Comment on the 2005 JBES Invited Lecture
“Realized Variance and Market Microstructure Noise”
by Peter R. Hansen and Asger Lunde *

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October, 2005

*This work was supported by grants from FQRSC, Hydro-Québec, MITACS, and SSHRCC. In addition, René Garcia thanks Bank of Canada while Nour Meddahi thanks Jean-Marie Dufour’s Econometrics Chair of Canada and CREST for their financial support. Many thanks to Bruno Feunou-Kamkui and Roméo Tedongap-Nguefack for their excellent research assistance. Thanks also to Silvia Gonçalves for helpful discussions.

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1 A historical review

The authors have to be congratulated for an excellent contribution, both from a theoretical and applied point of view, to the problem of measuring integrated variance in the presence of microstructure noise, an inescapable reality.

The negative effects of microstructure noise on measuring volatility have been well-known for years. For instance, French, Schwert and Stambaugh (1987) incorporate the cross-product of consecutive days in order to take into account the first-order autocorrelation of daily returns caused by non-synchronous trading of securities. Note that the estimator proposed by French, Schwert and Stambaugh (1987) coincides with the one proposed by Zhou (1996) and analyzed in Hansen and Lunde’s (HL) paper. Of course, we know from the current literature (e.g., Zang, Mykland, and Aït-Sahalia (2004), Bandi and Russell (2005a), and HL (2005)) that microstructure noise plays a crucial role in measuring daily integrated variance, but less so for measuring monthly integrated variance. This gives a theoretical foundation to French, Schwert and Stambaugh (1987) who find that the correction is not that important quantitatively.

Microstructure noise was also a well-known problem at the inception of the current literature. Fifteen years ago, Daniel Nelson (1958-1995) was among the pioneers who established the theoretical foundations for measuring instantaneous variance from high frequency data. Bollerslev and Rossi (1995) review his numerous contributions, in particular on measuring, filtering, and forecasting variance. Daniel Nelson’s research agenda, combined with the availability of high frequency data (provided most notably by the OLSEN group), lead several authors, in particular Torben Andersen and Tim Bollerslev to explore empirically the issue of measuring nonparametrically the instantaneous variance. They obviously ran into the problem of microstructure effects, especially the intraday periodicity; see for instance Andersen and Bollerslev (1997). At the same time, another research strand was developed under the leadership of Robert Engle with the goal of extracting the information on volatility contained in the durations between consecutive trades; see for instance Engle (2000).

Closer to the current literature, an important breakthrough can be attributed to Andersen and Bollerslev (1998), as well as to Barndorff-Nielsen and Shephard (2001). For instance, Andersen and Bollerslev (1998) clearly showed the importance of measuring daily integrated variance by summing squared intra-daily returns. Moving from measuring instantaneous variance to integrated variance was theoretically successful due to the availability of a central limit theorem (Jacod and Protter (1998) and Barndorff-Nielsen and Shephard (2002)). This move was also empirically successful because the intraday periodicity plays a less important role. Moving from instantaneous to integrated variance is indeed a temporal aggregation issue and it is well-known from the macroeconometric literature that aggregating helps to solve the seasonality problem. The empirical success of this approach
was clearly demonstrated in Andersen, Bollerslev, Diebold, and Labys (2001, 2003).

We finish this section by historical notes. Andersen and Bollerslev (1997) was presented at the World Congress of the Econometric Society in Tokyo (1995). In the same congress, Éric Renault gave an invited lecture on financial econometrics where he ended his talk by saying that the financial econometrics lecture in the next world congress should focus on the problems related to microstructure effects; see Renault (1997). It took in fact ten years to have two lectures on the topic by Yacine Aït-Sahalia and Neil Shephard at the last world congress in London. Engle (2000) was the Fisher-Schultz invited lecture of the European meeting of the Econometric Society in Istanbul (1996). One should also observe that at the end of Aït-Sahalia and Shephard’s talks, Robert Engle made the comment that the i.i.d. assumption of noise is clearly not empirically supported by the data because durations, i.e., the time between two consecutive trades, contain information on volatility.

The rest of the comment is organized as follows. Section 2 discusses several issues about statistical inference when one faces microstructure noise. Section 3 studies the time-series properties of the realized variance under the presence of microstructure noise. The last section discusses how important it is economically to account and correct for microstructure noise and for which questions it may be relevant to do so.

