslev, Diebold and Labys (2003, *Econometrica*)

- Predictive/conditional return distribution

$$f(r_{t+1} \mid \Omega_t) = f(r_{t+1} \mid RV_{t+1}(\Delta), \Omega_t) \cdot f(RV_{t+1}(\Delta) \mid \Omega_t)$$

– The standardized returns, $r_{t+1} / RV_{t+1}(\Delta)^{1/2}$, are approximately normal

– The dynamic dependencies in $RV_{t+1}(\Delta)$ are well described by simple reduced form AR-RV models

– One-day-ahead predictive distributions (VaR)

TABLE 4
One-Day Ahead Density Forecasts From Long-Memory Lognormal-Normal Mixture Model

Quantile:	*1%*	*5%*	*10%*	*90%*	*95%*	*99%*
In-Sample						
DM/$	0.016	0.059	0.105	0.895	0.943	0.987
¥/$	0.016	0.061	0.103	0.901	0.951	0.990
Portfolio	0.010	0.052	0.091	0.912	0.958	0.990
Out-of-Sample						
DM/$	0.005	0.045	0.092	0.893	0.941	0.990
¥/$	0.019	0.055	0.099	0.884	0.956	0.993
Portfolio	0.010	0.042	0.079	0.909	0.968	1.000

Andersen, Bollerslev, Diebold and Labys (2003, *Econometrica*)

Current Themes and New Directions

- Market Microstructure “Noise”
- Jumps and Distributional Implications
- Multivariate Measures
- Option Implied Volatilities and Variance Risk Premia

Market Microstructure "Noise"

- Fundamental (latent) logarithmic price

$$dp^*(t) \;=\; \mu(t)\, dt \;+\; \sigma(t)\, dW(t)$$

- Observed logarithmic price

$$p(t) \;=\; p^*(t) \;+\; u(t)$$

– $u(t)$ market microstructure "noise"

$$r_{t,\Delta} \;\equiv\; p^*(t) + u(t) - p^*(t-\Delta) - u(t-\Delta) \;\equiv\; r^*_{t,\Delta} + e_{t,\Delta}$$

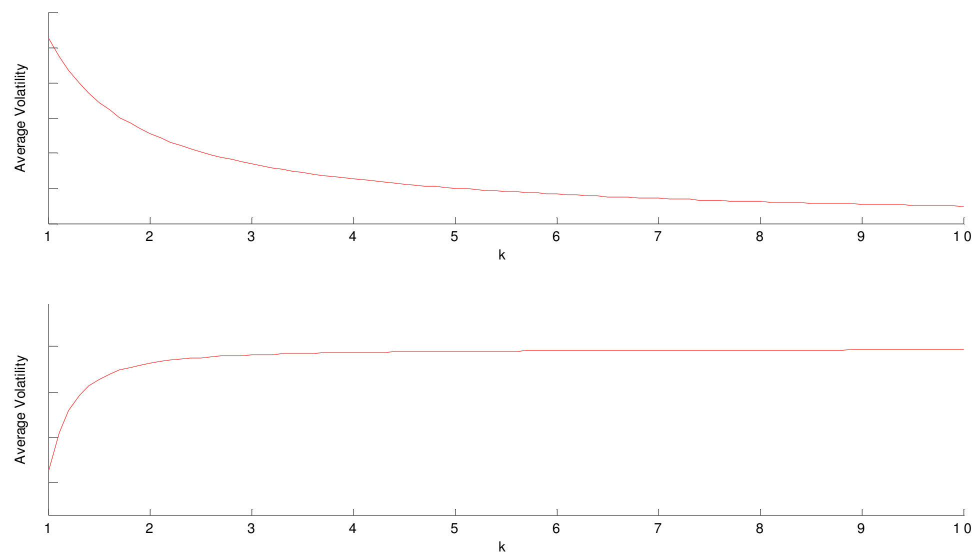
– $lim_{\Delta\to 0} E[(r^*_{t,\Delta})^2] = 0$, but $lim_{\Delta\to 0} E(e^2_{t,\Delta}) > 0$

– Noise term dominates for $\Delta \to 0$

– $RV_t(\Delta)$ is inconsistent for IV^*_t as $\Delta \to 0$

- Choose Δ "large enough" so that $E[r_{t,\Delta}^2] \approx E[(r_{t,\Delta}^*)^2]$
  - Sample mean of $RV_t(\Delta)$, $t=1, 2, ..., T$, as a function of Δ

Representative Volatility Signature Plots



Andersen, Bollerslev, Diebold and Labys (1999, *Risk*)

  - Shape depends on properties of $u(t)$ noise process
  - For $u(t)$ i.i.d. $RV_t(\Delta) \rightarrow \infty$ for $\Delta \rightarrow 0$

- "Optimal" choice of Δ

Aït-Sahalia, Mykland and Zhang (2005, *Re. Fin. Stud.*)
Bandi and Russell (2008, *ReStud.*)

  - $\min_{\Delta} MSE(RV_t(\Delta))$ conditional on $\{\sigma(s), t-1 < s \leq t\}$

$$h_t^* \approx (IQ_t/(4V_u^2))^{-1/3} \qquad IQ_t \equiv \int_{t-1}^{t} \sigma^4(s)\,ds$$

  - Unconditionally

Bandi and Russell (2006, *J.Fin.Econ.*)

$$h_1 = (E[IQ_t]/(4V_u^2))^{-1/3}$$

  - $\min_{\Delta} Var(RV_t(\Delta))$

Andersen, Bollerslev and Meddahi (2007, *wp*)

$$h_2 = (E[IQ_t]/(2V_u^2 K_u))^{-1/2}$$

  - Estimation of V_u and K_u

Bandi and Russell (2008, *ReStud.*)
Oomen (2005, *J. Fin Ect.*)
Hansen and Lunde (2006, *JBES*)

- Serial correlation adjustments

Andersen, Bollerslev, Diebold and Ebens (2001, *JFE*)
Zhou (1996, *JBES*)
Zumbach, Corsi and Trapletti (2002, *wp*)

$$RV_{t+1}^{Zhou}(\Delta) \equiv \sum_{j=1}^{1/\Delta} r_{t+j\cdot\Delta,\Delta}^2 + \sum_{j=2}^{1/\Delta} r_{t+j\cdot\Delta,\Delta} r_{t+(j-1)\cdot\Delta,\Delta} + \sum_{j=1}^{1/\Delta - 1} r_{t+j\cdot\Delta,\Delta} r_{t+(j+1)\cdot\Delta,\Delta}$$

  - For $u(t)$ *i.i.d.* $e_{t,\Delta}$ (and $r_{t,\Delta}$) is MA(1)
  - Unbiased, but inconsistent

- Kernel based methods

Barndorff-Nielsen, Hansen, Lunde and Shephard (2008, *Ecta.*)
Hansen and Lunde (2006, *JBES*)

$$RV_{t+1}^{Kernel}(\Delta) \equiv \gamma_0 + \sum_{h=1}^{H} k\left(\frac{h-1}{H}\right)[\gamma_h + \gamma_{-h}]$$

$$\gamma_h \equiv \sum_{j=1}^{1/\Delta} r_{t+j\cdot\Delta,\Delta} r_{t+(j-h)\cdot\Delta,\Delta}$$

  - Related to long-run variance and HAC type estimators

- Averaging, bias-correction, and sub-sampling

Zhang, AïtSahalia and Mykland (2005, *JASA*)

  - Accuracy of $RV_t(\Delta_{sparse})$ for $\Delta_{\text{sparse}} > 0$ may be improved by averaging over different sub-grids
  - For *u(t)* *i.i.d.* the bias in $RV_t(\Delta)$ grows like $2\Delta^{-1}V_u$ for $\Delta \to 0$
  - Combine realized volatility estimators based on all high-frequency data and sparsely sampled data to knock out the bias
  - Similar idea to Jackknife type estimators
  - Two-scale estimator consistent for IV_t^* as $\Delta \to 0$
  - Multi-scale estimators

Zhang (2006, *Bernoulli*)

- Pre-Averaging

Podolskij and Vetter (2007, *wp*)
Jacod, Li, Mykland, Podolskij and Vetter (2008, *wp*)

- "Business-time" sampling

Large (2006, *wp*)
Oomen (2006, *JBES*)

- Range-based estimators

Christensen and Podolskij (2007, *J.Ect.*)
Christensen, Podolskij and Vetter (2008, *Fin.Stoc.*)
Dobrev (2007, *wp*)
Martens and van Dijk (2007, *J. Ect.*)

- Duration-based estimation

Andersen, Dobrev and Schaumburg (2008, *wp*)

- Markov Chain-based estimation

Hansen and Horel (2008, *wp*)

