THE STABILITY OF EQUILIBRIUM: COMPARATIVE STATICS AND DYNAMICS

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INTRODUCTION

IT WAS AN achievement of the first magnitude for the older mathematical economists to have shown that the number of independent and consistent economic relations was in a wide variety of cases sufficient to determine the equilibrium values of unknown economic prices and quantities. Since their life spans were only of finite duration, it was natural that they should have stopped short at this stage of counting equations and unknowns. It remains to be explained, however, why in the first quarter of the twentieth century economists should have been content with what was after all only preliminary spade work containing in itself (at least explicitly) few *meaningful* theorems of observational significance such as could even ideally be empirically refuted under any conceivable circumstances.

It is the task of comparative statics to show the determination of the equilibrium values of given variables (unknowns) under postulated conditions (functional relationships) with various data (parameters) being specified. Thus, in the simplest case of a partial-equilibrium market for a single commodity, the two independent relations of supply and demand, each drawn up with other prices and institutional data being taken as given, determine by their intersection the equilibrium quantities of the unknown price and quantity sold. If no more than this could be said, the economist would be truly vulnerable to the gibe that he is only a parrot taught to say "supply and demand." Simply to know that there are efficacious "laws" determining equilibrium tells us nothing of the character of these laws. In order for the analysis to be useful it must provide information concerning the way in which our equilibrium quantities will change as a result of changes in the parameters taken as independent data.

In the above illustration let us consider "tastes" as a changing parameter influencing only the demand curve. Will an increase in demand raise or lower price? Clearly the statement that, before and after the assumed change, price is determined by the intersection of supply and demand gives us no answer to the problem. Nothing can be said concerning the movement of the intersection point of *any* two

plane curves as one of them shifts. And yet most economists would argue that in a wide variety of circumstances this question can be given a definite answer—namely that price will increase.

How is this conclusion derived? For few commodities have we detailed quantitative empirical information concerning the exact forms of the supply and demand curves even in the neighborhood of the equilibrium point. Not only would large amounts of time and money be necessary to get such information, but in many cases it is on principle impossible to derive useful empirical information concerning what would happen if virtual changes in price confronted the demanders or the suppliers.

This is a typical problem confronting the economist: in the absence of precise quantitative data he must infer analytically the qualitative direction of movement of a complex system. What little success he has hitherto achieved can be classified in large part under two headings: (1) theorems proceeding from the assumption of maximizing behavior on the part of firms or individuals, and (2) stability conditions relating to the interaction between economic units. Although inadequately explored until comparatively recently, the first type of conditions is best known and will not be dealt with here except incidentally. As will become evident later, however, from certain points of view they can be fitted in as special cases of the second set. It is the central task of this paper to show how the problem of stability of equilibrium is intimately tied up with the problem of deriving fruitful theorems in comparative statics.

COMPARATIVE STATICS

The problem may be approached in full generality by considering n unknown variables $(x_1, \cdots, x_n)$ whose equilibrium values are to be determined for preassigned values of a parameter, α. We assume n independent and consistent, continuously differentiable implicit relations involving some or all of the unknowns and the parameter α; or

$$(1) \qquad f^i(x_1, \cdots, x_n, \alpha) = 0, \qquad i = 1, \cdots, n.$$

These determine a set of equilibrium values[1]

$$(2) \qquad x_i^o = g_i(\alpha).$$

We wish to determine the sign of

$$(3) \qquad \frac{dx_i^o}{d\alpha} = g_i'(\alpha).$$

[1] If for a given value of $\alpha = \alpha_1$, there exists a solution $(x_1^o, \cdots, x_n^o)$, and if the matrix $\|\partial f^i/\partial x_j\|$ is of rank n in a neighborhood of (x^o), then by the implicit-function theorem equations (2) represent single-valued continuously differentiable functions in a sufficiently small neighborhood of (α_1, x^o).

Differentiating (1) totally with respect to α, we can express this as

$$(4) \qquad \frac{dx_i^o}{d\alpha} = - \frac{\sum_{j=1}^{n} f_\alpha^j \Delta_{ji}}{\Delta},$$

where the subscripts indicate partial differentiation,

$$\Delta = \begin{vmatrix} f_1^1 & f_2^1 & \cdots & f_n^1 \\ f_1^2 & f_2^2 & \cdots & f_n^2 \\ \cdot & \cdot & \cdot & \cdot \cdot \cdot \\ f_1^n & f_2^{r1} & \cdots & f_n^n \end{vmatrix} = |f_i^j|,$$

and Δ_{ji} is the cofactor of the element in the jth row and ith column of Δ.

Unless some a priori restrictions are placed upon the nature of the elements involved in these determinants, no useful theorems can be derived. Each unknown derivative depends upon an $n(n+1)$ infinity of possible values. If the various determinants were expanded out, a sum of $n!$ terms would appear in the denominator and in the numerator. Regarded simply as chance drawings taken at random from a hat, the probability that the signs of these would all agree would go rapidly to zero as the number of variables increased. Fortunately, as will be shown, the analysis of the stability of equilibrium will aid in evaluating these complicated expressions.