2 Microstructure noise and statistical inference

2.1 The signature plot

The HL paper is one of several papers in the current literature devoted to establishing the statistical properties of estimators under maintained assumptions about microstructure noise, mainly i.i.d. or serially correlated noise. One important tool for analyzing these estimators is the signature plot. However, these assumptions are not explicitly tested. The HL paper uses the behavior of the signature plots of various realized variance estimators to infer the properties of the noise, a flat signature plot being consistent with the absence of noise. This approach is very useful and informative, but it does not constitute a formal test.

In order to show the limitation of the signature plot as an inference tool, we consider a particular example where one can compute expressions in closed-form. We propose a model where the microstructure noise is stationary. Then, we show that for some specific values of the parameters, one can have a signature plot of realized variance that looks like one that corresponds to the i.i.d. microstructure noise case. Again, this example is provided in order to show that the current literature neglects to test formally the assumptions made on the microstructure noise and uses instead “visual” diagnostics.
Assume that the noise process \( u_t \) is stationary and given by

\[
u_t = \int_{-\infty}^{t} A(t-u) dW_u,\quad (1)
\]

where \( W_u \) is a standard Brownian process, independent of the efficient price process \( p_t^* \), and \( A(\cdot) \) is a real function such that \( \int_0^\infty A^2(u) du < \infty \). Observe that the Ornstein-Uhlenbeck (OU) process considered in A¨ıt-Sahalia, Mykland and Zhang (2005), i.e.,

\[
du_t = -\kappa u_t dt + \sigma dW_t,
\]

corresponds to the case

\[
A(s) = \sigma \exp(-\kappa s).
\]

Define the processes \( y^{(h)}_t, y^{*(h)}_t, \varepsilon^{(h)}_t, RV_t(h), \) and \( RV_t^*(h) \) by

\[
y^{(h)}_t = p_t - p_{t-h}, \quad y^{*(h)}_t = p^*_t - p^*_{t-h}, \quad \varepsilon^{(h)}_t = u_t - u_{t-h},
\]

\[
RV_t(h) \equiv \frac{1}{h^2} \sum_{i=1}^{1/h^2} y^{(h)}_{t-1+ih}, \quad RV_t^*(h) \equiv \frac{1}{h^2} \sum_{i=1}^{1/h^2} y^{*(h)}_{t-1+ih}.
\]

Then, one has:

\[
y^{(h)}_t = y^{*(h)}_t + \varepsilon^{(h)}_t, \quad y^{2(h)}_t = y^{*(h)}_t^2 + \varepsilon^{(h)}_t^2 + 2y^{(h)}_t \varepsilon^{(h)}_t.
\]

Feunou-Kamkui, Garcia, Meddahi, Tedongap-Nguefack (2005) show that:

\[
E[RV_t(h)] = E[RV_t^*(h)] + h^{-1} E[\varepsilon_t^{(h)}]^2, \quad (5)
\]

\[
E[\varepsilon_t^{(h)}]^2 = \int_0^h A^2(u) du + \int_h^\infty [A(u) - A(u-h)]^2 du, \quad (6)
\]

\[
Var[u_t] = \int_0^{\infty} A^2(u) du, \quad (7)
\]

\[
Cov[u_t, u_{t-h}] = \int_0^\infty A(u+h)A(u) du = \pi(-h), \quad (8)
\]

where \( \pi(\cdot) \) is defined in Assumption 2 of HL. Consequently, there is a connection between \( A(\cdot) \) and \( \pi(\cdot) \). For instance, one can show that the daily quadratic variation of the process \( u_t \) equals \( A^2(0) \) and therefore \( A^2(0) = -2\pi'(0) \).