- Market microstructure theory

Diebold and Strasser (2008, *wp*)
Engle and Sun (2007, *wp*)
Robert and Rosenbaum (2008, *wp*)

- Multivariate measures

  – Non-synchronous trading/quoting and the Epps-effect

  Epps (1979, *JASA*)

  – Alternative estimators

  Bandi and Russell (2006, *wp*)
  Bannouh, van Dijk and Martens (2008, *wp*)
  Barndorff-Nielsen, Hansen, Lunde and Shephard (2008, *wp*)
  Bauer and Vorkink (2006, *wp*)
  Corsi and Audrino (2007, *wp*)
  Griffin and Oomen (2006, *wp*)
  Hayashi and Yoshida (2005, *Bernoulli*)
  Martens (2004, *wp*)
  Palandri (2007, *wp*)
  Sheppard (2007, *wp*)
  Voev and Lunde (2007, *J.Fin.Ect.*)
  Zhang (2006, *wp*)

- Other variation measures

  – Realized quarticity

$$RQ_{t+1}(\Delta) \equiv \Delta^{-1}\mu_4^{-1}\sum_{j=1}^{1/\Delta} r_{t+j\Delta,\Delta}^4 \rightarrow \int_t^{t+1} \sigma^4(s)ds$$

  – Realized bi-power variation

$$BV_{t+1}(\Delta) \equiv \mu_1^{-2}\sum_{j=2}^{1/\Delta} | r_{t+j\cdot\Delta,\Delta} | \, | r_{t+(j-1)\cdot\Delta,\Delta} | \rightarrow \int_t^{t+1} \sigma^2(s)ds$$

  – Realized tri-power quarticity

$$TQ_{t+1}(\Delta) \equiv \Delta^{-1}\mu_{4/3}^{-3}\sum_{j=3}^{1/\Delta} | r_{t+j\Delta,\Delta} |^{4/3} | r_{t+(j-1)\Delta,\Delta} |^{4/3} | r_{t+(j-2)\Delta,\Delta} |^{4/3} \rightarrow \int_t^{t+1} \sigma^4(s)ds$$

Jumps

- Important to allow for dis-continuities, or jumps, in the price process

- Continuous-time jump diffusion

$$dp(t) = \mu(t)dt + \sigma(t)dW(t) + \kappa(t)dq(t)$$

  - $q(t)$ counting process (number of jumps up until time t)
  - $\kappa(t)$ jump size (if jump at time t)

Andersen, Benzoni and Lund (2002, *J. Fin.*)
Bates (1996, *RFS*; 2003, *J. Ect.*)
Chernov, Gallant, Ghysels and Tauchen (2002, *J. Ect.*)
Eraker (2004, *J. Fin.*), Johannes (2004, *J.Fin*)
Pan (2002, *J. Fin. Eco.*)

- More general Lévy processes

$$p(t) = p(0) + \int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dW(s) + \int_0^t \int_{-\infty}^{\infty} \kappa(x)q(ds,dx)$$

Barndorff-Nielsen and Shephard (2001, *JRSS B*; 2002, *JRSS B*)
Brockwell (2001, *Ann. ISM*)
Carr, Geman, Madan and Yor (2003, *Math. Fin.*)
Tauchen and Todorov (2006, *JBES*), Todorov (2008, *J.Ect.*)

- Continuous-time jump diffusion

$$dp(t) = \mu(t)dt + \sigma(t)dW(t) + \kappa(t)dq(t)$$

  - One period realized volatility/variation

$$RV_{t+1}(\Delta) \equiv \sum_{j=1}^{1/\Delta} r_{t+j\cdot\Delta,\Delta}^{2} \rightarrow \int_{t}^{t+1} \sigma^2(s)\, ds + \sum_{t<s\leq t+1} \kappa^2(s)$$

  - Is it possible to separate the two components?

- Realized bi-power variation

Barndorff-Nielsen and Shephard (2004, *J. Fin. Etc.*)

$$BV_{t+1}(\Delta) \equiv \mu_1^{-2} \sum_{j=2}^{1/\Delta} | r_{t+j\cdot\Delta,\Delta} | \, | r_{t+(j-1)\cdot\Delta,\Delta} | \rightarrow \int_t^{t+1} \sigma^2(s)\, ds$$

- Consistent (squared) jump measurements

$$RV_{t+1}(\Delta) - BV_{t+1}(\Delta) \rightarrow \sum_{t<s\leq t+1} \kappa^2(s)$$

- Statistically significant jumps

Barndorff-Nielsen and Shephard (2005, *J. Fin. Etc.*)
Huang and Tauchen (2005, *J. Fin Ect.*)

$$\Delta^{-1/2} \frac{RV_{t+1}(\Delta) - BV_{t+1}(\Delta)}{[\,(\mu_1^{-4} + 2\mu_1^{-2} - 5) \int_t^{t+1} \sigma^4(s)\, ds\,]^{1/2}} \rightarrow N(0,1)$$

– Feasible asymptotics based on realized quarticity

- Permits decomposition of realized volatility/variation into separate smooth/continuous and rough/jump components

$$RV_t(\Delta) = C_{t,\alpha}(\Delta) + J_{t,\alpha}(\Delta)$$

- Separate modeling, forecasting and pricing of jump and diffusive variation/risk

Andersen, Bollerslev and Huang (2009, *J.Ect.*)
Andersen, Bollerslev and Diebold (2007, *Re.Stat.*)
Bollerslev, Kretschner, Pigorsch and Tauchen (2008, *J.Ect.*)
Tauchen and Zhou (2007, *wp*), Wright and Zhou (2007, *wp*)
Todorov (2008, *J.Ect.*)

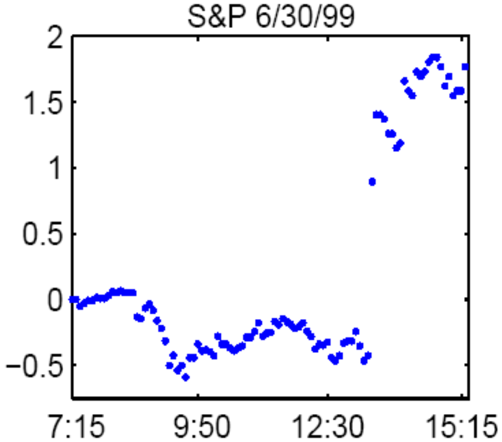
  - Little/no predictability coming from jumps

- What causes "significant" jumps?

  - Price discovery and market efficiency

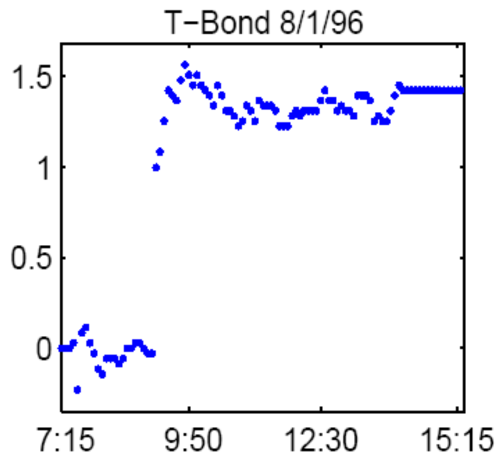
  - Macroeconomic announcements

Andersen and Bollerslev (1998, *J. Fin.*)
Andersen, Bollerslev, Diebold and Vega (2003, *AER*)
Andersen, Bollerslev, Diebold and Vega (2007, *JIE*)
Fair (2002, *J. Buss.*), Das (2002, *J.Etc.*)
Fleming and Remolona (1999, *J. Fin.*), Huang (2007, *wp*)
Johannes (2004, *J.Fin*), Rangel (2006, *wp*)



$Z_t(\Delta) = 7.659$

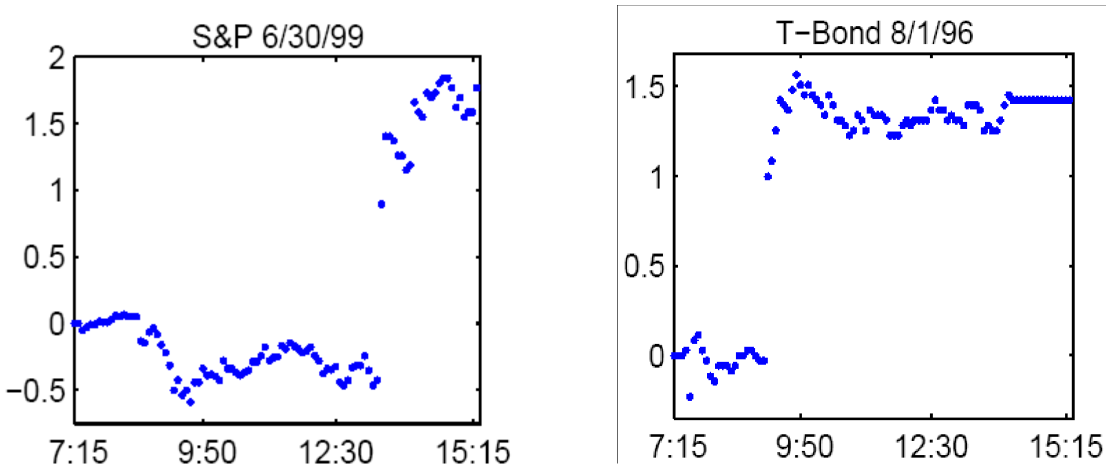
FED raised short term rate by ¼ percent at 13:15 CST (14:15 EST), but indicated that it "might not raise rates again in the near term due to conflicting forces in the economy."



