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Simulation Methods for Lévy-Driven Continuous-Time Autoregressive Moving Average (CARMA) Stochastic Volatility Models

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We develop simulation schemes for the new classes of non-Gaussian pure jump Lévy processes for stochastic volatility. We write the price and volatility processes as integrals against a vector Lévy process, which makes series approximation methods directly applicable. These methods entail simulation of the Lévy increments and formation of weighted sums of the increments; they do not require a closed-form expression for a tail mass function or specification of a copula function. We also present a new, and apparently quite flexible, bivariate mixture-of-gammas model for the driving Lévy process. Within this setup, it is quite straightforward to generate simulations from a Lévy-driven continuous-time autoregressive moving average stochastic volatility model augmented by a pure-jump price component. Simulations reveal the wide range of different types of financial price processes that can be generated in this manner, including processes with persistent stochastic volatility, dynamic leverage, and jumps.

KEY WORDS: Diffusions; Lévy process; Quadratic variation; Realized variance; Simulation; Stochastic volatility.

1. INTRODUCTION

Modeling the evolution of a financial price series as forced by a stochastic volatility process has a long history in financial econometrics. In most models, the underlying driving process(es) are locally Gaussian, or possibly locally Gaussian with occasional rare jumps, and positivity of the volatility process is ensured by the functional form assumptions. Recently, Barndorff-Nielsen and Shephard (2001a, b) suggested a completely new class of models, termed non-Gaussian Ornstein–Uhlenbeck (OU) models, in which the driving process for a volatility factor is a pure-jump Lévy process with non-negative increments; simple parametric sign restrictions ensure positivity. Brockwell (2001a, b) and Brockwell and Marquardt (2005) introduced a generalization to the Lévy-driven continuous-time autoregressive moving average (CARMA) class of volatility models. These newer classes of models based on more general Lévy processes can be expected to supplant traditional Brownian-based processes in serious efforts to model the movements of financial price data at very high frequencies. Tauchen (2004) reviewed the older classes of models and discussed some of the issues related to data and estimation methods for the newer classes of models.

Regardless of the estimation technique, simulation clearly will play a crucial role in the implementation of these newer classes of processes. With Bayesian methods, for example, simulation is part of the scheme to integrate out unobserved variables, including the unobserved values of the process between the sampling points. In frequentist likelihood-based approaches, simulation in conjunction with a cleverly chosen importance function can, in certain problems, make evaluation of the likelihood practicable. In method-of-moments-based approaches, simulation is used to evaluate predicted moments under the model, which are then compared with sample moments using a chi-squared criterion.

In this article we develop and assess practical schemes to simulate from Lévy-driven models for financial price dynamics. In the models considered herein, the volatility dynamics are governed by the Brockwell-style CARMA extension of the Barndorff-Nielsen and Shephard non-Gaussian OU setup. The returns process also contains a jump component, which is correlated with the jump innovations in volatility to accommodate the so-called “leverage effect.” We use simulation schemes based on series expansions that turn out to be considerably simpler to implement than schemes based on the tail mass function. The convenience of the series expansion is especially apparent in a bivariate (or multivariate) situation, because the need to determine the tail mass functions and copula function is completely circumvented. We introduce a two-dimensional mixture-of-gammas Lévy process that is extremely flexible while at the same time preserving positivity for the volatility Lévy increments and generating leverage-type correlations between the volatility increments and price jump increments.

The remainder of the article is organized as follows. Section 2 sets forth some notation and recalls some basic properties of the Lévy process. Section 3 sets out the series approximations to Lévy processes and also shows how to adapt the series approximations to simulate Lévy-driven processes as well as to assess accuracy. Section 4 develops the flexible mixture-of-gammas process, and Section 5 casts the material into a stochastic volatility framework. Section 6 contains examples illustrating the flexibility of the simulation schemes and the realistic financial price dynamics that can be generated with judicious choice of the parameters; it also discusses challenges in performing estimation with such processes on very high-frequency data. Section 7 contains some concluding remarks.
2. LÉVY PROCESSES

In this section we review some basic facts associated with the Lévy processes that we refer to throughout the article. (For more details, see Sato 1999; Bertoin 1996.)

Intuitively, the Lévy process can be described as a continuous-time analog of the random walk in discrete time. The following definition of the Lévy process is taken from Sato (1999): A stochastic process \( \{L_t : t \geq 0\} \) on \( (\Omega, \mathcal{F}, P) \) taking values in \( \mathbb{R}^d \) is a Lévy process if the following conditions are satisfied:

1. It has independent increments.
2. \( L_0 = 0 \) a.s.
3. The increments of the process are strictly stationary.
4. It is stochastically continuous.
5. There is \( \Omega_0 \in \mathcal{F} \) with \( P(\Omega_0) = 1 \) such that for every \( \omega \in \Omega_0 \), \( L_t(\omega) \) is càdlàg (i.e., right-continuous with left limits).

The last condition says that the Lévy process has a càdlàg version; this is the version of the process that we use throughout. If this condition is dropped, then the resulting process is the Lévy process in law.

There is an intimate link between the infinite divisible distributions and the Lévy processes in law. If \( \{L_t : t \geq 0\} \) is a Lévy process in law, then the distribution of \( L_t \) for every \( t \geq 0 \) is infinitely divisible. The converse is also true. For every infinitely divisible distribution \( \mu \), there exists a Lévy process in law \( L(t) \), such that the distribution of \( L(1) \) is equal to \( \mu \). Therefore, using the Lévy–Khinchine representation of the characteristic function of infinitely divisible distribution, we can write the following for the characteristic function of a \( d \)-dimensional Lévy process, \( L(t) \):

\[
E[e^{i\varepsilon \mathcal{L}(t)}] = e^{\ell(\varepsilon)}.
\]

where

\[
\ell(\varepsilon) = -\frac{1}{2} \varepsilon^T A \varepsilon + i \varepsilon^T a
+ \int_{\mathbb{R}^d} (e^{i\varepsilon y} - 1 - i\varepsilon^T y 1_{\{|y| \leq 1\}}) v(dy)
\]

and \( A \) is a symmetric positive semidefinite matrix. The measure \( v \), called the Lévy measure, on \( \mathbb{R}^d \) satisfies

\[
\int_{\mathbb{R}^d} (|y|^2 + 1) v(dy) < \infty.
\]

The characteristic triplet \( (A, a, v) \) of the measure \( \mu \) completely determines the Lévy process \( L \). The representation in (2) is not unique. Many other truncation functions could be used besides the \( 1_{\{|y| \leq 1\}} \) used here (see Sato 1999, p. 38, for details).

Two specific cases of the Lévy process are as follows:

1. \( v = 0 \). In this case the process reduces to Brownian motion and thus has a continuous version.
2. \( A = 0 \). In this case the process is pure jump.