Instead of considering the OU process, we assume in what follows a more general process given by

\[
A(s) = \exp(\kappa_1 s) \text{ if } 0 \leq s \leq s_0,
\]

\[
A(s) = \exp(\kappa_1 s_0 - \kappa_2 (s - s_0)) \text{ if } s_0 \leq s,
\]

where \( \kappa_1 > 0, \ \kappa_2 > 0, \ s_0 \geq 0 \). For this example, we gave in the Appendix the formulas of (6), (7) and (8).
The main goal of the rest of this subsection is to consider several values for \( \theta = (\kappa_1, \kappa_2, s_0) \) in order to study the behavior of

\[
f(h) \equiv h^{-1}E[e_i^{(h)^2}] = h^{-1} \left( \int_0^h A^2(u)du + \int_h^\infty [A(u) - A(u - h)]^2du \right),
\]

that is the difference between the ideal signature plot (no microstructure noise) and the observed one (with microstructure noise). Given that there is a scaling issue, instead of studying the function \( f(\cdot) \), we will study the function \( g(\cdot) \) defined by

\[
g(h) = \frac{f(h)}{Var[u_t]} = \frac{f(h)}{\int_0^\infty A^2(u)du}.
\]

In other words, we are assuming a unit variance for the noise \( u_t \). Also, the unit of time is not specified. As usual, the non-invariant measure is \( \kappa \Delta t \), i.e., by varying the parameters \( \kappa \) and \( t \) and maintaining the product \( \kappa t \) fixed, one gets the same results that we will show below.

The first scenario we consider is \((\kappa_1, \kappa_2) = (3.1, 2.9)\) while \( s_0 \) takes several values: \{0.01, 0.1, 0.25, 0.5, 1\}. Figure 1.a provides the plot of \( g(\cdot) \) while Figure 1.b provides the ACF of the microstructure noise \( u_t \). If one defines \( h^* \) as the ArgMax of \( g(\cdot) \), then the right part of Fig. 1.a (i.e., \( h > h^* \)) looks like the realized variance signature plot. Note that in practice, \( s_0 \) maybe very small and therefore one cannot observe the left part of the function \( g(\cdot) \), i.e., one cannot distinguish the noise considered here with an i.i.d. one.

Of course, one can argue that the magnitude of the maximum is not important, especially for a unit variance noise. Actually, one can readily increase this maximum by raising the values of \( \kappa_1 \) and \( \kappa_2 \) and lowering the value of \( s_0 \). We consider two other scenarios: \((\kappa_1, \kappa_2) = (500, 500)\) while \( s_0 \in \{0.001, 0.01\} \) (Scenario 2) and \((\kappa_1, \kappa_2) = (1000, 1000)\) while \( s_0 \in \{0.001, 0.01\} \) (Scenario 3). We provide in Figure 2.a and Figure 3.a. the plots of the function \( g(\cdot) \), where clearly the maximums are now very high.

We provide in Figures 1.b, 2.b and 3.b the autocorrelation functions of the microstructure noise in the three scenarios. Given that we take large values for \( \kappa_1 \) and \( \kappa_2 \), the autocorrelations vanish quickly to zero (especially for the second and third scenarios), meaning that the noise is almost “i.i.d.” but with finite variation. Again, this feature will increase the problem of identification.

We end this subsection by two remarks: 1) We focused in these examples on small value of \( h \) given that the quadratic variation of the microstructure noise equals \( A(0) \). However, for a given sample frequency, the behavior of \( A(h) \) for \( h \neq 0 \) plays a role in the signature plot. 2) One can argue that the examples provided here do not match the signature plot for different values of the sample frequency. A simple approach to handle this problem is to add to our examples a dependent noise (e.g. OU) in order to match the signature plot beyond the origin.
2.2 Nonnegative estimators

One desirable property of any estimator of the integrated variance is nonnegativity. There is no reason to believe that the current estimator as well as those that correct the bias are nonnegative. Of course, one can use more adequate kernels to ensure positivity as did the HAC literature (Newey and West (1987), Andrews (1991)), for instance the Bartlett Kernel. But it is not clear if one can find such a kernel and obtain an optimal estimator of the integrated variance as in Barndorff-Nielsen, Hansen, Lunde and Shephard (2004) (which is closely related to the optimal subsample estimator of Zhang (2004)).

Some papers considered the multivariate case, in particular Bandi and Russell (2005b) and Hayashi and Yoshida (2005). However, these papers focused on estimating the covariance and the variances independently, i.e. without imposing the Cauchy-Schwartz inequality, which leads to the possibility of having variance-covariance estimators that are not nonnegative. Again, an approach based on kernel may be helpful to solve this problem (but will still face the problem of non-synchronous trades).