$Z_t(\Delta) = 6.877$

National Association of Purchasing Managers (NAPM) index released at 9:00 CST (10:00 EST)

Andersen, Bollerslev and Diebold (2007, *ReStat*)



- Not all (significant) jumps map as "nicely" into readily identifiable macro news
  - What causes other jumps?

- Jumps in individual equities
  - Firm specific news
  - Apparent dis-connect with aggregate market
  - Test for non-diversifiable co-jumps

Bollerslev, Law and Tauchen (2008, *J.Ect.*)

- Alternative non-parametric jump detection schemes
  - Threshold type tests

    Mancini (2003, 2006, *wp*)
    Gobbi and Mancini (2007, *wp*)
    Lee and Mykland (2007, *RFS*)
    Corsi. Prino and Reno (2008, *wp*)

  - Arithmetic versus geometric returns (variance swap rate)

    Jiang and Oomen (2008, *J.Ect.*)

  - Wavelet based procedures

    Fan and Wang (2006, *JASA*)

  - Power variation

    AïtSahalia and Jacod (2008, *Annals Stat.*)
    Bollerslev and Todorov (2008, *wp*)
    Jacod and Todorov (2008, *Annals Stat.*)

Multivariate Measures

- Many/most issues in finance depend on covariance risk
  - ARCH/GARCH/SV modeling in high dimensions is challenging
  - Realized volatility concept extends directly to a multivariate setting

- Multivariate diffusion

$$dp(t) = \mu(t)\, dt + \Sigma(t)^{1/2} dW(t)$$

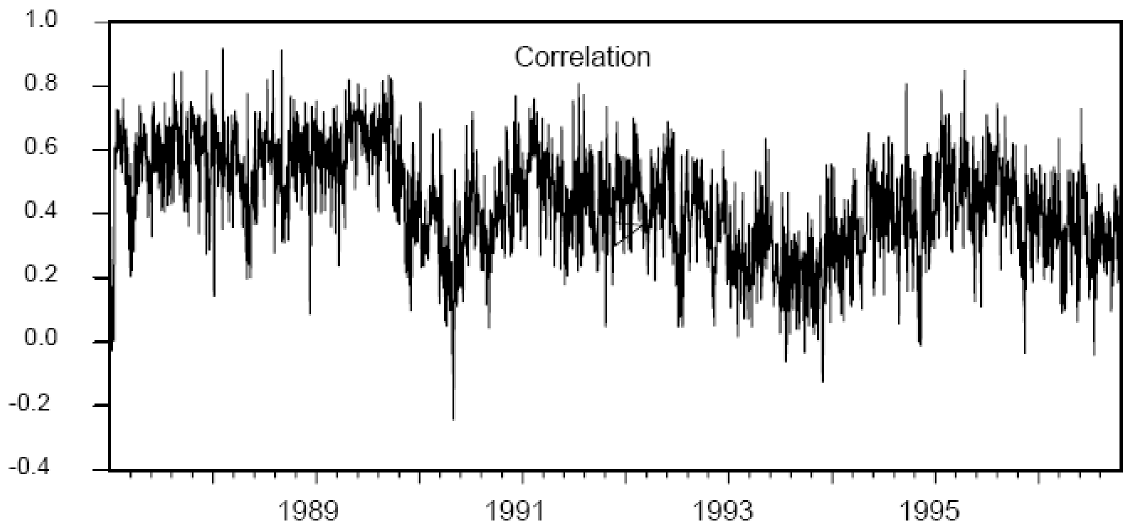
  - Realized volatility/covariation

$$RCov_{t+1}(\Delta) \equiv \sum_{j=1}^{1/\Delta} r_{t+j\cdot\Delta,\Delta} r'_{t+j\cdot\Delta,\Delta} \rightarrow \int_{t}^{t+1} \Sigma(s)\, ds$$

- Realized correlations

$$RCorr_{i,j,t} = \frac{\{RCov_t(\Delta)\}_{i,j}}{\{RCov_t(\Delta)\}_{i,i}^{1/2} \cdot \{RCov_t(\Delta)\}_{j,j}^{1/2}}$$

– DM/$ - ¥/$ daily correlations, 1986-1996



Andersen, Bollerslev, Diebold and Labys (2001, *JASA*)

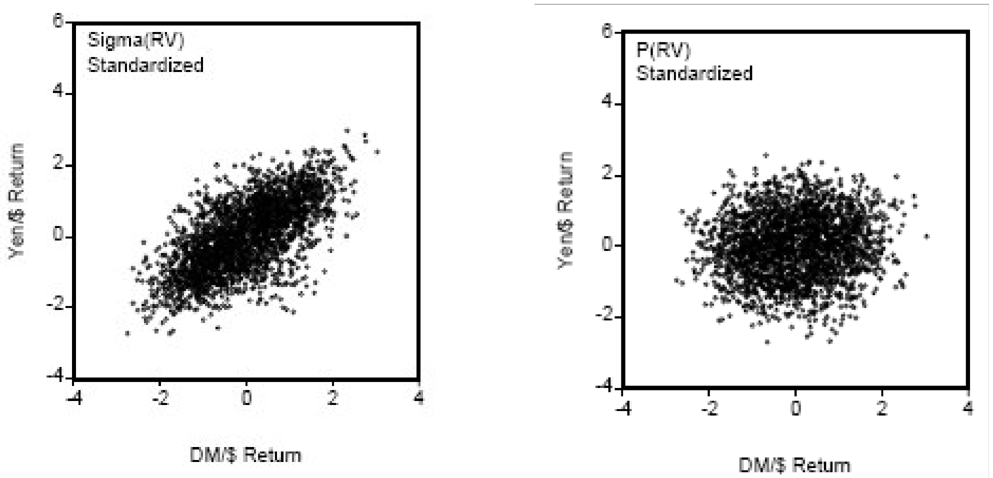
– Strong dynamic dependencies in correlations

– Volatility in correlation effect

- Standardized return distributions

$$r_t \cdot RCov_t(\Delta)^{-1/2}$$

  - DM/$ and ¥/$ daily returns, 1986-1996



Andersen, Bollerslev, Diebold and Labys (2001, *Mult.Fin.J.*)

  - Marginal standardized returns approximately Gaussian, but correlated
  - Multivariate standardized (by Cholesky decomposition) returns approximately *N(0,I)*

- Realized CAPM β's and factor loadings

  – Static CAPM

$$E(r_{i,t}) = r_{f,t} + \beta_i \lambda$$

$$\beta_i = \frac{Cov(r_{i,t}, r_{m,t})}{Var(r_{m,t})}$$

  – Conditional CAPM and time-varying β's

Ang and Chen (2006, *J.Emp.Fin.*)
Bollerslev, Engle and Wooldrige (1988, *JPE*)
Foster and Nelson (1996, *Ecta.*)