In the simple example of supply and demand alluded to above our unknowns are (p, q), and our equilibrium system can be written:

$$(5) \qquad \begin{cases} q - D(p, \alpha) = 0, & D_\alpha > 0, D_p < 0, \\ q - S(p) = 0, \end{cases}$$

where α is a parameter of shift representing "taste" and D_p is usually assumed to be less than zero. Also

$$(6) \qquad \frac{dp^o}{d\alpha} = D_\alpha^o \frac{1}{S_p^o - D_p^o},$$

$$(7) \qquad \frac{dx^o}{d\alpha} = D_\alpha^o \frac{S_p^o}{S_p^o - D_p^o}.$$

Whether or not price will increase when demand increases is seen to depend upon the algebraic difference between the slopes (referred to the price axis) of the demand and supply curves at the equilibrium point. Quantity will increase only if the slope of the supply curve is of

the same sign as this algebraic difference. If the system is stable in the sense of Walras, it can be shown that the supply curve must have a greater algebraic slope than the demand curve so that price will necessarily increase; the change in quantity is necessarily of ambiguous sign depending upon whether the supply curve is positively inclined or so-called "backward-rising."[2]

STABILITY AND DYNAMICS

Before deriving explicitly the Walrasian stability conditions referred to above, I turn to a discussion of the meaning of a stable equilibrium. This will be found to presuppose a theory of dynamics, namely a theory which determines the behavior through time of all variables from arbitrary initial conditions. If we have given n variables $[x_1(t), \cdots, x_n(t)]$, and n functional equations of the general form

$$(8) \qquad F^i\left[\underset{-\infty}{\overset{t}{x_1(\tau)}}, \underset{-\infty}{\overset{t}{x_2(\tau)}}, \cdots, \underset{-\infty}{\overset{t}{x_n(\tau)}}, t\right] = 0, \qquad i = 1, \cdots, n,$$

then their behavior is determined once certain initial conditions are specified.[3] Examples of functional equation systems are given by sets of differential, difference, mixed differential-difference, integral, integro-differential, and still more general systems. Following the excellent terminology of Professor Frisch,[4] stationary or equilibrium values of the variables are given by the set of constants $(x_1^o, \cdots, x_n^o)$ which satisfy these equations identically, or

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[2] The distinction suggested by Mr. Kahn between "forward-falling" and "backward-rising" negatively inclined supply curves while suggestive does not rest upon a dynamic analysis of the attainment of equilibrium and so does not adequately come to grips with the problem in all its complexity. Cf. R. F. Kahn, "The Elasticity of Substitution and the Relative Share of a Factor," *Review of Economic Studies*, October, 1933, pp. 72–78; also N. Kaldor, "A Classificatory Note on the Determinateness of Equilibrium," *Review of Economic Studies*, 1933, p. 122–136.

The suggestion of Professor Viner that the latter type of curve gives the maximum amount forthcoming at a given price while the former does not will receive amplification in the course of this argument.

[3] What constitute initial conditions depends upon the nature of the functional equations. For differential systems only values of the co-ordinates, velocities, and higher derivatives at some initial time need be specified. For difference equations defined only for integral values of t the same is true, where differences replace derivatives. In the general case values of the variables over a continuous time interval, possibly stretching back to $-\infty$, are required to constitute a complete set of initial conditions.

[4] R. Frisch, "On the Notion of Equilibrium and Disequilibrium," *Review of Economic Studies*, February, 1936.

(9) $$F^i\left[\underset{-\infty}{\overset{t}{x_1^o}}, \underset{-\infty}{\overset{t}{x_2^o}}, \cdots, \underset{-\infty}{\overset{t}{x_n^o}}, t\right] = 0,[5] \qquad i = 1, \cdots, n.$$

If the system has *always* been in equilibrium up until time t_o, it will subsequently continue in equilibrium. However, the equilibrium values $(x_1^o, \cdots, x_n^o)$ may be attained or even be maintained for a finite period of time, and yet because of generalized dynamical "inertia" it need not (and in general will not) remain in equilibrium subsequently, but may well "overshoot" the mark.

The equilibrium position possesses *perfect stability of the first kind* if from any initial conditions all the variables approach their equilibrium values in the limit as time becomes infinite; i.e., if

(10) $$\lim_{t \to \infty} x_i(t) = x_i^o,$$

regardless of the initial conditions. Alternatively, it is sometimes stated that an equilibrium is stable if a displacement from equilibrium is followed by a return to equilibrium. A displacement is equivalent to an arbitrary change in the initial conditions and is possible only if some of our functional equations are momentarily relaxed or if our system is enlarged to include impressed forces or shocks.

Stability of the first kind *in the small* exists if for sufficiently small displacements the equilibrium is stable. Stability in the small is contained within perfect stability but not vice versa. A system may be stable for small finite displacements but not for large displacements. Nevertheless, stability in the small is a necessary condition for perfect stability and will be analyzed here in greatest detail.

It should be pointed out that no conservative dynamical system of the type met in theoretical physics possesses stability of the first kind. If one displaces a frictionless pendulum, it will oscillate endlessly around the position of stable equilibrium.[6] Its motion is bounded, however, and it never remains on one side of the equilibrium position for more than a finite time interval. Such behavior may be characterized as *stability of the second kind* or as stability in the second sense. As before, a distinction can be made between stability of the second kind in

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[5] Of course, such a set need not exist. Thus, the simple system

$$\frac{dx}{dt} = e^x - x$$

has no stationary equilibrium values since $e^x - x = 0$ has no real roots. Similarly, $dx/dt = 1$ defines no stationary equilibrium position.

[6] A dynamical system into which friction is introduced via a dissipation function may enjoy stability of the first kind. On these and kindred matters see G. D. Birkhoff, *Dynamical Systems*.

the small and complete stability of the second kind. For the most part in the present investigation I shall be concerned with the problem of stability of the first kind.

The equations of comparative statics are then a special case of the general dynamic analysis. They can indeed be discussed abstracting completely from dynamical analysis. In the history of mechanics, the theory of statics was developed before the dynamical problem was even formulated. But the problem of stability of equilibrium cannot be discussed except with reference to dynamical considerations, however implicit and rudimentary.[7] We find ourselves confronted with this paradox: in order for the comparative-statics analysis to yield fruitful results, we must first develop a theory of dynamics.[8] This is completely aside from the other uses of dynamic analysis as in the studies of fluctuations, trends, etc. I turn now to some illustrations of these propositions.

i. In the literary explanations of the process by which supply and demand are equated, the assumption is usually made that if at any price demand exceeds supply, price will rise; if supply exceeds demand, price will fall. Let us state this more precisely as follows:

$$(11) \qquad \dot{p} = \frac{dp}{dt} = H(q_D - q_S) = H[D(p, \alpha) - S(p)],$$

where $H(0) = 0$, and $H' > 0$.