Every other Lévy process is a combination of these two. The continuous part of every Lévy process is the Brownian motion, which has unbounded variation and quadratic variation proportional to time. The pure jump part of every Lévy process is of finite activity when \( \nu(\mathbb{R}^d_0) < \infty \) and of infinite activity when \( \nu(\mathbb{R}^d_0) = \infty \). The finitely active pure jump Lévy processes (also known as compound Poisson) jump at most a finite number of times in every finite interval almost surely, whereas the infinitely active jump processes jump an infinite number of times on finite intervals. Further, the set of infinitely active pure jump processes can be subdivided into those with finite variation and those with infinite variation. For a pure jump process to be of finite variation it is necessary and sufficient that \( \int_{|y| \leq 1} |y| v(dy) < \infty \) (see Sato 1999, thm. 21.9). Intuitively, the pure jump processes of finite variation are characterized by the property that their trajectories are of finite length over finite intervals almost surely, unlike those of infinite variation.

Because we use Lévy processes to directly model the stochastic volatility, we are interested in those that are increasing; that is, for \( d = 1 \), \( L_t(\omega) \) is an increasing function of \( t \). Such Lévy processes are called Lévy subordinators. For all Lévy subordinators, \( v((-\infty, 0]) = 0 \) and \( \int_{|y| \leq 1} y v(dy) < \infty \) (see Sato 1999, thm. 21.5).

3. SERIES REPRESENTATION OF LÉVY PROCESSES AND THEIR SIMULATION

Here we introduce the series representation of pure-jump Lévy processes. This series representation offers a convenient way to simulate integrals with respect to these Lévy processes. Only in certain cases is the transition density of the Lévy process known in closed form (see, e.g., Li, Wells, and Yu 2004 and references therein). The series representation of Lévy processes offers a particularly useful way to simulate Lévy processes, given the fact that in most cases the law of the increments of the Lévy process is not known in closed form. Its implementation requires only a shot noise decomposition of the Lévy measure of the process (explained later) without the need for the transition density, which often will not be known explicitly and is rather expensive to evaluate numerically. Furthermore, working with the Lévy measure and its series representation is particularly useful for modeling and simulating dependence across multiple processes flexibly without imposing the extreme cases of perfectly dependent or independent arrival times. Cont and Tankov (2004, chap. 5) noted the importance of working directly with the Lévy measure in the multivariate case. Thus the series-based methods developed here provide a way to explore a much wider variety of interesting Lévy processes for financial applications.

3.1 General Theory

The fundamental result, which we make use of throughout the article, is the generalized shot noise method for series representation of infinitely divisible distributions, introduced by Rosiński (2001). Here we state in a theorem the implications for the series representation of Lévy processes. (For a proof of this theorem, see Rosiński 2001.)

**Theorem 1.** Let \( \{\Gamma_t : t \geq 1\} \) be a sequence of arrival times of a standard (unit intensity) Poisson process; that is, \( \{\Gamma_t : t \geq 1\} \) is the sequence of partial sums of standard exponential random variables. In addition, let \( \{V_t : t \geq 1\} \) be a sequence of iid random
variables with distribution \( F \) in a measurable space \( S \) and let \( \{ U_i \}_{i \geq 1} \) be a sequence of iid uniform on \([0, 1]\) random variables such that \( \{ \Gamma_i \}_{i \geq 1}, \{ V_i \}_{i \geq 1} \) and \( \{ U_i \}_{i \geq 1} \) are independent of one another. Let \( H(r, v) \) be a measurable function \( H: (0, \infty) \times S \to \mathbb{R}^d \).

In addition, make the following notation:

\[
\psi(r, B) = \mathbb{P}(H(r, V_i) \in B), \quad r > 0, B \in \mathcal{B}(\mathbb{R}^d),
\]

\[
A(s) = \int_0^s \int_{|x| \leq 1} x \psi(r, dx) dr \quad \text{for} \quad s \geq 0,
\]

and assume that the measure \( v \) can be decomposed as

\[
v(B) = \int_0^\infty \psi(r, B) dr, \quad r > 0, B \in \mathcal{B}(\mathbb{R}^d).
\]  

(a) If \( a := \lim_{s \to \infty} A(s) \) exists in \( \mathbb{R}^d \) and \( v \) is a valid Lévy measure, then

\[
\sum_{i=1}^{\infty} H(\Gamma_i, V_i) 1_{\{U_i \leq 0\}} \quad \text{(5)}
\]

converges almost surely, uniformly in \( t \) on \([0, 1]\) to a Lévy process with characteristic triplet \((0, a, v)\).

(b) If \( v \) is a valid Lévy measure, \( |H(r, v)| \) is nonincreasing in \( r \), and \( c_i = A(i) - A(i - 1) \), then

\[
\sum_{i=1}^{\infty} (H(\Gamma_i, V_i) 1_{\{U_i \leq 0\}} - ic_i) \quad \text{(6)}
\]

converges almost surely, uniformly in \( t \) on the interval \([0, 1]\) to a Lévy process with characteristic triplet \((0, 0, v)\).

This theorem provides a way to represent pure-jump Lévy processes and integrals with respect to them. All we need to do is find a shot noise decomposition of the corresponding Lévy measure. Many of the existing methods for simulating Lévy processes, such as the inverse Lévy measure method of Khintchine (Ferguson and Klass 1972) and the standard way of simulating compound Poisson processes, are special cases of shot noise decomposition with particular choices for \( H(r, v) \) and \( v \). More examples have been given by Rosiński (2001). Every Lévy process can have different shot noise decompositions of its Lévy measure. The convenience of every method will be determined by how easy it is to simulate from the distribution \( F \) and how fast \( H(r, v) \) can be evaluated. When the process is of finite variation, the first condition in part a of Theorem 1 is automatically satisfied. Note, however, that this condition could also be satisfied for processes of infinite variation. This will be the case when their Lévy measure is symmetric. An example would be a type G Lévy process (i.e., one that could be presented as Brownian motion subordinated by nonnegative Lévy processes; see Rosiński 1991 for their series representation). In the event that the first condition in part a of the theorem fails, it is still possible to derive series representation for the Lévy process; however, in this case the sum must be centered appropriately, as is shown in part b.

The shot noise decomposition of the Lévy measure in Theorem 1 could be used for deriving the series representation of integrals with respect to Lévy processes. In this article we are interested in the following integrals:

\[
X(t) = \int_0^t f(s^-) \, dL(s),
\]

where \( f: \mathbb{R}^+ \to \mathbb{R} \) is a càdlàg and bounded deterministic function and \( L(t) \) is a one-dimensional Lévy process of finite variation with the characteristic function

\[
\mathbb{E}(e^{i\xi L(t)}) = \exp \left( t \int_\mathbb{R}_+ (e^{i\xi y} - 1) v(dy) \right),
\]

where \( \nu \) satisfies (4). Using Theorem 1, we define the following approximations of the Lévy process \( L(t) \) and the integral with respect to it \( X(t) \):

\[
L^r(t) = \sum_{\Gamma_i \leq t} H(\Gamma_i, V_i) 1_{\{U_i \leq 0\}}
\]

and

\[
X^r(t) = \sum_{\Gamma_i \leq t} f(U_i) H(\Gamma_i, V_i) 1_{\{U_i \leq 0\}}.
\]

Using the result in Theorem 1, it is easy to see that the approximations \( L^r(t) \) and \( X^r(t) \) converge almost surely and uniformly in \( t \) on the interval \([0, 1]\) to \( L(t) \) and \( X(t) \).