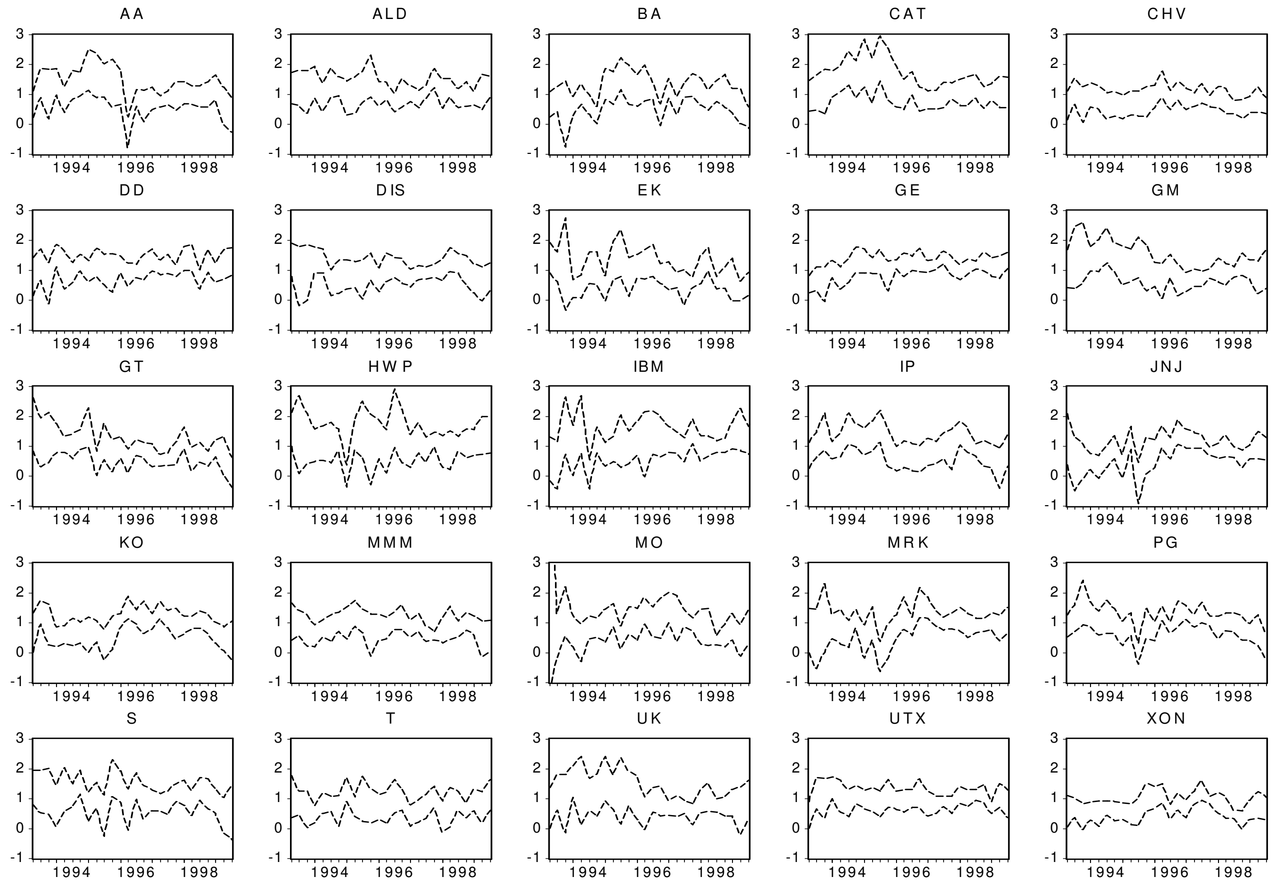
$$\mathrm{E}_{t-1}(r_{i,t}) = r_{f,t} + \beta_{i,t} \lambda_t$$

$$\beta_{i,t} = \frac{Cov_{t-1}(r_{i,t}, r_{m,t})}{Var_{t-1}(r_{m,t})}$$

  – Realized β's

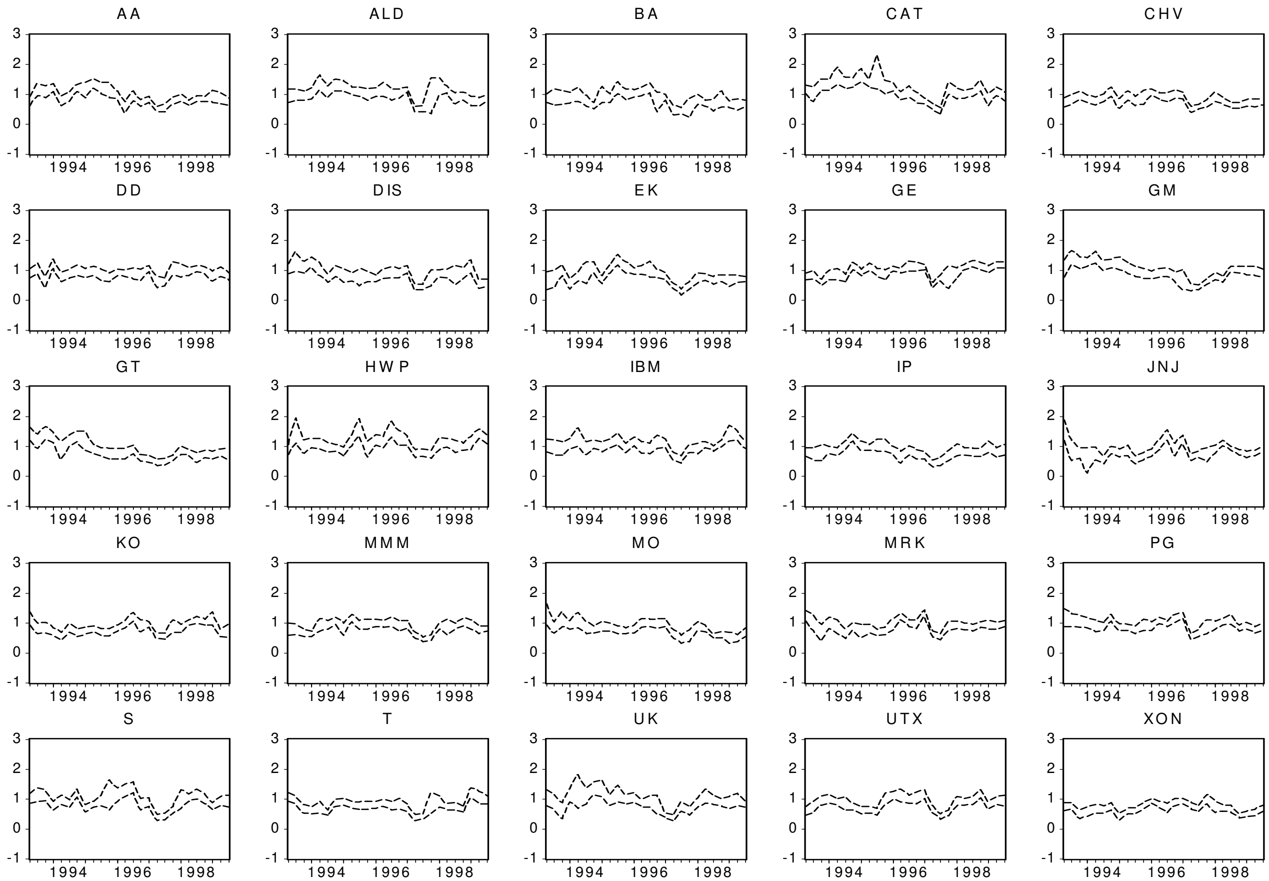
$$R\beta_{i,t} = \frac{\{RCov_t(\Delta)\}_{i,m}}{\{RCov_t(\Delta)\}_{m,m}}$$

- Ninety-five percent confidence intervals for quarterly realized β's based on daily returns for 25 DJ stocks, 1993 Q2 - 1999 Q3



Andersen, Bollerslev, Diebold and Wu (2006, *Book Ch.*)

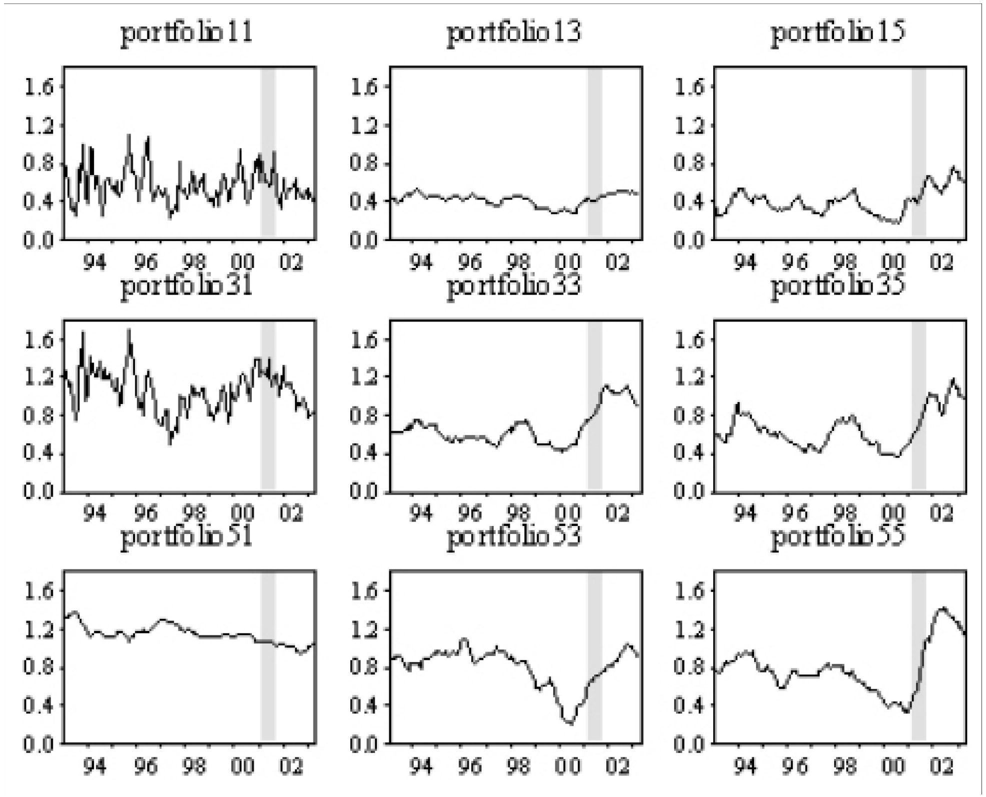
- Ninety-five percent confidence intervals for quarterly realized β's based on fifteen-minute returns for 25 DJ stocks, 1993 Q2 - 1999 Q3