A different approach would be, in a first step, to pre-whiten or filter the data as in Ebens (1999) and Andersen, Bollerslev, Diebold and Labys (2001). In a second step, one would compute the usual realized variance of the transformed data and therefore obtain nonnegative estimators. Under some assumptions, Bandi and Russell (2005a) studied the theoretical properties of such estimators. Their conclusions are somewhat mixed. We think however that this approach deserves more study because it is simple and provides nonnegative estimators (in both the univariate and the multivariate cases). A more fundamental argument is provided in Smith (2005) where a theoretical connection is drawn between the HAC estimators and estimators computed as the variance of transformed data. While these transformations are motivated by the literature on empirical likelihood, they can be interpreted as pre-whitening or filtering methods.

2.3 Calendar or transaction time sampling

As emphasized by Hasbrouck (2006), microstructure time series are distinctive because market data are discrete events realized in continuous time, so-called point processes, and are well-ordered. Macroeconomic time series are typically time aggregated and become positively correlated by construction. On the contrary, tick-by-tick data are well-ordered in time.

For the issue at hand in Hansen and Lunde, that is the influence of microstructure noise on the estimation of model-free realized variance, the distinction between transaction or quotation prices and “artificial” prices constructed from an equidistant calendar time sampling, is certainly of central interest. It is important both with respect to the bias introduced by increased sampling frequency in the measure of realized variance and for its influence on the bias correction by a kernel-type estimator.
In Oomen (2005), the efficient martingale component of the observed price is modeled as a compound Poisson process. In this setting, expressions for the bias and the mean-squared error of the bias-corrected realized variance (through the same Newey and West procedure as in this article) are available in closed-form. He shows that the bias correction allows to increase considerably the optimal sampling frequency and produces a sizable reduction in MSE of the realized variance. This is comparable to the results presented in this article. More interestingly though, transaction time sampling delivers a superior measure relative to calendar time sampling. The fact that the price process is sampled in transaction time reduces the MSE by about 20%. This is not apparent in the results of this article. Oomen attributes the error reduction most notably to the fact that returns sampled in transaction time are devolatized through an appropriate deformation of the time scale. In transaction time sampling, the time scale is deformed by the expected and realized number of transactions. This suggests an adjustment to the procedure for the computation of realized variance with calendar time sampling. Indeed, the main information missing in calendar time sampling is the level of activity in the market, and therefore the quantity of information potentially incorporated in the efficient price. Instead of weighting equally each time interval one can think of weighting the time intervals by the number of transactions that occurred during the intervals. This idea could serve also to construct a noise-to-signal ratio $\lambda$ that could vary from day-to-day according to the level of activity in the market.

We end this subsection with one remark about Lemma 1 in HL. This lemma suggests that one should avoid the linear-interpolation method. Maybe this is a good advice, but we find that Lemma 1 does not provide a theoretical foundation for such a recommendation. The critical point in the proof of Lemma 1 is the assumption on $N$, the number of trades, which is assumed constant. This assumption is quite peculiar. For instance, one should ask the same question, i.e. what is the limit of $RV^{(m)}$ as $m \to \infty$, if one uses the previous-tick method and maintains $N$ constant? It is certainly not the quadratic variation!

2.4 Durations

As noted in the introduction of our comment, the durations, i.e. the time between consecutive trades, contain information about the volatility. This was clearly established in Engle (2000). Consequently, it is of interest to extract this information. Of course, it might be difficult to do it in a nonparametric setting but even results in a parametric world will be useful to understand the interaction between volatility and the transaction process. Some preliminary and interesting results are provided by Renault and Werker (2004) in a semiparametric world who clearly showed that a causality between the durations and the variance has an impact on the characteristics of integrated variance, in particular its expectation (and therefore its forecast).
2.5 The loss function

The HL paper, as well as several papers in the literature, focused on the mean square error of the realized variance measures. However, in several economic examples, the object of interest is not the variance but a nonlinear function of it. For instance, in the context of optimal portfolio analysis, the object of interest is the inverse of the volatility. Of course, when one has a consistent estimator and a central-limit theorem, the Delta method allows one to study any nonlinear function of the variance.