The approximations \( L^r(t) \) and \( X^r(t) \) involve a random number of terms (but on average they will be \( \tau \)). This way of truncating the infinite series in (5) has the advantage that the approximation \( L^r(t) \) is itself a Lévy process of finite activity with Lévy measure \( \int_0^r \psi(r, \cdot) \, dr \).

In general, the approximation given in (8) does not necessarily truncate the small jumps. However, when \( H(r, v) \) is nonincreasing in \( r \), this will be true “on average”; that is, the approximation error, which is itself a jump process, will allocate less and less mass on bigger jumps.

The approximation error in the case in which the truncation involves cutting exactly the small jumps of the Lévy process (which will be the true in the case of the inverse Lévy measure method) was analyzed by Asmussen and Rosiński (2001); see also the work of Wiktorsson (2002) for the integrals with respect to type G Lévy processes.

### 3.2 Practical Implementation and Numerical Error

We now show the implementation of the foregoing Lévy shot noise decomposition for the relatively simple case of a gamma process with Lévy measure given by

\[
v(dy) = c e^{-\lambda y} \mathbb{I}(y > 0) \, dy,
\]

where \( \lambda \) is a scale parameter and \( c \) controls the overall intensity of the process, which is infinitely active. For the gamma process, a very convenient choice for the function \( H(\Gamma, V) \) in the representation (5) is

\[
H(\Gamma, V) = \frac{V}{\lambda} e^{-\Gamma/c},
\]

where \( V \) is standard exponential. We illustrate how to approximate a realization \( L(t), t \in [0, 1] \), and the integral

\[
X(t) = \int_0^t e^{-\rho s} \, dL(s), \quad t \in [0, 1],
\]
where the kernel function in (7) is \( f(s) = e^{-\rho s} \).

Table 1 shows an actual working FORTRAN 90 code segment written to look like pseudocode easily translatable into another language. For brevity, the table shows only the innermost part of the program that actually generates the approximations \( L^s(t) \) and \( X^s(t) \) in (8) and (9); the code segment is embedded in a larger main program that is available on request. Keep in mind that both \( L^s(t) \) and \( X^s(t) \) are defined over all \( t \in [0,1] \), but on a computer it is only possible to evaluate each over a finite number of points. After executing the code segment in Table 1, the arrays \( \text{levy} \) and \( \text{x} \) consist of \( L^s(t_j) \) and \( X^s(t_j) \) over the equispaced grid \( t_j = j/N, j = 1, 2, \ldots, N \), where \( N \) is the number of grid points, or bins. The process of computing the approximations simply consists of adding the shot noises (or weighted shot noises) to the appropriate bins corresponding to the times in the interval \([0, 1]\) at which the jumps occur.

Given values of \( c, \lambda, \rho \), and the cutoff \( \tau \), along with various control parameters, the code segment works as follows. The first main loop in Table 1 generates a random number \( nshot \) of shot noises, \( H(i) \), and integers, \( \text{bin}(i) \), corresponding to the jump times, where \( nshot \) is the cutoff \( \tau \) on average. The middle part of the code segment is simply checking that sufficient space was allocated and also doing some initialization. The final loops accumulate the shot noises to form \( \text{levy}(j) = L^s(t_j) \) and the weighted shot noises to form \( x(j) = X^s(t_j) \), \( j = 1, 2, \ldots, N \).

Because potential application is simulated method-of-moments estimation, we assess accuracy in that context. In particular, consider estimation through simulation of \( E \left[ \int_0^1 X(s) \, ds \right] \) and \( E \left[ \int_0^1 X(s)^2 \, ds \right] \). The simulation-based estimator would be obtained by generating many replicates of

\[
\frac{1}{N} \sum_{j=1}^{N} X^s(t_j)
\]
and

\[ \frac{1}{N} \sum_{j=1}^{N} [X^2(t_j)] = \] (15)

using the code displayed in Table 1, and then averaging the replications. Figure 1 displays the relative error, (expected – simulation average)/expected, as a function of \( \tau \), for \( \lambda = 1.0 \), with \( \rho \) set so that the half-life is .05, and values \( c = 5.0 \) and \( c = 15.0 \). Closed-form expressions are available for the expected values in (12) and (13), and we used a grid size of \( N = 2,000 \) and 10,000 replications of (14) and (15). Figure 1 suggests that quite accurate approximations can be obtained with rather modest values of \( \tau \), although the minimum \( \tau \) required for a given level of accuracy is higher for the more active process. The figure also indicates that the minimum required value of \( \tau \) can be determined in practice by trying different values and looking for the smallest value for which the computations stabilize.

4. MIXTURE OF GAMMAS LÉVY PROCESSES

4.1 The One-Dimensional Case

The one-dimensional Lévy processes that we use in this article are an extension of the gamma process (10). Our mixture-of-gammas process is a pure-jump, infinitely active and finite variation Lévy process with Lévy measure given by

\[
v(dy) = \left( \frac{c_1 e^{-\lambda_1 |y|}}{|y|} + \frac{c_2 e^{-\lambda_2 |y|}}{|y|} + \cdots + \frac{c_m e^{-\lambda_m |y|}}{|y|} \right) I(y < 0) dy
\]

\[
+ \left( \frac{c_{m+1} e^{-\lambda_{m+1} |y|}}{|y|} + \cdots + \frac{c_{m+n} e^{-\lambda_{m+n} |y|}}{|y|} \right) I(y > 0) dy,
\]

(17)

where \( c_1, c_2, \ldots, c_{m+n} \) and \( \lambda_1, \lambda_2, \ldots, \lambda_{m+n} \) are positive real numbers. Note that in this case, unlike in the gamma process, we allow for the possibility of negative jumps. This Lévy process is a superposition of the pure-jump Lévy processes used by Carr, Geman, Madan, and Yor (2002, 2003) with the tilting parameter set to 0.

For the mixture-of-gammas process, we can use the function

\[
H(\Gamma_i, V_i, J_i) = e^{-\Gamma_i/c} J_i V_i
\]

(19)

as the shot noise in the series representation, where \( V_i \) are iid standard exponential, \( c = c_1 + \cdots + c_{m+n} \), \( \Gamma_i \) are the arrival times of standard Poisson process, and \( J_i \) are iid random variables such that

\[
J_i = \begin{cases} 
-\lambda_1^{-1} & \text{with probability } c_1/c \\
\vdots & \\
-\lambda_m^{-1} & \text{with probability } c_m/c \\
-\lambda_{m+1}^{-1} & \text{with probability } c_{m+1}/c \\
\vdots & \\
-\lambda_{m+n}^{-1} & \text{with probability } c_{m+n}/c.
\end{cases}
\]

Verification of this claim follows from the fact that the measure given in (17) is easily shown to be a valid Lévy measure [i.e., it satisfies the integrability condition (3)] and is of finite variation. The rest of the proof follows directly from Theorem 1, part a. Using the proposed shot noise (19) along with (8) and (9), we can simulate a mixture-of-gammas process and a moving average of it.