In the neighborhood of the equilibrium point this can be expanded in the form

$$(12) \qquad \dot{p} = \lambda(D_{p^o} - S_{p^o})(p - p^o) + \cdots,$$

where $\lambda = (H')^o > 0$, and where terms involving higher powers of $(p - p^o)$ are omitted. The solution of this simple differential equation for initial price $\bar{p}$ at time zero can be written at sight:

$$(13) \qquad p(t) = p^o + (\bar{p} - p^o)e^{\lambda(D_{p^o} - S_{p^o})t}.$$

If the equilibrium is to be stable,

$$(14) \qquad \lim_{t \to \infty} p(t) = p^o.$$

This is possible if, and only if,

[7] This is seen to be involved in the virtual-work analysis and in the minimum-potential-energy condition characteristic of a stable statical ("stationary") equilibrium position.

[8] The point made here is not to be confused with the commonplace criticism of comparative statics that it does not do what it is not aimed to do, namely describe the transition paths between equilibria.

(15) $$D_{p^o} - S_{p^o} \leqq 0.$$

If in what follows we rule out neutral equilibrium in the large and in the small, the equality sign may be omitted so that

(16) $$D_{p^o} - S_{p^o} < 0.$$

If the supply curve is positively inclined, this will be realized. If it is negatively inclined, it must be less steep (referred to the price axis) than the demand curve. If our stability conditions are realized, the problem originally proposed is answered. Price must rise when demand increases.

ii. These so-called Walrasian stability conditions are not necessarily the only ones.[9] If alternative dynamic models are postulated, completely different conditions are deduced, which in turn lead to alternative theorems in comparative statics.

Thus, in Marshall's long-run theory of normal price the quantity supplied is assumed to adjust itself comparatively slowly. If "demand price" exceeds "supply price," the quantity supplied will increase. Preserving our notation of equations (5), remembering that quantity rather than price is regarded as the independent variable, and neglecting higher-order terms, we have the following differential equation

(17) $$\dot{q} = k\left(\frac{1}{D_{p^o}} - \frac{1}{S_{p^o}}\right)(q - q^o), \qquad k > 0,$$

whose solution is

(18) $$q(t) = q^o + (\bar{q} - q^o)e^{k(1/D_{p^o} - 1/S_{p^o})t}.$$

If the equilibrium is to be stable,

(19) $$\frac{1}{D_{p^o}} - \frac{1}{S_{p^o}} = \frac{1}{D_{p^o}}\frac{S_{p^o} - D_{p^o}}{S_{p^o}} < 0;$$

i.e., the demand-curve slope referred to the quantity axis is less algebraically than that of the supply curve. Since the demand curve is negatively inclined,

(20) $$\frac{S_{p^o}}{S_{p^o} - D_{p^o}} > 0.$$

Referring back to equations (7), we see that Marshallian stability con-

[9] An historical error is involved in the identification of the above stability conditions with Walras in alleged contrast to those of Marshall which are shortly to be discussed. Actually as far back as in the *Pure Theory of Foreign Trade* Marshall defined stable equilibrium, in which a so-called backward rising supply curve was involved, exactly as in the Walrasian case.

ditions require that quantity increases when demand increases in every case, while the change in price is necessarily ambiguous depending upon the algebraic sign of the supply curve's slope.

It is to be pointed out that this forward-falling supply curve is not a true supply curve in the sense of the amount forthcoming at each hypothetical price, although it is a true supply curve in the sense of being the locus of price-quantity points traced out by fluctuations in *sufficiently steep* demand curves. As such it is a reversible long-run relation.

iii. Still another dynamic model may be considered. It has been held that for some commodities supply reacts to price only after a given time lag, while price adjusts itself almost instantaneously. This leads to the familiar cobweb phenomenon. Using the same notation, our dynamic model takes the form of the following difference equations,

$$(21) \qquad \begin{aligned} q_t &= S(p_{t-1}), \\ q_t &= D(p_t, \alpha). \end{aligned}$$

In the neighborhood of equilibrium

$$(22) \qquad (q_t - q_o) = \left(\frac{S_{p^o}}{D_{p^o}}\right)(q_{t-1} - q^o)$$

with the solution

$$(23) \qquad q_t = q^o + (\bar{q} - q^o)\left(\frac{S_{p^o}}{D_{p^o}}\right)^t.$$

Stability requires that

$$(24) \qquad \left|\frac{S_{p^o}}{D_{p^o}}\right| < 1.$$

If the supply curve is positively inclined, it must be steeper absolutely than is the demand curve, reference being made to the quantity axis. In this case the approach to equilibrium is of a damped oscillatory nature, every other observation being on one side of the equilibrium value.

If the supply curve is negatively inclined, it must be steeper referred to the quantity axis than is the demand curve, precisely as in the case of Walrasian stability. The approach to equilibrium is asymptotic. As in the Walrasian case, we can deduce the theorem in comparative statics that price will necessarily increase even though the change in quantity is indefinite.