2.6 Estimation of the optimal frequency

Here we focus on the empirical application of Corollary 6, in particular the estimation of the optimal frequency for an estimator of the integrated variance (e.g. $RV^{(m)}$ or $RV^{(m)}_{AC1}$).

As pointed out in the paper, an approximation of the optimal frequency for the estimator $RV^{(m)}_{AC1}$ is $m^*_1 \approx \sqrt{3}(2\omega^2/IV)^{-1}$ where $\omega^2$ is the variance of the process $u_t$. Consequently, the optimal length of returns is $\frac{2\omega^2}{\sqrt{3} IV}$. Two main remarks are in order. First, given that the integrated variance is time-varying, the optimal length/frequency of observations is time-varying. Second, the optimal length/frequency is a function of the object of interest, i.e. the integrated variance. Therefore, the estimation of this length/frequency is difficult.

The current literature followed two approaches to handle this problem. The first approach consists in estimating the daily integrated variance at a lower frequency where microstructure noise is small. However, at such frequencies, the discretization noise is quite important; see for instance Barndorff-Nielsen and Shephard (2002), Meddahi (2002), and Gonçalves and Meddahi (2005).

The second approach consists in estimating the empirical mean of the realized variance (computed again at a low frequency) over several days. In other words, this approach ignores the time-variation in the integrated variance. In so doing, this approach neglects the effect of Jensen’s inequality. The daily optimal length is proportional to $IV^{-1}$ and not $IV$. Therefore, one gets a bias. Actually, given that the function $f(x) = x^{-1}$ is convex for $x > 0$, one will underestimate the optimal length of returns. A similar analysis maybe done for the other measures.

In a Monte Carlo experiment, we quantify the magnitude of the bias. We consider two models with no drift and leverage effect from Bollerslev and Zhou (2002):

\[ dp^*_t = \sigma_t dW_t, \quad d\sigma_t^2 = 0.03(0.25 - \sigma_t^2)dt + 0.1\sigma_t dW_{1,t}, \]

and

\[ dp^*_t = \sigma_t dW_t, \quad \sigma_t^2 = \sigma_{1,t}^2 + \sigma_{2,t}^2, \quad d\sigma_{1,t}^2 = 0.5708(0.3257 - \sigma_{1,t}^2)dt + 0.2286\sigma_{1,t} dW_{1,t}, \]

\[ d\sigma_{2,t}^2 = 0.0757(0.1786 - \sigma_{2,t}^2)dt + 0.1096\sigma_{2,t} dW_{2,t}, \]
The results based on 5,000 replications are reported in Table 1. We provide the mean, the standard deviation as well as some quantiles of $IV_t$ and $IV_{t-1}$. There is a clear and non negligible Jensen’s effect. For instance, if one considers Model 2 that was estimated by Bollerslev and Zhou (2002) on DM-$ exchange rates where the sample period extends from December 1, 1986 through December 1, 1996, $(E[IV_t])^{-1}$ equals 1.98 while $E[IV_{t-1}]$ equals 2.20.

### 2.7 Cointegration

A way to improve the quality of the efficient price component is to use several prices - transaction, ask or bid quote, mid quote - to extract the efficient price and noise. In this article, Hansen and Lunde propose to use a cointegration method to decompose the prices into a common stochastic trend and transitory components. Therefore, one can evaluate the contributions of innovations in the various price series to the efficient price series. The results confirm the importance of the information content in transaction prices, at least for the securities on the New York Stock Exchange (a specialist market), since larger instantaneous correlations between the transaction price and the efficient price are observed.

The methodology used by Hansen and Lunde is based on Hasbrouck (1995) who uses prices of the same security on several exchanges to better estimate the efficient price. Another way to improve the quality of the efficient price component is to use information on several securities through correlations between the efficient prices or the microstructure noises. A multivariate analysis of efficient prices and noises raises at least two issues: synchronization and specification of the variance-covariance matrices. Bandi and Russell (2005c) avoids the first problem by constructing equally-spaced continuously-compounded returns using mid-quotes and the previous tick method; see Hayashi and Yoshida (2005) for a different approach. Moreover Bandi and Russell (2005c) make simplifying assumptions about the correlations over time and across securities. One important characteristic of the covariance matrix is that it should be semi-definite positive. As pointed out in Section 2.2, the estimator of Bandi and Russell (2005c) does not impose by construction this property.