4.2 A Bivariate Mixture of Gammas

To accommodate the leverage effect in the stochastic volatility model, we need a two-dimensional pure-jump Lévy process with linked individual processes. One process pertains to the price; the other, to the volatility. To capture the possibility of common jumps and separate jumps, we consider the joint Lévy measure. Intuitively, if the two processes almost always jump together and their jumps are in a constant ratio, then this will manifest itself in the Lévy measure being concentrated predominantly on the set \( \{(x, y) : x = \text{constant} \times y\} \). This is an example of a completely dependent Lévy process, such as those used by Barndorff-Nielsen and Shephard (2001b) or Carr and Wu (2004), among others. Note that the processes jumping in a fixed proportion is one of the many possible instances of complete dependence. Another example is the jumps of one process being recovered from the jumps of the other process by raising them to the third power. If the processes in a two-dimensional Lévy process never jump together, then the Lévy measure is concentrated on the set \( \{(x, y) : xy = 0\} \); in this case, the two Lévy processes are independent.

One approach to capture dependence is to use the Lévy copula introduced by Tankov (2003) and Cont and Tankov (2004) and studied further by Barndorff-Nielsen and Lindner (2004). A possible advantage of this approach is that the dependence between the individual processes captured by the Lévy copula is disentangled from the marginal distributions. However, the simulation from Lévy processes linked by a Lévy copula is not so easy, because it uses the inverse Lévy measure method, and
the inverse of the tail of the Lévy measure is known in closed form in only very few cases.

To bridge between the two extremes of complete dependence and independence, we directly model the joint Lévy measure. Theorem 1 is already stated in multidimensional framework. All we need to do is to specify the Lévy measure in a way that allows us to capture the dependence structure parsimoniously.

We propose the following multidimensional Lévy measure, \( \nu \):

\[
\nu(A) = \int_{\mathbb{R}^d_+} \int_0^\infty \int_0^\infty I_A(e^{-r}v^2)e^{-r}drdv \mu(dz),
\]

(20)

where the measure \( \mu \) on \( \mathbb{R}^d_+ \) satisfies

\[
\int |z| \mu(dz) < \infty.
\]

(21)

It is easy to verify that when the condition in (21) is satisfied, the measure in (20) is a valid Lévy measure of finite variation but infinite activity. It could be shown that this measure is the same as the tempered stable measure introduced by Rosiński (2002) for value of the tempering parameter 0. However, note that the analysis of Rosiński (2002) is restricted only to the case of the tempering parameter taking values in the interval (0, 2), because only in this case could the measure be analyzed as tempering of a stable process (see Rosiński, 2002, 2004).

We specialize the measure \( \mu \) to the following sums of atoms in \( \mathbb{R}^2_+ \):

\[
\mu = c_1 \delta_{(\lambda_{11},0)} + c_2 \delta_{(0,\lambda_{22})} + c_3 \delta_{(\lambda_{11},\lambda_{22})} + \cdots + c_m \delta_{(\lambda_{1m},\lambda_{2m})},
\]

(22)

with \( c_1, \ldots, c_m \) nonnegative numbers. This could be viewed as a generalization of the mixture-of-gamma processes to the bivariate case. The \( \lambda \)'s may be of either sign, because there is no way to separate out the positive and negative jumps in the bivariate case as was done in (17). Note that \( \lambda_{22}/\lambda_{11} \) determines the ratio of the jumps.

We can enforce complete independence of the two individual processes with the measure

\[
\mu = c_1 \delta_{(\lambda_{11},0)} + c_2 \delta_{(0,\lambda_{22})}.
\]

In this case, the measure will be concentrated on the set \( \{(x,y) : xy = 0\} \), and the marginal distribution will be gamma processes. An example of two independent gamma processes is illustrated in Figure 2(a).

The other extreme of complete dependence—with the jumps of the two individual processes in fixed proportion—can be achieved with the measure

\[
\mu = c \delta_{(\lambda_{11},\lambda_{22})}.
\]

In this case, the marginal distributions are again gamma processes. This case is illustrated in Figure 2(b).

In general, we want to permit patterns of dependence between the two individual Lévy processes that are flexible and admit complete independence and dependence as special cases. Following the reasoning for the one-dimensional case, we can specify a series representation in the two-dimensional case where the Lévy measure is specified in (20). In particular, \( H(\Gamma_i, V_i, J_i) \) is given by

\[
H(\Gamma_i, V_i, J_i) = e^{-\Gamma_i/c} V_i J_i,
\]

(25)
The scaled logarithm of the price at time $t$ of the financial price is denoted by $p(t)$; scaling of the logarithm is often used so that the increments represent a geometric return. The general Lévy-driven stochastic volatility model is written as

\[ dp(t) = \mu_p(t) dt + \sigma(t-) dL_{p1}(t) + dL_{p2}(t), \]  

\[ \sigma^2(t) = h(c'X(t)), \]  

\[ dX(t) = a(X(t-), t) dt + b(X(t-), t) dL_X(t), \]

where $L_{p1}(t)$ and $L_{p2}(t)$ are two independent Lévy processes, $L_{p2}(t)$ and $L_X(t)$ are potentially dependent Lévy processes, and $h: \mathbb{R} \rightarrow \mathbb{R}^+$. Equation (28) specifies $\sigma^2(t)$ as driven by an $n$-dimensional vector of factors $X(t)$. The $(n \times 1)$ vector of constants $c$ defines the weights of the factors in the variance. The process $\sigma^2(t)$ is not the variance of the return process, but only part of it. However, the other part, which comes from the variance of the pure-jump component, is not time-varying. Throughout, we keep the term stochastic variance for $\sigma^2(t)$. The vector of factors $X(t)$ driving the stochastic variance of the price process follows a Lévy-driven SDE, specified in (29), where $L_X(t)$ is a $k$-dimensional Lévy process.

Most stochastic volatility models can be viewed as particular cases of this generic model. For example, the model introduced by Barndorff-Nielsen and Shephard (2001a) can be obtained by setting $L_X(t)$ to be pure-jump Lévy subordinator, $L_{p2}(t)$ to be a centered version of it, $L_{p1}(t)$ to be Brownian motion, and the factor $X(t)$ to follow an OU process.

The time-changed Lévy-driven stochastic volatility models for the risk-neutral dynamics of Carr et al. (2003) are also embedded in the foregoing, provided that the time change is of a Brownian motion. In these models the time change is a continuous process, which is locally deterministic with activity rate following an SDE. The models of Carr et al. (2003) are different when the time change is of a pure-jump Lévy process; however, under certain conditions on the Lévy measure (see Jacod 1979), these models are nested in the more general Markov jump diffusions. The activity rate of the economy in these models plays the same role as the volatility in our models here.

Many of the jump-diffusion models in the finance literature can be nested in the generic stochastic volatility model (27)–(29). In these models jumps are rare events and are modeled as finitely active processes. The stochastic volatility is usually driven by Brownian motion; that is, in the framework here, $L_X(t)$ is a Brownian motion. The leverage effect is captured by specifying $L_{p1}(t)$ as another Brownian motion correlated with those driving the factors of the volatility. Among others, the affine jump-diffusion models of Duffie, Pan, and Singleton (2000) with time-homogeneous Lévy measure are nested in this framework.

There are three main features of the stock market returns that every stochastic volatility model should address: (a) volatility clustering and persistence with possible jumps in volatility, (b) jumps in price, and (c) the leverage effect. Here we elaborate on how these features can be captured by the generic stochastic volatility model defined in (27)–(29).