It is to be noted that a first-order difference equation is richer in

solution than the corresponding first-order differential equation. Not only does it admit of oscillatory solutions, but the stability conditions relate to the absolute value of the root of an equation, implying two distinct inequalities. Remembering that $D_p{}^o$ is negative, the inequality of (24) can be written

$$D_p{}^o < S_p{}^o < - D_p{}^o.$$

The new inequality tells us that any increase in output as a result of an increase in demand cannot be so large as the increase in output resulting from an "equivalent" increase in supply.
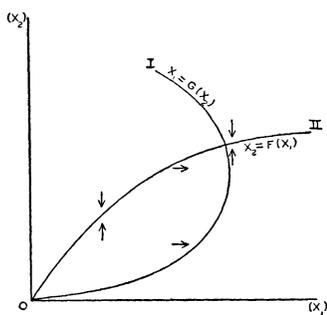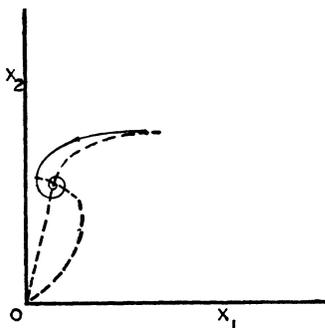

Figure 1   Figure 2

iv. Still a fourth dynamical model that has been considered is that of Marshall in the *Pure Theory of Foreign Trade*. Let Figure 1 represent the familiar offer curves of two trading bodies (suppliers and demanders) respectively. Equilibrium is attained at the intersection (not necessarily unique) of two such curves. If equilibrium is displaced, country I is to act in such a way as to change the amount of x_1 in the horizontal direction of its offer curve (as indicated by the pointed horizontal arrows. Similarly country II adjusts x_2 vertically in the direction of its offer curve. Mathematically,

$$(25) \quad \begin{aligned} \dot{x}_1 &= H_1[G(x_2) - x_1], \\ \dot{x}_2 &= H_2[F(x_1) - x_2], \end{aligned}$$

where $H_i{}' > 0$, $H_i(0) = 0$, and $G(x_2) - x_1 = 0$, $F(x_1) - x_2 = 0$ represent the statical offer curves of countries I and II respectively. If units are properly chosen, the following system of differential equations will hold in the neighborhood of equilibrium,

$$(26) \quad \begin{aligned} \dot{x}_1 &= -(x_1 - x_1{}^o) + (G')^o(x_2 - x_2{}^o), \\ \dot{x}_2 &= (F')^o(x_1 - x_1{}^o) - (x_2 - x_2{}^o). \end{aligned}$$

The solution takes the form:

$$x_1(t) = x_1{}^o + k_{11}e^{\lambda_1 t} + k_{12}e^{\lambda_2 t},$$
(27)
$$x_2(t) = x_2{}^o + k_{21}e^{\lambda_1 t} + k_{22}e^{\lambda_2 t},$$

where the k's depend upon the initial values $(\bar{x}_1, \bar{x}_2)$ and the λ's are roots of the characteristic equation

$$(28) \qquad D(\lambda) = \begin{vmatrix} -1 - \lambda & (G')^o \\ (F')^o & -1 - \lambda \end{vmatrix} = 0.$$

Clearly

$$(29) \qquad \lambda = -1 \pm \sqrt{(G')^o(F')^o}.$$

The equilibrium will be stable if the real part of λ is necessarily negative, or

$$(30) \qquad R(\lambda) < 0.$$

If both $(G')^o$ and $(F')^o$ are of opposite sign (e.g., if one has an elastic demand, the other an inelastic demand), this condition will necessarily be satisfied. The solution will be oscillatory, but damped, approaching equilibrium in a spiral as shown above in Figure 2, and obeying an equation of the form

$$(31) \qquad x_i = x_i{}^o + e^{-t}(a_i \sin \theta t + b_i \cos \theta t), \qquad i = 1, 2.$$

If both are positive, however (each with elastic demands),

$$(32) \qquad \sqrt{(G')^o(F')^o} < 1,$$

$$(33) \qquad (G')^o(F')^o < 1,$$

and

$$(34) \qquad (G')^o < \frac{1}{(F')^o} \cdot$$

In terms of the slopes of both offer curves referred to the x_1 axis

$$(35) \qquad \left(\frac{dx_2}{dx_1}\right)_I > \left(\frac{dx_2}{dx_1}\right)_{II} \cdot$$

The equilibrium is approached asymptotically.

If both curves are negatively inclined, stability requires

$$(36) \qquad \left(\frac{dx_2}{dx_1}\right)_I < \left(\frac{dx_2}{dx_1}\right)_{II} \cdot$$

Clearly, the general condition when the curves are of like sign can be written

$$(37) \qquad \left|\frac{dx_2}{dx_1}\right|_{\mathrm{I}} > \left|\frac{dx_2}{dx_1}\right|_{\mathrm{II}},$$

and the approach to equilibrium will in every case be asymptotic.[10] The stability conditions derived here will be found, if translated into terms of supply and demand curves rather than offer curves, to imply differing and inconsistent conditions from those of the preceding cases.

v. In the four cases considered I have been concerned with problems of stability of the first kind. Following a suggestion of Dr. Francis Dresch of the University of California, let us suppose that price falls not when the instantaneous supply exceeds demand, but only when accumulated stocks exceed some normal value, Q^o, or

$$(38) \qquad \dot{p} = \lambda(Q^o - Q) + \cdots = \lambda Q^o - \lambda \int_0^t (q_S - q_D)dt, \qquad \lambda > 0,$$

since the stock equals the accumulated difference between the amount produced and the amount consumed. Differentiating with respect to t, neglecting terms of higher power, and adhering to our previous notation, we have

$$(39) \qquad \ddot{p} = (D_{p^o} - S_{p^o})(p - p^o),$$

whose solution is

$$(40) \qquad p(t) = p^o + c_1 e^{(\sqrt{D_{p^o} - S_{p^o}})\,t} + c_2 e^{-(\sqrt{D_{p^o} - S_{p^o}})\,t},$$

the c's depending upon the initial price and price change.

Only if

$$(41) \qquad D_{p^o} - S_{p^o} < 0,$$

can explosive behavior be avoided. If the above inequality is realized, however, the square root will be a pure imaginary number so that the solution takes the form of an undamped harmonic:

$$(42) \quad p(t) = b_1 \cos \sqrt{S_{p^o} - D_{p^o}}\, t + b_2 \sin \sqrt{S_{p^o} - D_{p^o}}\, t + p^o.$$

Thus, if we require our second-order differential equation to have at least stability of the second kind, we come to the same theorems in comparative statics as in case i.