Andersen, Bollerslev, Diebold and Wu (2006, *Book Ch.*)

– High-frequency data allows for much improved β measurements

- The returns on many financial assets are counter cyclical
  - Risk and/or risk aversion?
  - Monthly realized β's for Fama-French portfolios 1993.1 - 2003.5



Andersen, Bollerslev, Diebold and Wu (2005, *American Economic Review*)

  - Realized β's tend to increase in bad times, especially for value (high book-to-market) portfolios

- Stronger comovements in "extreme" markets

Andersen, Bollerslev, Diebold and Labys (2001, *JFE*)
Ang and Chen (2002, *JFE*)
Poon, Rockinger and Tawn (2004, *RFS*)

  - One-factor (market) model too simplistic

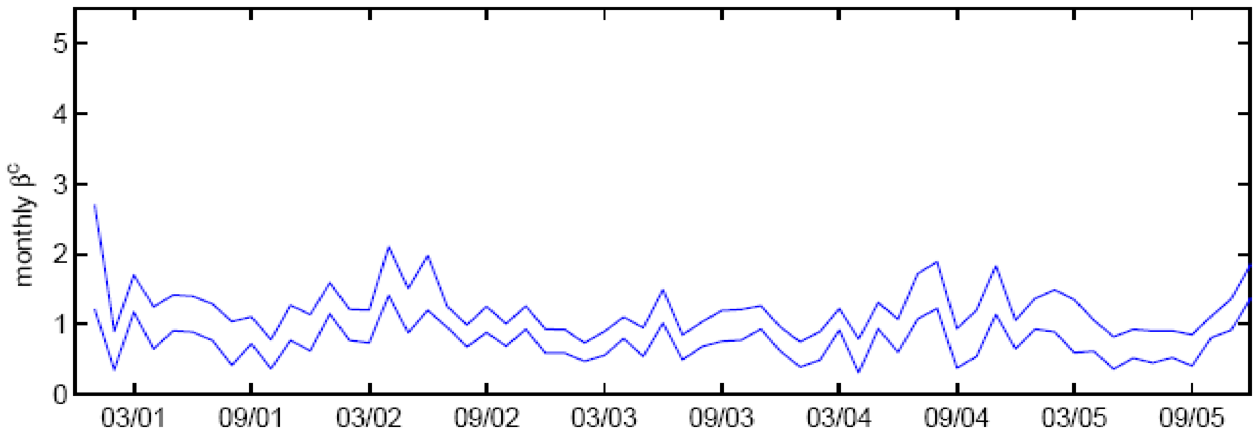
- Realized continuous and jump β's

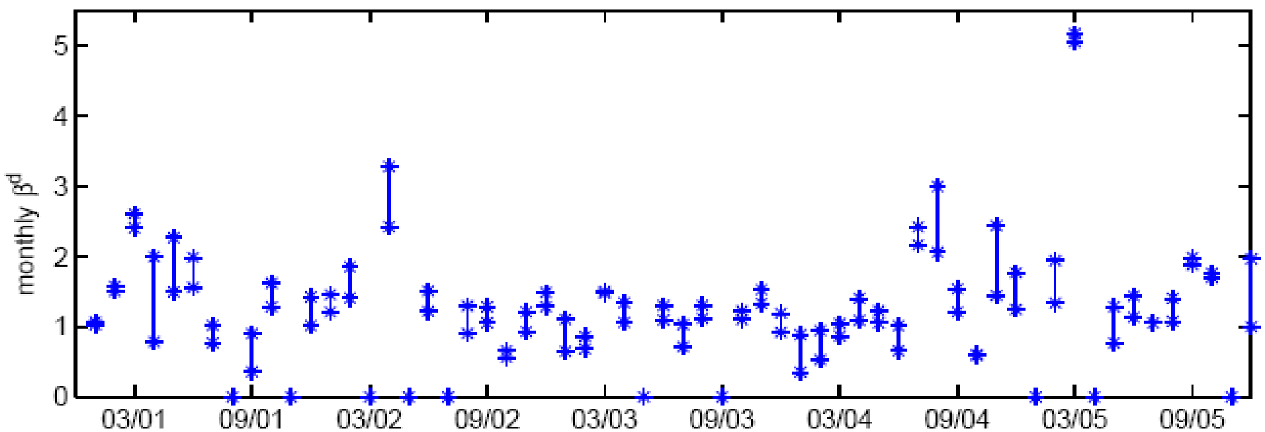
Todorov and Bollerslev (2008, *wp*)

$$r_{i,t} = \alpha_i + \beta_i^c r_{m,t}^c + \beta_i^d r_{m,t}^d + \varepsilon_{i,t}$$

  - Different β_i^c and β_i^d important for hedging
  - Risk premia may differ

– Ninety-five percent confidence intervals for β_i^c and β_i^d for Genentech





Todorov and Bollerslev (2008, *wp*)

- Multivariate issues
  - Non-linear (fractional) cointegration and β's

    Andersen, Bollerslev, Diebold and Wu (2006, *Book Ch.*)

  - Multi-factor models and realized factor loadings

    Bollerslev and Zhang (2003, *J.Emp.Fin.*)

  - Modeling and forecasting realized covariation

    Andersen, Bollerslev, Diebold and Labys (2003, *Ecta.*)
    Bauer and Vorkink (2007, *wp*), Bonate, Caporin and Ranaldo (2008, *wp*)
    Chiriac and Voev (2007, *wp*), Corsi and Audrino (2007, *wp*)
    Engle, Shephard and Sheppard (2008, *wp*), Palandri (2007, *wp*)

  - Realized covariation and dynamic portfolio choice

    Bandi, Russell and Zhou (2008, *Ect.Rev.*)
    De Pooter, Martens and van Dijk (2008, *Ect.Rev.*)
    Fleming, Kirby and Ostdiek (2003, *JFE*)

  - Multivariate measures and market microstructure "noise"

    Bandi and Russell (2007, *wp*)
    Barndorff-Nielsen, Hansen, Lunde and Shephard (2008, *wp*)
    Voev and Lunde (2007, *J.Fin.Ect.*), Zhang (2006, *wp*)

  - Co-jumps

    Bollerslev, Law and Tauchen (2008, *J.Ect.*)
    Bollerslev and Todorov (2008, *wp*)
    Jacod and Todorov (2008, *Annals Stat.*)

Option Implied Volatilities

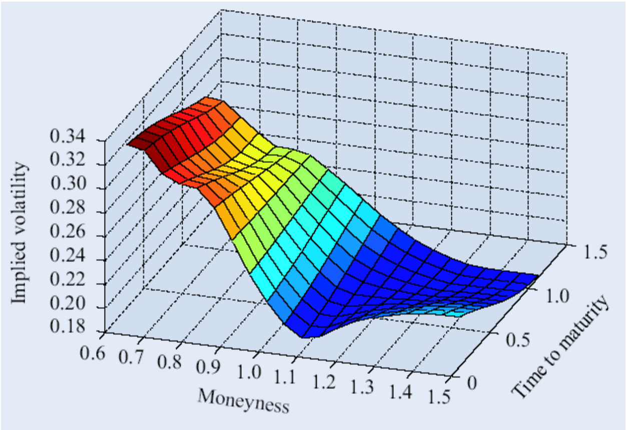
- Options trading is essentially equivalent to trading volatility
- Black-Scholes-Merton pricing formula

$$dp(t) = \mu + \sigma\, dW(t)$$

  - Implied volatility

$$C_t^{BS}(\, p(t) \,,\, K_i \,,\, T_i \,,\, r \,,\, \sigma \,) = C_t^{Market}(\, p(t) \,,\, K_i \,,\, T_i \,,\, r \,)$$

  - Market-based forecast of (average) volatility over the life of the option
  - Implied volatilities varies across strikes, K_i, and maturities, T_i