Also, as pointed out earlier for a different issue, there are some identification problems when one wants to disentangle the efficient price from the noise. Assumptions are needed. The assumptions made in the cointegration part are not clear, at least to us. More details may be helpful.
3 Time series properties of realized variance under microstructure noise

3.1 Dynamics of the realized variance under microstructure noise

Barndorff-Nielsen and Shephard (2002) and Meddahi (2003) studied the time series properties of the realized variance when there is no microstructure noise. More precisely, these authors derived the state space and ARMA representations of the realized variance when one specifies the process of the instantaneous variance as a linear combination of autoregressive processes. We study now these representations under the presence of microstructure noise.

For simplicity, we assume that the microstructure noise $u_t$ is i.i.d.; the rest of this subsection provides results given in Andersen, Bollerslev and Meddahi (2005) who showed:

**Proposition 1** Let $J_t \equiv \sigma(p, f, u, t, \tau \leq t, \tau \in \mathbb{R})$ where $f$ is the Markov state variable that drives the volatility process $\sigma_t$, then

\[
E[RV_{t+1}(h) | J_t] = E[RV^*_{t+1}(h) | J_t] + V_u \left( \frac{2}{h} - 1 \right) + u_t^2
\]

$\forall n \geq 2$, \hspace{1mm} $E[RV_{t+n}(h) | J_t] = E[RV^*_{t+n}(h) | J_t] + \frac{2V_u}{h}$

\[
Var[RV_t(h)] = Var[RV^*_t(h)] + 2V_u^2 \left( \frac{2K_u}{h} - K_u + 1 + 4 \frac{E[\sigma^2_t]}{V_u} \right)
\]

\[
Cov[RV^*_{t+1}(h), RV^*_t(h)] = Cov[RV_{t+1}(h), RV_t(h)] + (K_u - 1)V_u^2
\]

$\forall n \geq 2$, \hspace{1mm} $Cov[RV^*_{t+n}(h), RV^*_t(h)] = Cov[RV_{t+n}(h), RV_t(h)]$

where $V_u = Var[u_t^2]$ and $K_u = E[u_t^4]/(E[u_t^2])^2$.

In this subsection, we will focus on the ARMA representation of the realized variance. Equation (13) implies that the covariance of the true realized variance and the observed one have quite similar ARMA representations in the settings of Barndorff-Nielsen and Shephard (2002) and Meddahi (2003): the orders of the ARMA are the same, as well as the autoregressive roots. However, there is a difference in the moving average roots, as well as between the unconditional means (which is equal to $2V_u h^{-1}$; see Eq. (10)). There is also an important difference between the variance of the two variables. In addition, the variance of the noisy realized variance goes to infinity when $h \to 0$ given that the quadratic variation of the noise is unbounded (see Eq. (11)). This suggests that the autocorrelation function of the noisy realized variance will vanish to zero when $h \to 0$.

Table 2 provides these ACF for the DM/$ spot rates that were used by Andersen, Bollerslev and Diebold (2005). The results on the S&P500 and the US T-Bond futures are similar and are not reported. For the DM/$, $h=288$ corresponds to 5 minutes, $h=144$ to
10 minutes, and so on. Even though we are not providing a formal test, a look at Table 2 suggests that the i.i.d. assumption of the microstructure noise is not supported by the data. In particular, it is not at all clear that the ACF goes to zero when $h \to 0$. Actually, for a given lag, the autocorrelation increases when $h$ decreases, i.e., as in the non microstructure case, the persistence of the instantaneous variance dominates when $h$ decreases.

The calculations we derived here can be extended to the non i.i.d. case. In particular, it is clear that the dynamics of the noisy realized variance contains information about the dynamics of the microstructure noise. In other words, it will be interesting to explore this avenue in future research.

### 3.2 Forecasting integrated variance based on realized variance under microstructure noise

Andersen, Bollerslev, and Meddahi (2004) gave an analytical explanation for the empirical finding of Andersen, Bollerslev, Diebold and Labys (2003) who showed that realized variance can be well forecasted. The work of Andersen, Bollerslev, and Meddahi (2004) was based on analytical formulas for the autocovariance functions of the realized variance under non microstructure noise when one assumes that the data generating process is the eigenfunction stochastic volatility model of Meddahi (2001). Proposition 1 allows Andersen, Bollerslev, and Meddahi (2005) to extend their work to the presence of microstructure noise.