There is at least one serious objection to assuming a nondamped

[10] Somewhat paradoxically, in this case positions of stable equilibrium need not be separated by positions of unstable equilibrium because of the possibility of complex roots.

system of this kind. If there are superimposed upon our system random shocks or errors, these will tend to accumulate so that the expected amplitude of the cycles will increase with time. This is well illustrated by the familiar Brownian movement of large molecules under the impact of random collisions. The molecule "takes a random walk," and its mean variance increases with the observation time.[11] Before adopting a similar hypothesis in economic analysis, some statistical evidence of its possible validity should be adduced.[12]

We have now surveyed five different dynamic setups and related stability conditions, all referring to a simple one-commodity market. Except possibly for cases iv and v, all are mathematically trivial. Unaided intuition or simple geometrical methods serve to reveal sufficient conditions for stability. They are of significance, however, because each has played an important part in the history of economic science; and precisely because of their simplicity, they provide a useful illustration of the general principle involved. In the following sections I shall be concerned with more complex problems.

THE STABILITY OF MULTIPLE MARKETS

While it might be more elegant at this stage to develop formally for general systems the fundamental principles illustrated thus far, our

[11] Random shocks are not necessarily to be regarded as a nuisance. In their absence friction might imprison the system at some fixed level other than the "true" equilibrium level (friction being disregarded). Often random shocks serve to insure the realization of *average* values nearly equal to the equilibrium ones, just as iron filings placed upon a piece of paper over a magnet assume the lines of force of the magnetic field when gently tapped.

[12] One can avoid an undamped system by assuming that price tends to fall not only when stocks are large, but also when current supply exceeds current demand; i.e., when stocks tend to accumulate. Then we have

$$\dot{p} = \alpha\left[Q^o - \int_0^t (q_S - q_D)dt \right] - \beta(q_S - q_D)$$

or

$$\ddot{p} = \alpha(D_{p^o} - S_{p^o})p + \beta(D_{p^o} - S_{p^o})\dot{p}, \qquad \alpha, \beta > 0.$$

The equilibrium is stable only if

$$R(\lambda) < 0,$$

or if

$$\lambda^2 - \alpha(D_{p^o} - S_{p^o})\lambda - \beta(D_{p^o} - S_{p^o}) = 0,$$

or if

$$D_{p^o} - S_{p^o} < 0.$$

This agrees with the condition of case i and the one just derived. In fact, each of these is a special case when one of the coefficients vanishes. For intermediate values, the solutions range continuously between damped harmonic motion and exponential approach to equilibrium.

foregoing discussion provides a very convenient opening for an examination of a problem which has received considerable attention lately at the hands of Professor Hicks. In *Value and Capital*, Chapter VIII and Mathematical Appendix, §21, he has attempted to generalize to any number of markets the stability conditions of a single market. The method of approach is postulational; stability conditions are not deduced from a dynamic model except implicitly.[13] Propositions which are deduced here as theorems are assumed as definitions of stability.

For a single market, according to Professor Hicks, equilibrium is stable if an increase in demand raises price. (This rules out in the beginning cases ii and iv.) For multiple markets equilibrium is *imperfectly* stable if an increase in demand for a single good raises its price after all other prices have adjusted themselves; the equilibrium is *perfectly* stable if an increased demand for a good raises its price even when *any* subset of other prices is arbitrarily held constant (by means of a relaxation of other equilibrium conditions).

To test the necessity or sufficiency of these criteria in terms of a more fundamental definition of stability of equilibrium let us make a natural generalization of the Walrasian conditions of the following form: the price of any good will fall if its supply exceeds its demand, *these each being regarded as functions of all other prices.*

Mathematically,

$$\dot{p}_i = - H(q_S{}^i - q_D{}^i)$$
$$= - H\left[q_S{}^i(p_1, \cdots, p_n) - q_D{}^i(p_1, \cdots, p_n)\right]$$

$$(43) \qquad = \sum_{j=1}^{n} a_{ij}{}^o(p_j - p_j{}^o) + \cdots,$$

where

$$(44) \quad 0 = q_S{}^i(p_1, \cdots, p_n) - q_D{}^i(p_1, \cdots, p_n) = - a_i(p_1, \cdots, p_n)$$

represent statical equations of supply and demand, $a_{ij}{}^o$ represents the partial derivative of a_i with respect to the jth price evaluated at the equilibrium set of prices. In general, $a_{ij}{}^o \neq a_{ji}{}^o$.[14] It is instructive to

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[13] It is true that on page 70 a hint of a dynamical process creeps into the discussion. The approach to equilibrium seems to be regarded as taking place in finite steps at discrete intervals of time; i.e., in accordance with certain difference equations. Correctly stated, this argument would not lead to essentially different stability conditions from my system of differential equations discussed below, as the later general discussion will disclose.

[14] If the demand and supply were each drawn up with reference to firms maximizing profit, well-known integrability conditions would guarantee this equivalence. On the consumer's side there need be no such equivalence, and if we

consider first, however, the symmetrical case (such as characterizes markets made up exclusively of entrepreneurs). The solution of equations (43) can be written

$$(45) \qquad\qquad p_i(t) = p_i^o + \sum_{j=1}^{n} k_{ij} e^{\lambda_j t},$$

where $(\lambda_1, \cdots, \lambda_n)$ are latent roots of the characteristic equation and

$$(46) \qquad f(\lambda) = \begin{vmatrix} a_{11}^o - \lambda & a_{12}^o & \cdots a_{1n}^o \\ a_{21}^o & a_{22}^o - \lambda & a_{2n}^o \\ \cdots & \cdots & \cdots \\ a_{n1}^o & a_{n2}^o & \cdots a_{nn}^o - \lambda \end{vmatrix}$$

$$= \left| a - \lambda I \right| = \left| a_{ij}^o - \lambda \delta_{ij} \right| = 0,$$

the k's depend upon the matrix a and upon the initial conditions.[15] As before, stability requires $R(\lambda_j) < 0$.