- Extensive literature devoted to reconciling smiles/smirks and term structure in BS implied volatilities by allowing for stochastic volatility and/or jumps

  - Continuous-time jump-diffusions

$$dp(t) = \mu(t)dt + \sigma(t)dW(t) + \kappa(t)dq(t)$$

Andersen, Benzoni and Lund (2002, *JF*)
Bakshi, Cao and Chen (1997, JF), Bates (1996, *RFS*; 2003, *JoE*)
Chernov, Gallant, Ghysels and Tauchen (2002, *JoE*)
Duffie, Pan and Singleton (2000, *Ect*)
Eraker (2004, *JF*), Eraker, Johannes and Polson (2003, *JF*)
Pan (2002, *JFE*)

  - More general Lévy processes

$$p(t) = p(0) + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dW(s) + \int_0^t \int_{-\infty}^{\infty} \kappa(x) q(ds, dx)$$

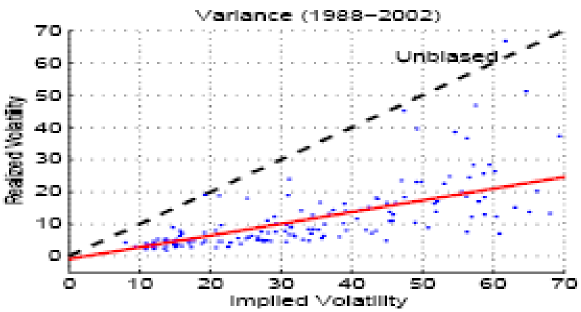
Carr and Wu (2004, *JFE*)
Cont and Tankov (2004, *Book*)
Carr, Geman, Madan and Yor (2003, *Math. Fin.*)
Todorov (2008, *J.Ect.*)

- Extensive literature devoted to testing whether option implied volatilities are unbiased forecasts of future (realized) volatilities

Canina and Figlewski (1993, *RFS*)
Day and Lewis (1998, *JFE*)
Lamoureux and Lastrapes (1993, *RFS*)

$$RV_{t+1}(\Delta) = b_0 + b_1 ImpV_t + \varepsilon_{t+1}$$

  - Typically $b_0 > 0$ and $b_1 < 1$



Bollerslev and Zhou (2006, *J.Ect.*)

  - Regression plagued by errors-in-variables, strong persistence, overlapping date, ...

Andersen, Frederiksen and Staal (2007, *wp*)
Bandi and Perron (2006, *J.Fin.Ect.*)
Chernov (2007, *JBES*)
Christensen and Prabhala (1998, *JFE*)

  - But what to expect in population?

- One-factor affine SV model

Heston (1993, *RFS*)

$$dp(t) = \mu(t)\,dt + \sigma(t)\,dB(t)$$

$$d\sigma^2(t) = \kappa(\theta - \sigma^2(t))\,dt + \sigma\,\sigma(t)\,dW(t)$$

- Corresponding risk-neutral dynamics

$$dp(t) = r^*(t)\,dt + \sigma(t)\,dB^*(t)$$

$$d\sigma^2(t) = \kappa^*(\theta^* - \sigma^2(t))\,dt + \sigma\,\sigma(t)\,dW^*(t)$$

$$\kappa^* = \kappa + \lambda \qquad \theta^* = \kappa\theta/(\kappa + \lambda)$$

- Volatility risk premium λ (<0)

- Corresponding integrated-implied volatility regression

Bollerslev and Zhou (2006, *J.Ect.*)

$$\int_t^{t+1} \sigma^2(s)\,ds = b_0 + b_1 \cdot E_t^*\left(\int_t^{t+1} \sigma^2(s)\,ds\right) + u_{t+1}$$

$$b_1 = \frac{1 - e^{-\kappa}}{1 - e^{-\kappa^*}} \cdot \frac{\kappa^*}{\kappa} < 1$$

– Integrated-implied volatility relationship

$$\int_t^{t+1} \sigma^2(s)\,ds = b_0 + b_1 \cdot E_t^*\left(\int_t^{t+1} \sigma^2(s)\,ds\right) + u_{t+1}$$

$$b_1 = \frac{1 - e^{-\kappa}}{1 - e^{-\kappa^*}} \cdot \frac{\kappa^*}{\kappa} < 1 \qquad \kappa^* = \kappa + \lambda$$

allows for the estimation of the volatility risk premium λ (and other model parameters)

Bollerslev and Zhou (2006, *J.Ect.*)
Bollerslev, Gibson and Zhou (2006, *wp*)
Carr and Wu (2008, *RFS*)
Todorov(2007, *wp*)

– IV_{t+1} may be accurately measured by $RV_{t+1}(\Delta)$ for $\Delta \to \infty$

– But, how do you actually measure $E_t^*(IV_{t+1})$ without a model?

- "New" model-free implied volatility

Carr and Madan (1998, *Book Ch.*)
Demeterfi, Derman, Kamal and Zou (1999, *J.Deriv.*)
Britten-Jones and Neuberger (2000, *JF*)

$$E_t^*(IV_{t+1}) = \int_0^\infty \frac{C_t(t+1, K) - \max\{P(t) - K, 0\}}{K^2} dK$$

  - Risk-neutral density $\partial^2 C_t(t+1, K)/\partial K^2$

Breeden and Litzenberger (1978, *J.Buss.*)

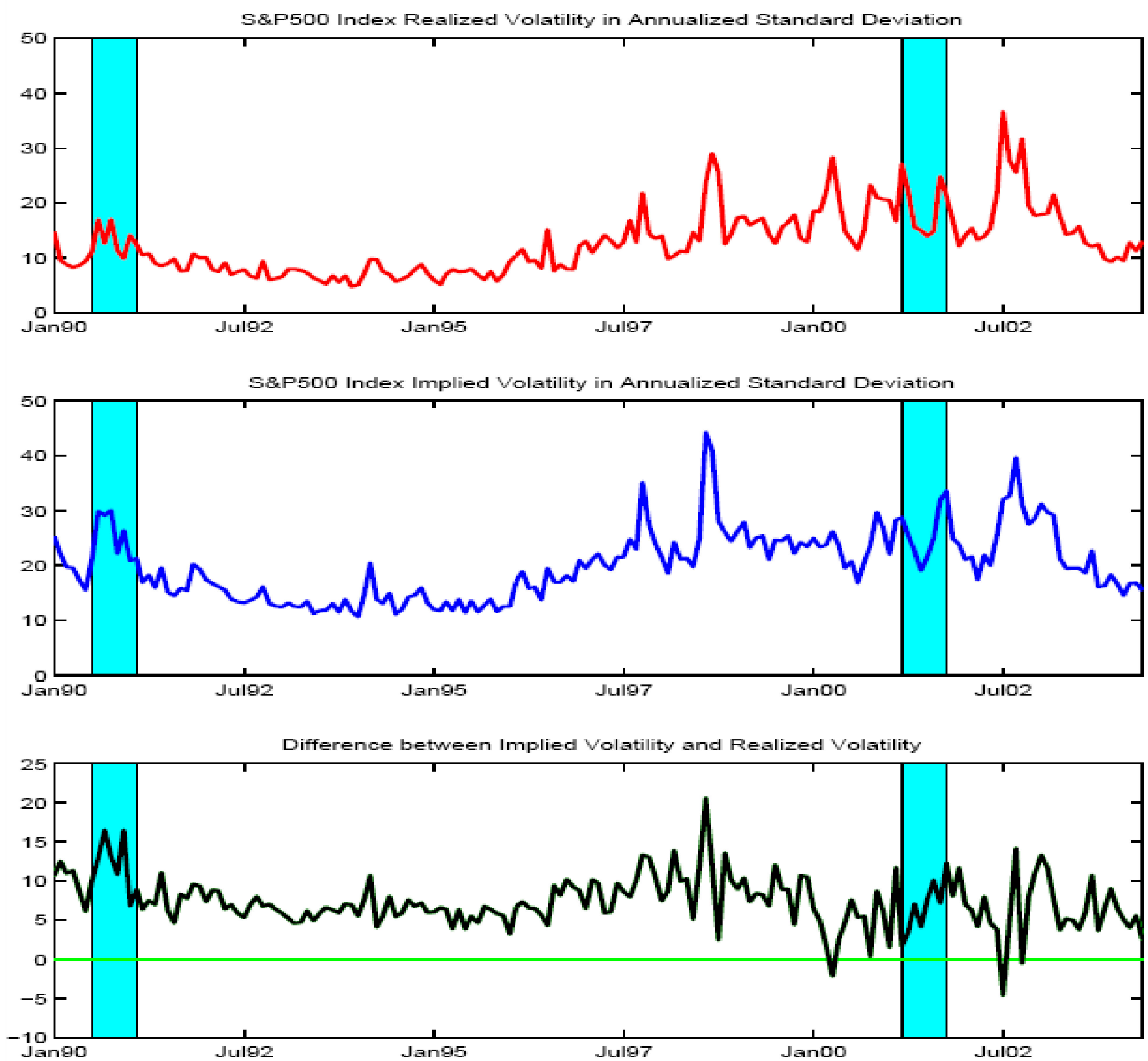
  - Also works with jumps

Carr and Wu (2008, *RFS*)
Jiang and Tian (2005, *RFS*)