As shown in the previous proposition, the noisy realized variance is less predictable than the realized variance. In addition, the empirical ACF of the noisy realized variance are clearly high. Therefore, the predictability of realized variance and integrated variance is higher than suggested by Andersen, Bollerslev, Diebold and Labys (2003). Of course, some problems have to be fixed. For instance, the forecast is biased given that the means of the noisy realized variance and realized variance are different. Again, a simple approach to fix this problem is to adjust the mean.

Of course, there are several estimators of the integrated variance under microstructure noise, like those that minimize the mean square error of the estimator or those that are consistent. Andersen, Bollerslev and Meddahi (2005) are also studying their forecasting power; in particular, results like those in Proposition 1 will be derived for these estimators for i.i.d. noise as well as stationary and serially correlated noise.

### 3.3 Estimation of continuous time stochastic volatility models based on realized variance under microstructure noise

the QML combined with the Kalman filter. In contrast, Bollerslev and Zhou (2002) derived some moments fulfilled by the integrated variance when one assumes a square-root process for the variance and then used an instrumental variable method to estimate the parameters by using the realized variance data instead of the unobservable integrated variance data.


In Feunou-Kamkui et al. (2005), we are exploring in detail the impact of the presence of microstructure noise on estimating continuous time models (by the Kalman filter as in Barndorff-Nielsen and Shephard (2002) and by GMM as in Bollerslev and Zhou (2002)). We obviously find an impact, but not very important, that one can easily explain. We showed above that the microstructure noise implies that the noisy realized variance has the same autoregressive part as the non-noisy one, but with a different unconditional mean and moving-average part. The autoregressive part identifies the mean reverting parameters and is therefore not sensitive to the microstructure noise. The unconditional mean can be easily adjusted. The moving average part of the noisy realized variance is a function of the mean reverting parameters, the unconditional mean, and the variance of the variance parameters. As a consequence, the variance of the variance is overestimated. However, the variance of the variance is not well estimated even in the absence of microstructure noise (see Barndorff-Nielsen and Shephard (2002) and Bollerslev and Zhou (2002)).

4 The economic importance of microstructure noise

An important motivation invoked by Bandi and Russell (2005c) for a multivariate analysis is a portfolio allocation problem. This raises the issue of the economic significance of optimally sampling the returns at higher frequencies than say the 5-minute or 15-minute sampling that has typically been used in the literature. It is a central issue that is not analyzed in the present article. Their article uses the methodology of Fleming et al. (2001, 2003) to measure the fees that an investor would be willing to pay to switch from a covariance forecast based on 5-minute or 15-minute sampling to optimally-sampled realized covariances. They consider an investor who uses a conditional mean-variance optimization rule to allocate funds in several classes of assets with daily rebalancing. Given a mean return target, the gains in variance reduction will be translated into an economic quantity by simply scaling them by a risk aversion factor. One needs a high risk aversion parameter and a high mean return target to arrive at economic significant fees (at most 67 basis points per year with a mean target of
Another economic application for realized variance has been the extraction of a volatility risk premium. This has been proposed by Garcia, Lewis and Renault (2001). The methodology is based on estimating jointly the objective and risk-neutral parameters of stochastic volatility diffusion models. This procedure is based on series expansions of option prices and implied volatilities and on a method-of-moment estimation that uses analytical expressions for the moments of the integrated variance and realized variance measures. Recently, Bollerslev et al. (2004) have followed a similar course with nonparametric measures of implied volatility instead of series expansions. Once these parameters are estimated one can imagine an application of the stochastic volatility model to say pricing options and one can measure the economic impact of using various frequencies for computing realized variance. However, the frequency at which these applications are conducted are typically a month, since a main interest is to compare the dynamics of the risk premium to several macroeconomic variables. Therefore the statistical refinements developed in Hansen and Lunde cannot be thought as very useful for these types of applications.