**Volatility Persistence.** The generic stochastic volatility model proposes a multifactor structure for the spot variance, allowing different degrees of persistence in the spot volatility. In addition, the pure-jump Lévy component generates jumps in volatility.

**Jumps in the Price.** We observe the price process only at discrete intervals, so it is natural to ask whether we can disentangle the jump part, if it exists, from the diffusion part. The answer to that question is yes. We can do so using the non-parametric tests proposed by Barndorff-Nielsen and Shephard (2004, 2006a). Using these statistics, Andersen, Bollerslev, and Diebold (2005) and Huang and Tauchen (2006) found considerable evidence for jumps in exchange rate returns and asset returns. Therefore, we model the price process by allowing for jumps in the prices.

**Leverage Effect.** The leverage effect pertains to a negative correlation between the volatility and the return process. It can be captured in the generic stochastic volatility model in various ways. One way is to link the continuous parts in the price and the variance; another way to create leverage effect is to link the jump parts of the price and the variance; and a third way is a combination of the previous two. We only have these three possibilities, because the pure-jump Lévy process is always orthogonal to the Brownian motion. A common approach is to let $L_{p1}(t)$ and $L_X(t)$ have Brownian motions that are correlated. In this way we can produce the leverage effect through the diffusion parts of the variance and the price process. This approach is followed in most of the jump-diffusion literature. An alternative modeling approach is to link the jumps in the price with the jumps in the variance, that is, let $L_{p2}(t)$ and the pure-jump part of $L_X(t)$ be dependent pure-jump processes. Eraker, Johannes, and Polson (2003) introduced a leverage effect through a combination of these two modeling alternatives. Another approach is to capture the leverage effect by linking the pure-jump components of $L_{p1}(t)$ and $L_X(t)$. This could also avoid the need for an additional jump component $[L_{p2}(t)$ here] for capturing the leverage effect; because $L_{p1}(t)$ is specified as a process-containing jump part, we will have jumps in the price. This is similar in spirit to the approach proposed by Carr and Wu (2004) in which the pure-jump Lévy process of the price and the instantaneous business activity rate are linked.

### 5.2 The Lévy-Driven CARMA Stochastic Volatility Model

In this section we adapt the work of Barndorff-Nielsen and Shephard (2001a) and Brockwell (2001a, b) to introduce a Lévy-driven CARMA stochastic volatility model. The model is nested within the generic stochastic volatility introduced in the previous section, and it accommodates jumps in both the price process and the variance. In fact, the variance process is driven solely by jumps.

The price process is modeled as having a diffusion part and a pure-jump component. The pure-jump component is a Lévy process that can be of infinite activity (even infinite variation) and can have positive and negative jumps. The diffusion part of the price displays time-varying variance, and the pure-jump part is of constant variance.

The model is

\[ dp(t) = \mu_p(t) dt + \sigma(t-) dW(t) + dL_p(t), \]

\[ \sigma(D)\sigma^2(t) = b(D) DL_\sigma(t). \]
driving the stochastic variance, \( D \) is a differential operator, and 
\[
a(z) = z^p + a_1 z^{p-1} + \cdots + a_p
\]
and 
\[
b(z) = b_0 + b_1 z + \cdots + b_q z^q, \quad q < p.
\]
In (31), \( \sigma^2(t) \) is a Lévy-driven CARMA\((p, q)\) process of Brockwell (2001a, b). Equivalently, we can write 
\[
\sigma^2(t) = b'X(t),
\]
where \( X(t) \) is a solution to the SDE, 
\[
dX(t) = AX(t) \, dt + e \, dL(t),
\]
and where 
\[
A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-\alpha_p & -\alpha_{p-1} & -\alpha_{p-2} & \cdots & -\alpha_1 \end{pmatrix}, \quad e = \begin{pmatrix} b_0 \\
\vdots \\
0 \\
1 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} b_0 \\
b_1 \\
b_2 \\
\vdots \\
b_{p-2} \\
b_{p-1} \end{pmatrix}.
\]
The foregoing shows that the Lévy-driven stochastic volatility model is nested in the generic framework introduced in the previous section.

If the eigenvalues of \( A \), denoted as \( \zeta_j, j = 1, 2, \ldots, p \), have negative real parts [i.e., \( \Re(\zeta_j) < 0 \) for \( j = 1, 2, \ldots, p \)], then the CARMA\((p, q)\) is a causal stationary process (Brockwell 2001b). In this case, for \( t \) large enough, the contribution of the starting value is negligible and 
\[
\sigma^2_t = \int_0^t g(t - u) \, dL(u), \quad (33)
\]
where \( g(h) \) is the kernel for the corresponding CARMA process and (from Brockwell 2001b) 
\[
g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iu} \frac{b(j\lambda)}{a(j\lambda)} \, d\lambda.
\]
The kernel function \( g(u) \) determines the memory of the process \( \sigma^2(t) \). It gives the weights with which the past observations enter the Lévy functional. Because we are modeling the spot variance of the return process by a CARMA process, we require that its kernel would be nonnegative everywhere. The CARMA approach provides a rich variety of processes that generalize the Lévy-driven OU case [a CARMA\((1, 0)\)] analyzed by Barndorff-Nielsen and Shephard (2001b) and also used by Nicolato and Venardos (2003) for options pricing.

In Section 6.2 we concentrate on a CARMA\((2, 1)\) process for the stochastic variance, which we parameterize as 
\[
a(z) = (z - \rho_1)(z - \rho_2), \quad b(z) = 1 + b_1 z,
\]
for real \( \rho_1 < 0 \) and \( \rho_2 < 0, \rho_1 \neq \rho_2 \). The kernel is 
\[
g(h) = \frac{1 + b_1 \rho_1 e^{\rho_1 h} + \rho_2 e^{\rho_2 h}}{\rho_1 - \rho_2}, \quad h \geq 0 . \quad (34)
\]
An necessary and sufficient condition that guarantees nonnegativity of the kernel is \( 0 \leq \rho_1 \leq \max\left(-\frac{1}{\rho_1}, -\frac{1}{\rho_2}\right) \), as shown in the Appendix.

The kernel, given in (34), will be decreasing for \( \forall h \geq 0 \) if 
\[
b_1 \in \left[-\frac{1}{\rho_1 + \rho_2}, \max\left(-\frac{1}{\rho_1}, -\frac{1}{\rho_2}\right)\right],
\]
whereas for 
\[
b_1 \in \left(0, -\frac{1}{\rho_1 + \rho_2}\right)
\]
the kernel increases initially, reaches a maximum, and then decreases.

This Lévy-driven CARMA\((2, 1)\) setup accommodates the three key properties defined in Section 5.1 as follows.