By a well-known theorem of Hermitian matrices, in the symmetrical case all the roots are necessarily real. If the equilibrium is to be stable, they must all be negative. According to a classical theorem, this is possible if and only if a is the matrix of a negative definite quadratic form; i.e., only if all principal minors alternate in sign as follows:

$$(47) \qquad \left| a_{ii}^o \right| < 0; \quad \begin{vmatrix} a_{ii}^o & a_{ij}^o \\ a_{ji}^o & a_{jj}^o \end{vmatrix} > 0; \quad \begin{vmatrix} a_{ii}^o & a_{ij}^o & a_{ik}^o \\ a_{ji}^o & a_{jj}^o & a_{jk}^o \\ a_{ki}^o & a_{kj}^o & a_{kk}^o \end{vmatrix} < 0,$$

$$i \neq j \neq k \neq i.$$

Any ratio of the form

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consider a consumer whose total purchases balance his total sale of productive services, such an equality for every combination of goods and services would, strictly interpreted, lead to an absurdity; it would imply expenditure proportionality and, hence, zero consumption of every good and zero offering of every service! For the general demand or supply function we need not expect a cancelling off of "income effects" since individuals usually face firms in consumption and factor markets. Contrast in this connection Hicks' views, Chapters V and VIII.

[15] If the roots are not distinct, polynomial terms of the form $te^{\lambda t}, t^2 e^{\lambda t}, \cdots, t^s e^{\lambda t}$ appear where $(s+1)$ is the multiplicity of a repeated root. In any case the problem of stability depends only upon the λ's and is unaffected by such multipliers since the exponential always governs the asymptotic behavior of the solution whenever dampening does occur.

$$(48) \qquad \frac{\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{jj}^{o} & \cdots & a_{jk}^{o} \\ \cdot & \cdot & \cdot \cdot \cdot \cdot \cdot \\ 0 & a_{kj}^{o} & \cdots & a_{kk}^{o} \end{vmatrix}}{\begin{vmatrix} a_{ii}^{o} & a_{ij}^{o} & \cdots & a_{ik}^{o} \\ a_{ji}^{o} & a_{jj}^{o} & \cdots & a_{jk}^{o} \\ \cdot & \cdot & \cdot \cdot \cdot \cdot \cdot \\ a_{ki}^{o} & a_{kj}^{o} & \cdots & a_{kk}^{o} \end{vmatrix}}$$

is necessarily negative in sign. But such ratios are precisely equal to the change in the price of the ith good with respect to a unit increase in its own supply when appropriate subsets of other prices are held constant, so that for this case the stability criteria of Professor Hicks are seen to be correct theorems. In the symmetrical case more than this can be said: *Imperfect stability in the Hicks sense necessarily implies perfect stability and conversely.* Imperfect stability involves the same n inequalities as does perfect stability, no more and no less.

Where perfect symmetry is not present (and in business-cycle analysis it is almost always absent), the Hicks criteria are not at all necessary conditions and in many cases not sufficient.[16] A system may possess stability of the first kind even though neither *perfectly* nor *imperfectly* stable in Hicks' sense. I suspect but have not yet devised a proof to show that perfect stability is a sufficient condition for stability

[16] A word of warning may be in order concerning Hicks' indiscriminate use of either prices or quantities as independent variables. This leads to contradictory definitions of complementarity in the literary discussion on page 44 and the mathematical definitions on page 311, the inconsistency between which can lead to opposite signs. Such an interchange of independent variables (as between the literary definition of stability on page 63 and the mathematical conditions on 315 and 325) is particularly important where nonsymmetrical matrices are concerned. Does

$$(49) \qquad \frac{dx_i}{dp_i} < 0, \quad \begin{vmatrix} \dfrac{dx_i}{dp_i} & \dfrac{dx_i}{dp_j} \\ \dfrac{dx_j}{dp_i} & \dfrac{dx_j}{dp_j} \end{vmatrix} < 0, \cdots, \quad \text{where} \quad \frac{dx_i}{dp_j} \neq \frac{dx_j}{dp_i},$$

imply

$$(50) \qquad \frac{dp_i}{dx_i} < 0, \quad \begin{vmatrix} \dfrac{dp_i}{dx_i} & \dfrac{dp_i}{dx_j} \\ \dfrac{dp_j}{dx_i} & \dfrac{dp_j}{dx_j} \end{vmatrix} > 0, \cdots ?$$

Even with symmetry the product $(dp_i/dx_i)(dx_i/dp_j)$ need not be of positive sign if more than two variables are involved.

of the first kind. In any case it is too strict a condition, while the requirement of imperfect stability is not strict enough; only in the case of symmetry do these limits converge. Why any system should be expected to possess *perfect* stability, or why an economist should be interested in this property is by no means clear. Not working with an explicit dynamical model, Professor Hicks probably argued by analogy from well-known maximum conditions, whereby a maximum must hold for arbitrary displacements and through any transformation of variables. As a result, some variables may be made constants, and with respect to the remaining arbitrary subsets the definiteness of various quadratic forms must be insured. On the other hand, in terms of a truly dynamic process the equilibrium must be stable for arbitrary initial conditions or displacements and for arbitrary nonsingular transformations of variables, but *not* necessarily for arbitrary modifications of the dynamic equations of motion such as are involved in the Hicks procedure of holding subsets of other prices constant (by violating or relaxing true dynamical relations). In principle the Hicks procedure is clearly wrong, although in some empirical cases it may be useful to make the hypothesis that the equilibrium is stable even without the "equilibrating" action of some variable which may be arbitrarily held constant. (In connection with the Keynesian model later discussed, an example of this is presented.)