  - VIX and many other (now actively traded) indexes are based on this idea

Andersen and Bondarenko (2007, *wp*)
Jiang and Tian (2007, *J.Deriv.*)

- Monthly S&P500 VIX and realized volatility 1990.1 - 2004.5



Bollerslev, Gibson and Zhou (2006, *wp*)

- Estimated volatility risk premium $-\lambda_t$ based on macro/finance explanatory variables and one-factor affine SV model

Bollerslev, Gibson and Zhou (2006, *wp*)

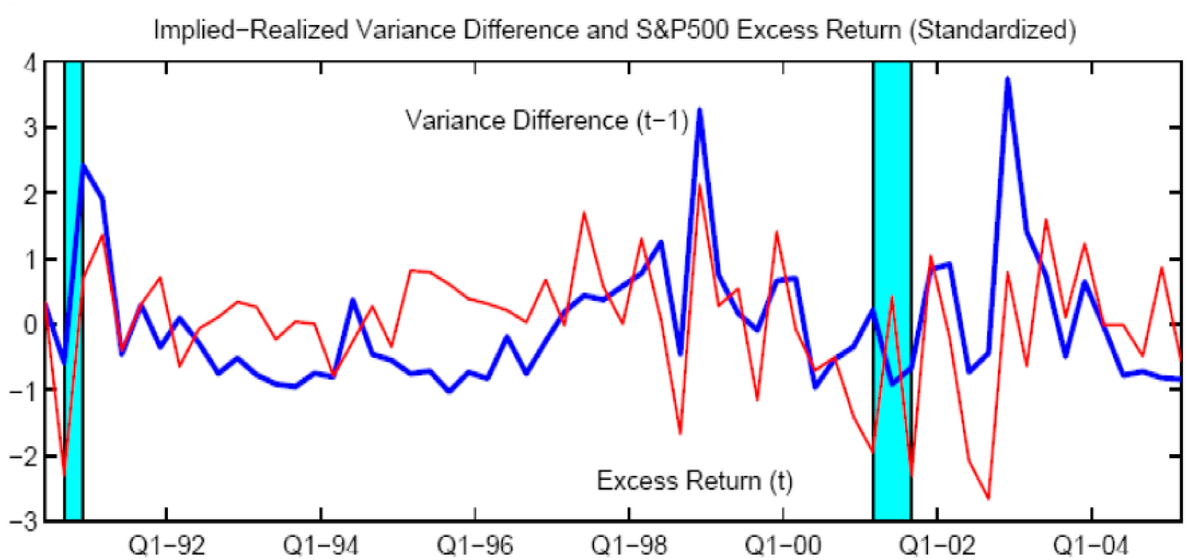
- Higher premium associated with financial market "crises" or "bad" times
- Generally associated with higher volatility-of-volatility
- Premium may also be linked to notions of aggregate risk aversion

Bakshi and Kapadia (2003, *RFS*)
Gordon and St-Amour (2004, *JBES*)
Rosenberg and Engle (2002, *JFE*)

- What about return predictability?

- Simple variance difference

$$E_t^*(IV_{t+1}) - IV_t \approx VIX_t - RV_t(\Delta)$$



  – Striking coherence with (quarterly) returns

- S&P500 monthly return regressions 1990.1 - 2007.12

	Simple									Multiple				
Constant	-0.55	-0.19	4.88	92.72	75.51	14.23	7.73	6.67	5.45	101.13	-2.52	78.02	91.20	101.86
	(-0.13)	(-0.04)	(1.20)	(2.22)	(1.87)	(0.91)	(1.32)	(2.12)	(1.53)	(2.42)	(-0.55)	(1.49)	(1.74)	(2.28)
$IV_t - RV_t$	0.39									0.49	0.42		0.50	0.57
	(1.76)									(2.16)	(1.87)		(2.10)	(2.34)
IV_t		0.20												
		(1.30)												
RV_t			0.11											
			(0.41)											
$\log(P_t/E_t)$				-2.30						-2.76		-1.90	-2.49	-2.93
				(-2.02)						(-2.40)		(-1.36)	(-1.76)	(-2.31)
$\log(P_t/D_t)$					-1.47									
					(-1.68)									
$DFSP_t$						-0.77								
						(-0.50)								
$TMSP_t$							-0.72							2.87
							(-0.28)							(0.96)
$RREL_t$								1.63						3.29
								(0.43)						(0.76)
CAY_t									3.71		3.94	1.78	1.46	
									(2.04)		(2.20)	(0.87)	(0.72)	
Adj. R^2 (%)	1.07	0.57	-0.34	1.80	1.11	-0.31	-0.43	-0.40	1.44	3.77	2.78	1.89	3.86	3.34

Bollerslev, Tauchen and Zhou (2009, *RFS*)

  – Variance difference marginally significant, but low R^2's

- S&P500 (overlapping) quarterly return regressions 1990.Q1 - 2007.Q4

	Simple									Multiple				
Constant	-2.08	0.24	6.60	92.41	73.35	20.63	7.39	6.92	5.53	101.89	-4.12	85.93	100.06	98.21
	(-0.56)	(0.06)	(1.60)	(2.17)	(1.81)	(1.32)	(1.24)	(2.18)	(1.54)	(2.40)	(-1.00)	(1.67)	(1.93)	(2.18)
$IV_t - RV_t$	0.47									0.58	0.51		0.59	0.70
	(2.86)									(3.43)	(3.02)		(3.38)	(4.01)
IV_t		0.19												
		(1.41)												
RV_t			0.00											
			(0.00)											
$\log(P_t/E_t)$				-2.28						-2.82		-2.11	-2.77	-2.95
				(-1.97)						(-2.42)		(-1.54)	(-1.98)	(-2.33)
$\log(P_t/D_t)$					-1.42									
					(-1.62)									
$DFSP_t$						-1.39								
						(-0.90)								
$TMSP_t$							-0.46							4.08
							(-0.17)							(1.42)
$RREL_t$								3.27						6.39
								(0.88)						(1.56)
CAY_t									3.23		3.52	1.08	0.74	
									(1.78)		(1.99)	(0.53)	(0.37)	
Adj. R^2 (%)	6.82	2.49	-0.47	6.55	4.19	1.18	-0.43	0.43	4.13	16.76	11.87	7.21	17.42	19.74

Bollerslev, Tauchen and Zhou (2009, *RFS*)

- Striking R^2's
- Variance difference dominates other traditional predictor variables

- S&P500 (overlapping) annual return regressions 1990 - 2005

	Simple									Multiple				
Constant	4.62	7.62	9.49	78.47	79.83	15.59	5.37	7.29	5.42	81.00	1.91	52.85	55.11	74.04
	(1.50)	(2.44)	(3.20)	(2.05)	(2.17)	(1.13)	(0.90)	(2.33)	(1.47)	(2.15)	(0.53)	(1.03)	(1.08)	(1.88)
$IV_t - RV_t$	0.12									0.19	0.18		0.20	0.33
	(1.00)									(1.68)	(1.51)		(1.74)	(2.96)
IV_t		-0.02												
		(-0.21)												
RV_t			-0.17											
			(-1.20)											
$\log(P_t/E_t)$				-1.90						-2.06		-1.24	-1.40	-2.14
				(-1.80)						(-2.00)		(-0.91)	(-1.03)	(-1.92)
$\log(P_t/D_t)$					-1.55									
					(-1.92)									
$DFSP_t$						-0.87								
						(-0.64)								
$TMSP_t$							0.88							4.53
							(0.35)							(1.69)
$RREL_t$								4.09						6.29
								(1.11)						(1.75)
CAY_t									3.48		3.62	2.13	2.12	
									(1.99)		(2.12)	(0.99)	(0.99)	
Adj. R^2 (%)	1.23	-0.37	2.89	16.34	19.53	1.79	0.01	4.54	18.15	20.12	21.18	21.46	25.52	32.58

Bollerslev, Tauchen and Zhou (2009, *RFS*)