**Volatility Persistence and Ability of the Volatility to Move Quickly.** The studies of Chernov, Gallant, Ghysels, and Tauchen (2003) and Alizadeh, Brandt, and Diebold (2002) argue in favor of a two-factor structure in the volatility process. One of the factors should slowly mean-revert, allowing for volatility persistence. The second factor, on the other hand, should quickly mean-revert to allow the volatility to move quickly. Motivated by this two-factor structure of volatility, here we propose (following Brockwell 2001b) a Lévy-driven CARMA\((2, 1)\) model for the stochastic variance. One of the autoregressive roots is high in magnitude, corresponding to quick mean reversion, whereas the other is low in magnitude and thus allows for slow mean reversion. This structure of the autoregressive part of the CARMA\((2, 1)\) allows for fairly flexible autocorrelation in the spot variance process. Furthermore, because the driving Lévy process of the CARMA\((2, 1)\) process is a pure-jump process, the Lévy-driven CARMA\((2, 1)\) process could naturally produce the desired effect of quick moves in the variance, as argued by Alizadeh et al. (2002). An alternative approach, which resembles the diffusion factor stochastic volatility models analyzed by Chernov et al. (2003), is the superposition of Lévy-driven OU processes as proposed by Barndorff-Nielsen and Shephard (2001a) and further evaluated empirically in the context of subordinated Levy processes without leverage by Barndorff-Nielsen and Shephard (2006b). This alternative allows more flexible autocorrelation structures for the spot variance over the exponential one implied by the single OU process in a similar way to the CARMA\((2, 1)\) kernel. The potential advantage of CARMA modeling is an even more flexible autocorrelation structure, which is achieved with just a single driving process.

**Jumps in Price.** The model proposed here allows for jumps in the price process. In addition, the price contains a diffusion component, making this model different from the models analyzed by Carr et al. (2003), in which price has no diffusion component. The modeling here resembles the jump-diffusion models. However, unlike the jump-diffusion models, here the jumps in the price need not be rare events, but can be infinitely many in any finite time interval.

**Leverage Effect.** Because in our model the variance is driven solely by a pure-jump process, \( L_0(t) \), the only way that we can capture the leverage effect is by linking the two jump components, \( L_0(t) \) and \( L_0(t) \). That is, the leverage effect in our
model is captured by the jumps in the price process. Note that in our model the jumps in both the variance and in the price need not be rare events (as in the jump-diffusion models); there could be an infinite number of small jumps in a given interval of time. Thus the model here allows for a link between variance and returns, not only in the rare case of large changes in the price, but also for small changes in the price and variance. A similar approach of modeling the leverage effect was taken by Barndorff-Nielsen and Shephard (2001a) and Carr et al. (2003). In these models, however, the jump components in the price and in the variance are perfectly correlated. In terms of the notation adopted here in these models, \( L_p(t) = \text{constant} \times L_o(t) \). This implies that the jumps in the variance and in the price process arrive at the same time and are proportional. In this article we relax the dependence structure between the jumps in the price and in the variance by using the two-dimensional generalization of the mixture-of-gammas process proposed in Section 4.2. As discussed in that section, we allow for various degrees of dependence between the two jump processes; in some cases the jumps arrive at the same time, whereas in other cases they arrive at different times. In addition, in a case in which the jumps arrive at the same time, the jump sizes can be in different proportions.

6. EXAMPLES

In this section we demonstrate through several examples that the proposed stochastic volatility model can produce reasonable dynamics. In practice we observe the price process at discrete intervals. For simplicity, we assume that the observational intervals are equally spaced. We let \( a \) denote the observational interval. Throughout the article, time is measured in terms of number of trading days. In the examples here, we work with 1/2-hour returns and assume an 8-hour trading day, and thus we have that \( a = 1/16 \).

6.1 Simulating From \( \text{Lévy-Driven CARMA} \)
Stochastic Volatility Models

The stochastic volatility model specified in (30) and (31) implies that the geometric return, \( r_o(t) \), over the interval \( (t - a, t] \) is

\[
r_o(t) = p(t) - p(t - a) = \int_{t-a}^{t} \sigma(s^-)dW(s) + L_p(t) - L_p(t - a),
\]

where for now we assume that the drift in (30) is 0. Using the facts that \( \int_{t-a}^{t} \sigma(s^-)dW(s) \) is Gaussian conditional on the pure-jump processes in the price and the variance and that under the \( \text{Lévy-driven} \) model the variance process has no fixed time of discontinuity \( \{\sigma^2(s) = \sigma^2(s^-) \text{ almost surely}\} \), we can write

\[
p(t) - p(t - a) \overset{d}{=} Z(t) \sqrt{\int_{t-a}^{t} \sigma^2(s) ds + L_p(t) - L_p(t - a)},
\]

where \( Z(t) \) is standard normal distribution independent of \( L_p(t) \) and \( L_o(t) \) and \( \int_{t-a}^{t} \sigma^2(s) ds \) is the integrated variance over the time interval \( a \).

Applying the Fubini theorem to the general case (33), the integrated variance is

\[
\int_{t-a}^{t} \sigma^2(s) ds = \int_{t-a}^{t} \int_{0}^{t} g(s-u) dL(u) ds
= \int_{t-a}^{t} \int_{0}^{t} g(s-u) dL(u) + \int_{0}^{t} \int_{t-a}^{t} g(s-u) ds dL(u)
= \int_{0}^{t} g^*(t, u) dL(u),
\]

where the functional form of \( g^* \) can be obtained from that of \( g \).

In the case of CARMA(2, 1),

\[
g^*(t, u) = \begin{cases} 
\phi_1(t-u) \phi_2(t-u) \frac{1 + \rho_1 \rho_2}{\rho_2(\rho_2 - \rho_1)} & \text{if } 0 < u < t - a \\
\phi_2(t-u) \frac{1 + \rho_1 \rho_2}{\rho_2(\rho_2 - \rho_1)} & \text{if } t - a < u < t.
\end{cases}
\]

The equations in (41) can be viewed as extensions of the general case observed by Barndorff-Nielsen and Shephard (2001b, c) that in the non-Gaussian OU model the integrated variance is linear in the \( \text{Lévy} \) innovations, and (42) gives the functional form for the CARMA(2, 1) model. The proposed stochastic volatility model suggests a relatively easy way to simulate from it using the results in Section 3.

Suppose that we want to simulate from the price process at intervals with length \( a \). Conditional on the integrated variance and the increment from the pure-jump component of the price, the price increment is Gaussian with variance equal to the integrated variance. Therefore, once we develop a way to jointly simulate from the integrated variance and the pure-jump component of the price, simulation of the price increments is trivial.

The generic simulation scheme is as follows:

1. Simulate jointly from the two pure-jump \( \text{Lévy} \) processes \( L_p(t) \) and \( L_o(t) \).
2. Generate the implied integrated variance using (42).
3. Simulate the price increment by drawing from a normal distribution with mean 0 and variance equal to the integrated variance computed in the preceding step and adding the increments of \( L_p(t) \) that occur within the interval \( (t - a, t] \).

Given the discussion in Section 3, we know that we are simulating approximately from the model specified earlier and exactly from a stochastic volatility model with pure-jump \( \text{Lévy} \) processes \( L_p(t) \) and \( L_o(t) \) substituted with high-activity compound Poisson processes, which closely resemble the infinite-activity ones. Also note that the proposed simulation scheme does not involve Euler discretization. Finally, it generalizes in a direct way if there is a volatility risk premium and the drift.
in (30) is an affine function of the spot variance, because the cumulated drift is thereby an affine function of the integrated variance.