To summarize: for every case true necessary and sufficient stability conditions are that $R(\lambda_j) < 0$, where λ_j represents the latent roots of the matrix a. This is not equivalent to the Hicks conditions.[17]

Before leaving the problem of stable multiple markets, I should like to sketch the effect of the introduction of stocks and its relevance to stability of the second kind. Let price fall not when current supply exceeds current demand, but when existing stocks (accumulated over time from the divergence of current production and consumption) exceed an equilibrium amount. Then neglecting terms of higher powers, we write

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[17] The following illustrations bear this out: The system

$$(51) \quad \begin{aligned} \dot{p}_1 &= -2p_1 + 4p_2, \\ \dot{p}_2 &= -p_1 + p_2, \end{aligned}$$

possesses stability of the first kind, but is neither perfectly nor imperfectly stable. The system

$$(52) \quad \begin{aligned} \dot{p}_1 &= p_1 - p_2, \\ \dot{p}_2 &= 2p_1 + p_2, \end{aligned}$$

is imperfectly stable, but departs ever further from equilibrium.

$$\dot{p}_i = Q_i{}^o - \int_0^t (q_S - q_D)d\tau = Q_i{}^o + \int_0^t \sum_{j=1}^n a_{ij}{}^o(p_j - p_j{}^o)d\tau + \cdots$$

or

$$(53) \qquad \ddot{p}_i = \sum_{j=1}^n a_{ij}{}^o(p_j - p_j{}^o) + \cdots,$$

whose solution takes the form

$$(54) \qquad p_i(t) = p_i{}^o + \sum_{j=1}^n (k_{ij}e^{\sqrt{\overline{\lambda_j}}\,t} + h_{ij}e^{-\sqrt{\overline{\lambda_j}}\,t})$$

where $\left| a_{ij}{}^o - \lambda_j\delta_{ij} \right| = 0$, and where for unrepeated roots the k's and the h's are constant depending upon initial conditions. Clearly the motion will be explosive and undamped unless $\sqrt{\lambda_j}$ are all pure imaginary numbers; i.e., unless λ_j is real and negative.

If the system is symmetrical, this clearly leads to the same conditions as those for stability of the first kind. If not symmetrical, the substitution of second derivatives everywhere for first derivatives (through the hypothesis of dependence upon accumulated stocks rather than instantaneous flows) implies more rigid conditions upon the coefficients to insure stability of the second kind than were previously required to insure stability of the first kind. This is of course because of the requirement that the roots be real as well as negative.[18]

ANALYSIS OF THE KEYNESIAN SYSTEM

Up until now I have considered examples drawn from the field of economic theory. The techniques used there are of even more fruitful applicability to problems of business cycles. To illustrate this I shall analyze in some detail the simple Keynesian model as outlined in the *General Theory*. Various writers, such as Meade, Hicks, and Lange, have developed explicitly in mathematical form the meaning of the Keynesian system.[19] The three fundamental relationships stressed by

[18] One could consider the generalization of the intermediate hypothesis of footnote 12 where price change depends upon stocks and flows, namely,

$$(55) \qquad \ddot{p}_i = \sum_{j=1}^n a_{ij}{}^o \lfloor \alpha\dot{p}_j + \beta(p_j - p_j{}^o) \rfloor, \qquad \begin{array}{l} \alpha > 0, \\ \beta > 0. \end{array}$$

If stable for $\beta > 0$, $\alpha = 0$, and also for $\beta = 0$, $\alpha > 0$, it can perhaps be proved to be stable for all intermediate cases.

[19] J. E. Meade, "A Simplified Model of Mr. Keynes' System," *Review of Economic Studies*, February, 1937, pp. 98–107; J. R. Hicks, "Mr. Keynes and the 'Classics'; A Suggested Interpretation," ECONOMETRICA, Vol. 5, 1937, pp. 147–159; Oskar Lange, "The Rate of Interest and the Optimum Propensity to Consume," *Economica*, 1938, pp. 12–32.

Keynes are (1) *the consumption function* relating consumption (and hence savings-investment) to income, and for generality to the interest rate as well; (2) *the marginal efficiency of capital* relating *net* investment to the interest rate and to the level of income (as of a fixed level of capital equipment, fixed for the short period under investigation); (3) *the schedule of liquidity preference* relating the existing amount of money to the interest rate and the level of income.

Mathematically, these may be written as follows:

$$(56) \qquad C(i,\ Y) - Y + I = -\alpha,$$

$$(57) \qquad F(i,\ Y) - I = -\beta,$$

$$(58) \qquad L(i,\ Y) = M,$$

where i, Y, I stand respectively for the interest rate, income, and investment; C, F, L stand respectively for the consumption function, the marginal-efficiency-of-capital schedule, and the schedule of liquidity preference. M stands for the existing amount of money, taken as a parameter; α is a general parameter representing an upward shift in the propensity-to-consume schedule; similarly as the parameter β increases, the marginal-efficiency schedule shifts upward.

We have three relations to determine the three unknowns in terms of three parameters, viz.:

$$i = i(\alpha,\ \beta,\ M),$$

$$(59) \qquad Y = Y(\alpha,\ \beta,\ M),$$

$$I = I(\alpha,\ \beta,\ M).$$

As explained in the first section of this paper, the usefulness of the Keynesian equilibrium system lies in the light it throws upon the way our unknowns will change as a result of changes in data. More specifically, what are the signs of

$$\frac{di}{d\alpha},\quad \frac{dY}{d\alpha},\quad \frac{dI}{d\alpha},$$

$$\frac{di}{d\beta},\quad \frac{dY}{d\beta},\quad \frac{dI}{d\beta},$$

$$\frac{di}{dM},\quad \frac{dY}{dM},\quad \frac{dI}{dM}\ ?$$

Differentiating totally with respect to our parameters and evaluating the resulting linear equations, we find