  – Variance difference dominated by P/E, P/D, and CAY

  – Predictability maximized at quarterly horizon

- Traditional risk-return tradeoff

	Monthly Returns			Quarterly Returns			Annual Returns		
Constant	-0.83	112.29	134.16	-0.88	113.52	121.71	6.84	78.82	81.17
	(-0.18)	(2.88)	(2.85)	(-0.21)	(2.77)	(2.71)	(2.22)	(2.28)	(2.16)
RV_t	0.03	0.32	0.46	-0.11	0.17	0.33	-0.22	-0.03	0.10
	(0.10)	(1.22)	(1.58)	(-0.41)	(0.65)	(1.11)	(-1.25)	(-0.23)	(0.75)
$IV_t - RV_t$	0.38	0.45	0.54	0.50	0.56	0.69	0.17	0.20	0.33
	(1.68)	(2.02)	(2.43)	(2.91)	(3.42)	(4.22)	(1.43)	(1.73)	(2.94)
$\log(P_t/E_t)$		-3.43	-4.06		-3.18	-3.76		-1.99	-2.39
		(-2.91)	(-2.93)		(-2.80)	(-2.87)		(-2.13)	(-2.23)
$TMSP_t$			4.95			5.52			4.97
			(1.48)			(1.68)			(1.82)
$RREL_t$			5.09			7.55			6.62
			(1.17)			(1.82)			(1.87)
Adj. R^2 (%)	0.61	4.17	4.45	6.78	17.11	21.81	5.80	19.80	33.00

  - RV and VIX both insignificant when included in isolation

- Use of "new" model-free realized and implied volatility measures crucial

- Why does it work so well?

- Stochastic volatility in general equilibrium

Bollerslev, Sizova and Tauchen (2008, *wp*)
Bollerslev, Tauchen and Zhou (2009, *RFS*)

  - Endowment economy with Epstein-Zin-Weil recursive preferences

Bansal and Yaron (2004, *JF*)

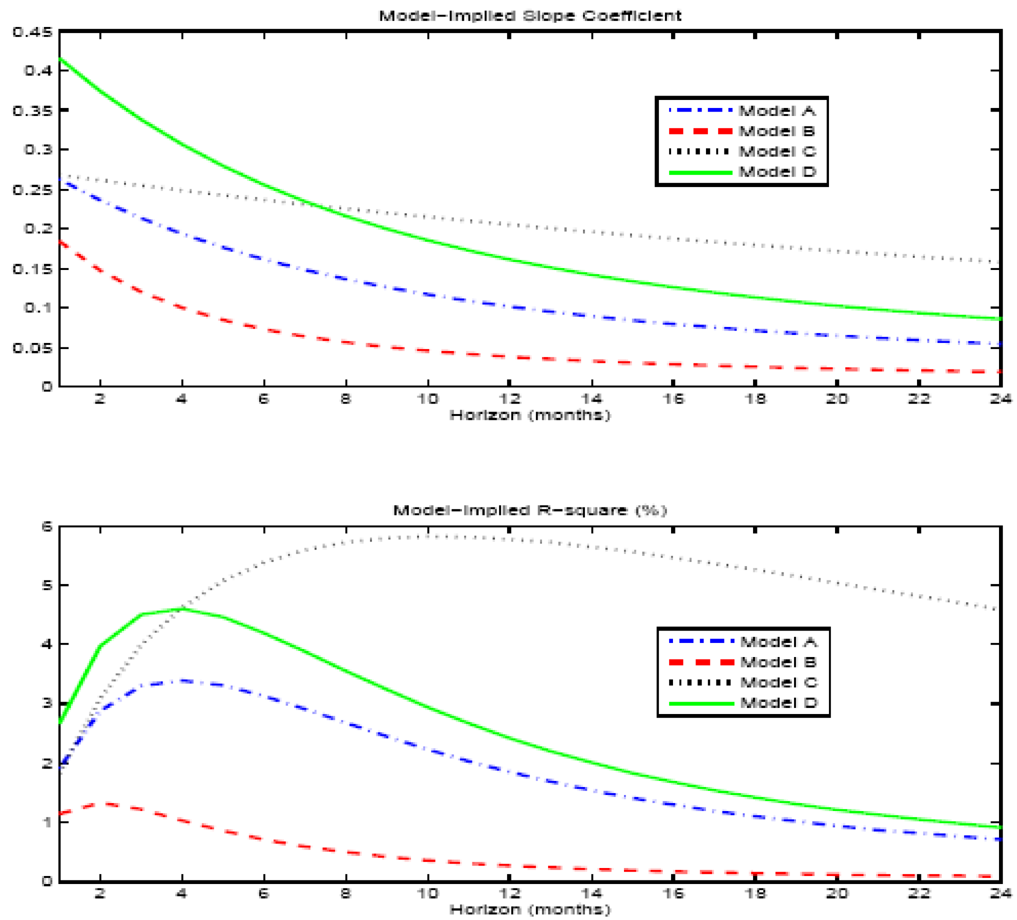
  - Richer volatility dynamics

  - The variance risk premium, or variance difference

$$E_t^*(IV_{t+1}) - IV_t \approx VIX_t - RV_t(\Delta)$$

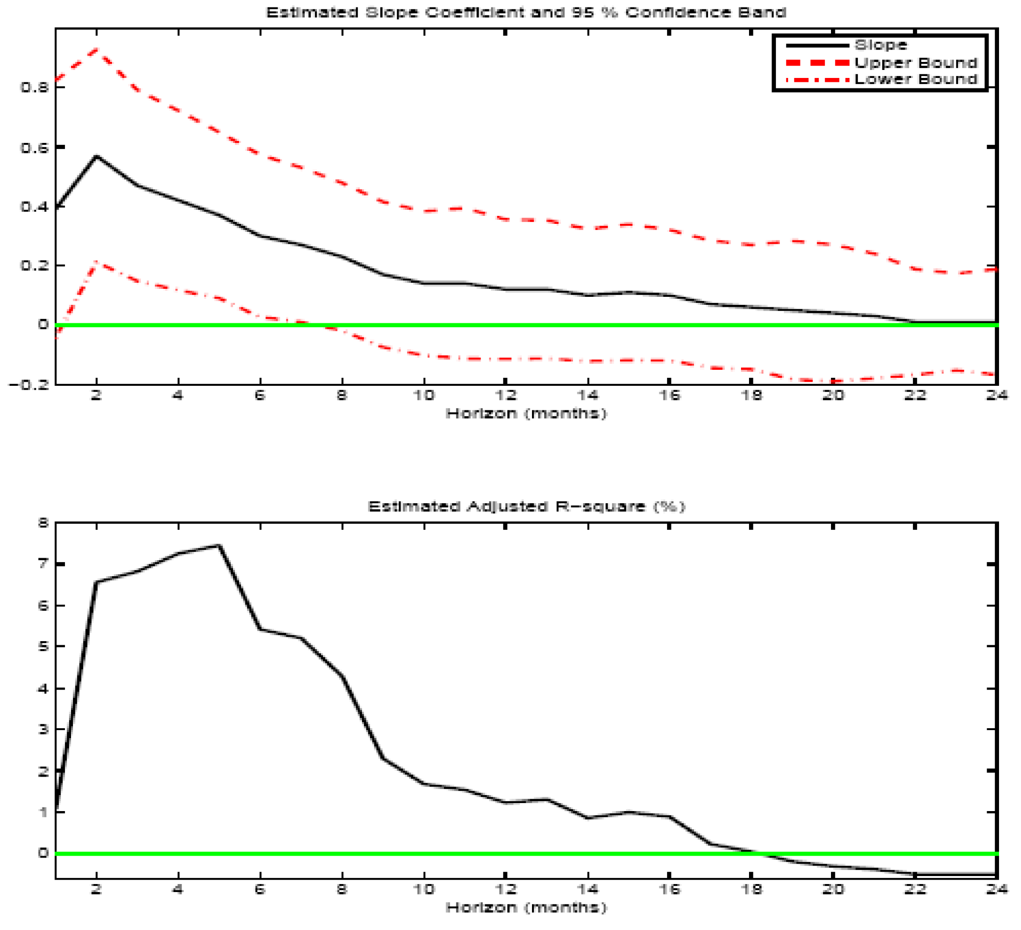
effectively isolates a systematic risk factor associated with time-varying economic uncertainty, or volatility-of-volatility in consumption growth

– Calibrated GE model slope coefficients and R^2's



Bollerslev, Tauchen and Zhou (2009, *RFS*)

– Estimated slope coefficients and R^2's



Bollerslev, Tauchen and Zhou (2009, *RFS*)

- Current themes
  - Dependent “noise” structures and improved variation measures
  - Jumps, co-jumps and price discovery
  - Distributional implications
  - Realized covariation and multivariate volatility models
  - Realized betas and factor loadings
  - Model-free options implied volatilities and variance risk premia
  - Pricing of volatility risk
  - Much exciting work ahead ...