6.2 The MG–CARMA(2, 1) Model

To illustrate the properties of the proposed simulation schemes, we show the implied dynamics of an MG–CARMA(2, 1) model, a mixture-of-gammas Lévy process that jointly drives a pure jump in the price and a CARMA(2, 1) stochastic volatility specification. We show characteristics of the simulated realizations for four different settings of the parameters as listed in Table 2. In all cases, the simulation was run for 1,000 days (4 trading years) with a burn-in period of 250 days. We work with 1/2-hour returns, which correspond to 16 equally spaced intervals in every trading day. Thus in each of Figures 3–6 there is a total of 16,000 observations on 1/2-hour returns. The tolerance parameter \( \tau \) is fixed at 20, implying that on average 20 jumps are generated per trading day. As seen from Table 2, in each case \( c = \sum c_i \approx 0.2 \). From the analysis of Section 3.2, the value of \( \tau = 20 \) is well above that needed for a reasonable level of accuracy for this value of \( c \) and much larger values as well. This truncation entails discarding jumps of extremely small magnitude, which, in the presence of continuous component in the price process, are immaterial. In all four cases we chose the parameter values so that the variance of the daily returns is equal to 1.00, which is about the variance of the daily S&P 500 Index return, and the drift in the price equation is always identically 0.

The parameters governing the two-dimensional mixture-of-gammas process, which is the background driving Lévy process, are set as follows. The number of terms in the mixture-of-gammas process in (22) is \( m = 4 \). The price and volatility jump independently if \( J_1 \) or \( J_2 \) occurs. If \( J_3 \) or \( J_4 \) occurs, then price and volatility jump together but in different proportions.

For the parameters governing the CARMA(2, 1) kernel, the moving average parameter \( b_1 \) is set equal to .001 in all four cases. Following the discussion in Section 5, \( \rho_2 \) has a very short half-life (.03 days for all cases) and so corresponds to the rapidly mean-reverting autoregressive root. The other root, \( \rho_1 \), is relatively slowly mean-reverting. In the first two cases (Figs. 3 and 4), the half-life is 50 days; in the other two (Figs. 5 and 6),

![Figure 3](image1.png) Simulated Half-Hour Realizations From the MG–CARMA(2, 1) Stochastic Volatility Model With Parameter Setting Specified in Table 2. (a) The spot variance. (b) The Lévy subordinator driving the variance. (c) The pure-jump part of the price process. (d) The half-hour price change.

![Figure 4](image2.png) Simulated Half-Hour Realizations From the MG–CARMA(2, 1) Stochastic Volatility Model With Parameter Setting Specified in Table 2. (a) The spot variance. (b) The Lévy subordinator driving the variance. (c) The pure-jump part of the price process. (d) The half-hour price change.

<table>
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<th>Parameter</th>
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<th>Figure 4</th>
<th>Figure 5</th>
<th>Figure 6</th>
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**NOTE**: Shown are the parameter choices for the CARMA(2, 1) model (34) and those controlling the distribution of the \( J_\tau \) in (26).
the half-life is 10 days. The effects of the persistence of the root $\rho_1$ are quite noticeable in the dynamics of the spot variance. A jump in the driving Lévy process of the CARMA(2, 1) dies off much more quickly when $\rho_1 = 10$ than when $\rho_1 = 50$. The spot variances shown in Figures 3–6 indicate that the MG–CARMA(2, 1) model can produce the persistence of the variance and at the same time its ability to change quickly. This translates automatically in clustering of the volatility, which can be seen most clearly in the return dynamics in Figures 3(d), 4(d), 5(d), and 6(d).

Finally, the parameters in the stochastic volatility model were chosen in such a way that the pure-jump component of the returns has a fixed proportion in the total variance of the returns: .3 in the cases corresponding to Figures 3 and 5 and .1 in the cases corresponding to Figures 4 and 6. Because the variance of the jump component of the returns is not varying, we would expect the clustering of the volatility to be less pronounced if the jumps contributed a higher proportion in the total variance of the returns. Indeed this is the case, as can be easily confirmed by comparing Figures 3 with 4 and 5 with 6.

Figures 7–9 provide additional characteristics of the simulated returns corresponding to the case plotted in Figure 6. Figure 7 shows the realized daily variance, the daily bipower variation of Barndorff-Nielsen and Shephard (2004), and the integrated daily variance. We define the bipower variation as

$$\frac{\pi}{2} \sum_{j=1}^{16} |x_{t+j}\beta||x_{t+(j-1)}\beta|$$

where we work with 1/2-hour returns so that $\beta = 1/16$ and the scale factor $\pi/2$ makes the measure directly comparable to the integrated variance. It is well known that the realized daily variance is not a consistent estimator of the integrated variance in the presence of jumps in the price process; using squared returns picks up the variation of the jump component along with
that of the diffusive component. This is readily confirmed by comparing Figures 7(a) and 7(c). In the presence of relatively large jumps during a given day, the realized variance is significantly higher than the integrated variance.

The bipower variation is robust to jumps and provides a consistent estimator of the integrated variance as the sampling interval goes to 0. Barndorff-Nielsen et al. (2006) contains sufficient conditions under which the presence of jumps in the price does not affect the consistency and asymptotic normality of the bipower variation. Whereas consistency of the bipower variation is preserved in the case of price jumps, this article suggests that its asymptotic distribution will be affected. Comparing Figures 7(b) and 7(c) indicates that, as expected, the bipower variation does an excellent job of tracking the integrated variance despite the infinitely active character of the underlying BDLP.

Figure 8(a) shows the difference between the realized variance and the bipower variation, which is a measure of the pure-jump component of the price process. Figure 8(b) shows this difference divided by the realized variance, which is the measure of the relative share of jumps in total variance considered by Huang and Tauchen (2006). The underlying mixture of gammas is infinitely active, but generally with only a few large jumps and many small jumps. Comparing Figures 8(a) and 8(b) to Figure 6(d) suggests that these jump measures are large on days with large jumps, but the measures are unable to separate the very small jumps from the Brownian component of the price. This contrast suggests that the simulation methods developed in this article could be used for a far more extended analysis of the properties of the jump-detection tests of Barndorff-Nielsen and Shephard (2006a) than that conducted by Huang and Tauchen (2006).

Finally, Figure 9 shows the autocorrelation in the absolute and the squared returns along with the cross-correlation of the increment of the price jump process, \( L_p(t) - L_p(t - \alpha) \), and the integrated variance, \( \int_{t-\alpha}^{t} \sigma_s^2 \, ds \). The plots extend for 160 lags, corresponding to 10 trading days. In Figures 9(a) and 9(b) the persistence in the absolute and squared returns is quite evident, even with \( \rho_1 \) having a half-life of 10 days. Figure 9(c) shows the leverage effect, which in this setup is generated by the negative covariance between price jumps and variance increments in the bivariate gamma mixture model. The contemporaneous correlation is negative and then slowly dissipates in a manner consistent with that for observed data (Bollerslev, Litvinova, and Tauchen 2005b; Litvinova 2004; Tauchen 2005).

6.3 Discussion of Estimation

The continuous-time MG–CARMA (2, 1) model of the previous section can capture many of the known features of financial returns data. Nonetheless, taking such a model directly to very high-frequency returns presents serious challenges pertaining to the data and to the estimation strategy.