The potential economic importance of the statistical refinements proposed by Hansen and Lunde and others ought to be gauged in market microstructure models. Hasbrouck (2006) provides a list of important questions in market microstructure, such as the incorporation of information in prices, the relationship between market structure and the valuation of securities, and the mechanisms to better aggregate information into prices. In these models, it is often the case that sampling is determined by information arrivals. Recently, Owens and Steigerwald (2005) develops a microstructure model in order to estimate the frequency and quality of private information. The frequency of private information is measured by the potential revelation of the information at each arrival. In our view, it is more in these settings that the potential gains associated with high frequency sampling and with bias corrections must be assessed.
Appendix

One can show that:

• If \( h \leq s_0 \), then

\[
hf(h) = \frac{1}{2k_1} \left[ \exp(2k_1 h) - 1 \right] + \frac{1}{2k_2} \left[ 1 - \exp(-k_1 h) \right]^2 \left[ \exp(2k_1 s_0) - \exp(2k_1 h) \right] \\
+ \frac{1}{2k_2} \exp(2k_1 s_0) \left[ 1 - \exp(-2k_2 h) \right] \\
- \frac{2}{k_1 - k_2} \exp(2k_1 s_0) \left[ \exp(-k_2 h) - \exp(-k_1 h) \right] \\
+ \frac{1}{2k_1} \exp(2k_1 (s_0 - h)) \left[ \exp(2k_1 h) - 1 \right] \\
+ \frac{1}{2k_2} \exp(2k_1 s_0 - 2k_2 h) \left[ 1 - \exp(k_2 h) \right]^2 ;
\]

• if \( s_0 \leq h \), then

\[
hf(h) = \frac{1}{2k_1} \left[ \exp(2k_1 s_0) - 1 \right] + \frac{1}{2k_2} \exp(2(k_1 + k_2) s_0) \left[ \exp(-2k_2 s_0) - \exp(-2k_2 h) \right] \\
+ \frac{1}{2k_2} \exp(2(k_1 + k_2) s_0 - 2k_2 h) \left[ 1 - \exp(-2k_2 s_0) \right] \\
- \frac{2}{k_1 - k_2} \exp(k_1 s_0 - 2k_2 h) \left[ \exp(k_1 s_0) - \exp(k_2 s_0) \right] \\
+ \frac{1}{2k_1} \left[ \exp(2k_1 s_0) - 1 \right] \\
+ \frac{1}{2k_2} \exp(2k_1 s_0 - 2k_2 h) \left[ 1 - \exp(k_2 h) \right]^2 .
\]

• In addition, one has

\[
Var(u_t) = \frac{1}{2} \left( \frac{1}{k_1} + \frac{1}{k_2} \right) \exp(2k_1 s_0) - \frac{1}{2k_1}.
\]

• Finally, if \( h \leq s_0 \), then

\[
cov[u_t, u_{t-h}] = \frac{1}{2k_1} \left[ \exp(k_1 (2s_0 - h)) - \exp(k_1 h) \right] \\
+ \frac{1}{k_1 - k_2} \left[ \exp(2k_1 s_0 - 2k_2 h) - \exp(2k_1 s_0 - k_1 h) \right] \\
+ \frac{1}{2k_2} \exp[2k_1 s_0 - k_2 h] ;
\]

• if \( s_0 \leq h \), then

\[
cov[u_t, u_{t-h}] = \frac{1}{k_1 - k_2} \exp(s_0 (k_1 + k_2) - 2k_2 h) \left[ \exp(s_0 (k_1 - k_2)) - 1 \right] \\
+ \frac{1}{2k_2} \exp[2k_1 s_0 - k_2 h] .
\]

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### Table 1: Characteristics of $IV_t$ and $IV_t^{-1}$

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<th>Median</th>
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### Table 2: ACF of the DM/$ realized variance

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References


Figure 1a

scenario 1: $k_1 = 3.1$, $k_2 = 2.9$

Figure 1b

scenario 1: $k_1 = 3.1$, $k_2 = 2.9$
Figure 2a
scenario 2: \( k_1 = 500, k_2 = 500.1 \)

Figure 2b
scenario 2: \( k_1 = 500, k_2 = 500.1 \)
Figure 3a

scenario 3: k1 = 1000, k2 = 1000.1

Figure 3b

scenario 3: k1 = 1000, k2 = 1000.1