In terms of the data, the sampling interval cannot be too small, or the returns will be dominated by microstructure noise, which often limits the sampling interval to no finer than 5 minutes. Complications such as bid–ask bounce overwhelm the information in the data at the highest frequencies. Microstructure noise is an area of intense current research, and a full review is beyond the scope of this article; a comprehensive recent survey of the issues was given by Barndorff-Nielsen and Shephard (2005). The noise issue notwithstanding, another
problem is that return volatility shows a deterministic pattern over the course of the day, with high volatility in the early part of the day, lower volatility in the middle of the day, and higher volatility again later in the day. Moreover, the overnight return has a different variance than very short-term returns. Therefore, raw returns must be adjusted for diurnal and overnight patterns. The result is an adjusted returns series that is not actually the return on any traded security and is not sampled as finely as one might think possible.

These data issues might be considered minor and addressable by data transformations, but the estimation problems associated with application to very-high-frequency returns are truly formidable. As is widely known, the challenge is that the model-implied transition density of returns given past returns is not available in simple closed form. Direct analytical likelihood-based methods are thus intractable. Except in the simplest situations, simulation can be expected to play a role somewhere to integrate out the unobserved volatility variable(s). Simulated likelihood in the manner of Durham and Gallant (2002) is applicable only in certain cases, because the joint density of the observed and unobserved variables is generally not available, and so there is no convenient way to define an importance function for efficient simulation. Indirect estimation techniques as discussed by Tauchen (1997) potentially can be applied in this context, but that approach requires a good statistical description of the high-frequency returns, and it is not clear whether the existing models are adequate for this purpose.

MCMC in the style of Eraker et al. (2003) and Li et al. (2004) potentially can be adapted for estimating models like the MG–CARMA(2, 1) but there are obstacles to overcome. With exceptions (e.g., Eraker 2001; Elerian, Chib, and Shephard 2001), MCMC applications typically assume that the sampling interval is the same as a tick on the continuous-time clock, which greatly reduces the number of unobserved variables that must be dealt with. In effect, this rephrases the task to that of estimating a discrete-time stochastic volatility model. Doing this makes it impossible to explore the model’s implications for the price series at intervals finer than the sampling interval, and there is discretization bias. Roberts, Papaspiliopoulos, and Dellaportas (2004) avoided discretization bias in an MCMC context by using a shot noise decomposition as analyzed here, but only for OU models applied to low-frequency data.

Another matter pertaining to any estimation strategy is the matter of compounding of specification error bias. Regardless of the estimation technique used, small specification errors at the highest frequency can be expected to accumulate if the model’s dynamics are spun out to daily, weekly, or monthly intervals.

An alternative route to direct estimation on the high-frequency returns is to work at the daily level, while retaining from the high-frequency summary measures such as the realized variance and the bipower variation studied by Andersen et al. (2005) and others. An underlying continuous-time model [e.g., the MG–CARMA(2, 1)], could be forced to confront the dynamics of the vector of the daily summary variables. One empirical effort in this style was made by Barndorff-Nielsen and Shephard (2006b), who used quasi-maximum likelihood to generate semiparametric estimates of continuous-time volatility dynamics but did not generate estimates of the entire continuous-time law of motion of the price process. In special cases, the law of motion might be estimable using likelihood-based techniques, but in general, computation of the transition density of the daily summary measures is intractable. Simulated method-of-moments or indirect estimation techniques, appear to be more directly applicable to the task, although that approach requires a good statistical description of the dynamics of the daily summary measures. A starting point in this direction is the work of Bollerslev, Kretschmer, Pigorsch, and Tauchen (2005a), which is an extensive statistical modeling effort on its own. In ongoing work, we and other researchers are in the initial stages of implementing both method-of-moments–type estimators and simulated indirect estimation methods using the daily summary statistics.

Development of an appropriate estimation strategy that effectively uses the information in the large high-frequency datasets is a topic of current research. Regardless of the strategy, used however, it appears clear that simulation schemes from more general Lévy processes such as those discussed here can be expected to play a central role.

7. CONCLUSION

We have developed simulation schemes for the new classes of non-Gaussian pure-jump Lévy processes for stochastic volatility. We showed how to write the price and volatility processes as integrals against a vector Lévy process, which then makes the series approximation methods directly applicable. These methods entail simulation of the Lévy increments and formation of weighted sums of the increments; they do not require a closed-form expression for a tail mass function or specification of a copula function. We have also presented a new, and apparently quite flexible, bivariate mixture of gammas model for the driving Lévy process. Within this setup, it is quite straightforward to generate simulations from a Lévy-driven CARMA stochastic volatility model augmented by a pure-jump price component. The simulations reveal the wide range of different types of financial price processes that can be generated in this manner. Through an appropriate choice of parameters, the resulting simulated price series displays many of the same features of observed price data, including persistent stochastic volatility, dynamic leverage, and jumps.

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APPENDIX: A NECESSARY AND SUFFICIENT CONDITION FOR NONNEGATIVITY OF THE CARMA(2, 1) KERNEL

The CARMA(2, 1) kernel is given by

$$g(h) = \frac{1 + b_1 \rho_1}{\rho_1 - \rho_2} e^{\rho_1 h} + \frac{1 + b_1 \rho_2}{\rho_2 - \rho_1} e^{\rho_2 h}, \quad h \geq 0.$$ 

We want to find a condition on the parameters $\rho_1$, $\rho_2$, and $b_1$ that will guarantee nonnegativity of the kernel for every $h \geq 0$. We assume that $\rho_2 < \rho_1$ without loss of generality. Then this is equivalent to conditions on the parameters such that

$$(1 + b_1 \rho_1) e^{\rho_1 h} \geq (1 + b_1 \rho_2) e^{\rho_2 h}, \quad h \geq 0.$$ 

First, note that this inequality will be violated for $b_1 < 0$ for values of $h$ of 0 and sufficiently close to 0. In addition, the inequality is trivially satisfied for $b_1 = 0$. We concentrate on giving the necessary and sufficient condition for the nonnegativity of the CARMA(2, 1) kernel in the case where $b_1 > 0$.

The first derivative with respect to $\rho$ of the function $f(\rho) = (1 + b_1 \rho) e^{\rho h}$ is

$$f'(\rho) = e^{\rho h} (h + b_1 \rho h + b_1).$$

We have three cases:

1. $b_1 \rho_1 + 1 > 0$ and $b_1 \rho_2 + 1 \geq 0$. In this case $f(\rho)$ is increasing in the interval $[\rho_2, \rho_1]$ for every $h \geq 0$, which implies the positivity of the kernel.

2. $b_1 \rho_1 + 1 \geq 0$ and $b_1 \rho_2 + 1 < 0$. In this case the kernel is trivially nonnegative.

3. $b_1 \rho_1 + 1 < 0$ and $b_1 \rho_2 + 1 < 0$. In this case there exist values for $h \geq 0$ such that $f(\rho)$ is decreasing in the interval $[\rho_2, \rho_1]$ which will produce negative values for the kernel for those values of $h$.

Taken together, the foregoing results imply the following necessary and sufficient condition for the nonnegativity of the CARMA(2, 1) kernel: $0 \leq b_1 \leq \text{max} \{-\frac{1}{\rho_1}, \frac{1}{\rho_2}\}$.

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