(60)

$$\frac{di}{d\alpha} = \frac{-L_Y}{\Delta}, \quad \frac{dY}{d\alpha} = \frac{L_i}{\Delta}, \quad \frac{dI}{d\alpha} = \frac{F_Y L_i - F_i L_Y}{\Delta},$$

$$\frac{di}{d\beta} = \frac{-L_Y}{\Delta}, \quad \frac{dY}{d\beta} = \frac{L_i}{\Delta}, \quad \frac{dI}{d\beta} = \frac{(1 - C_Y)L_i + C_i L_Y}{\Delta},$$

$$\frac{di}{dM} = \frac{1 - C_Y - F_Y}{\Delta}, \quad \frac{dY}{dM} = \frac{F_i + C_i}{\Delta},$$

$$\frac{dI}{dM} = \frac{F_Y(F_i + C_i) + (1 - C_Y - F_Y)F_i}{\Delta},$$

where

$$(61) \quad \Delta = \begin{vmatrix} C_i & C_Y - 1 & 1 \\ F_i & F_Y & -1 \\ L_i & L_Y & 0 \end{vmatrix} = L_Y(F_i + C_i) + L_i(1 - C_Y - F_Y).$$

On the basis of a priori, intuitive, empirical experience the following assumptions are usually made:

$$(62) \qquad C_Y > 0, \quad F_Y > 0, \quad F_i < 0, \quad L_Y > 0, \quad L_i < 0,$$

while

$$C_i \gtreqless 0$$

and is usually assumed in modern discussions to be of minor quantitative importance.

In order to evaluate our nine derivatives we must be able to determine unambiguously the signs of all numerators as well as the common denominator, Δ. Δ consists of five terms, two of which are of positive sign, two of negative sign, and one ambiguous. On the basis of deductive analysis along strictly statical lines nothing can be inferred concerning its sign. Moreover, even if the sign of Δ were determined, all but four of the nine would be found to have numerators of indeterminable sign.

This is a typical case. If we are to derive useful theorems, we must clearly proceed to a consideration of a more general dynamic system which includes the stationary Keynesian analysis as a special case. This can be done in a variety of alternative ways. I shall consider two, the first of which is based upon a differential system and yields quite definite results.

Case 1. Let us assume as before that the second and third relations of marginal efficiency and liquidity preference work themselves out in so short a time that they can be regarded as holding instantaneously. Let us assume, however, that I now represents "intended" investment, and

this magnitude equals savings-investment only in equilibrium, i.e., when all the variables take on stationary values. If, however, because of some change, consumption (say) should suddenly increase, national income not having a chance to change, actual savings-investment would fall short of "intended" investment because of inventory reduction, etc. Consequently, income would tend to rise. Similarly an excess of actual savings-investment over intended investment would tend to make income fall. Mathematically, this hypothesis may be stated as follows: *the rate of change of income is proportional to the difference between intended savings-investment and actual savings-investment.* The discussion here is unrelated to the controversy over the equality of savings and investment despite possible appearances to the contrary. The superficial resemblance between my formulation and the Robertson-Ohlin identities whereby the difference between investment and savings is the time difference of income should not mislead the careful reader.

Equations (56), (57), and (58) are replaced by the dynamical ones:

$$\dot{Y} = I - [Y - C(i, Y) - \alpha], \tag{63}$$

$$0 = F(i, Y) - I + \beta, \tag{64}$$

$$0 = L(i, Y) - M. \tag{65}$$

The solution of these is of the form:

$$Y = Y^o + a_1 e^{\lambda t},$$
$$i = i^o + a_2 e^{\lambda t}, \tag{66}$$
$$I = I^o + a_3 e^{\lambda t},$$

where

$$\Delta(\lambda) = \begin{vmatrix} C_i & C_Y - 1 - \lambda & 1 \\ F_i & F_Y & -1 \\ L_i & L_Y & 0 \end{vmatrix} = \Delta + \lambda L_i = 0. \tag{67}$$

The equilibrium is stable only if

$$\lambda = -\frac{\Delta}{L_i} < 0. \tag{68}$$

But $L_i < 0$; therefore,

$$\Delta < 0 \tag{69}$$

unambiguously.

This establishes four theorems: an increased marginal efficiency of capital will (1) raise interest rates and (2) raise income; an increased

propensity to consume will (3) raise interest rates and (4) raise income. But how will the creation of new money affect interest rates? This can be answered by considering more stringent stability conditions. Let us suppose that the interest rate were kept constant (say) by appropriate central bank action. This assumption is equivalent to dropping the liquidity preference equation (65) and treating i as a constant in the remaining equations. If the equilibrium is stable under these conditions, we must have

$$(70) \qquad \begin{vmatrix} C_Y - 1 - \lambda & 1 \\ F_Y & -1 \end{vmatrix} = 0 = (1 - C_Y - F_Y) + \lambda,$$

or

$$(71) \qquad -\lambda = (1 - F_Y - C_Y) > 0.$$

This leads to another important theorem: (5) *the marginal propensity to consume plus the marginal propensity to invest cannot exceed unity or the system will be unstable* (as of a fixed interest rate).[20] It also tells us (6) that an increase in the amount of money must, *ceteris paribus*, lower interest rates.

We are left with four ambiguities of sign. Two of them depend upon the fact that savings may vary in any direction with respect to a change in interest rates. If we assume that normally savings out of a given income increase with the interest rate, or, if they do decrease, do so not so much as does investment, then three more theorems become true: an increase in the amount of money (7) increases income and (8) increases investment; (9) an increase in the marginal-efficiency schedule increases investment. There remains a final ambiguous term. What is the effect upon investment of an increased propensity to consume? This is seen to be essentially ambiguous depending upon the quantitative strengths of the liquidity-preference slopes and the marginal-efficiency slopes. As income increases, money becomes tight because of the need for financing more transactions. This tends to depress investment. As an offset, the increase in income tends to increase investment through the marginal propensity to invest. Which effect will be the stronger cannot be decided on a priori grounds.

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[20] If we take investment also as an independent parameter (say through government action), we lose equation (57) and have for stability the condition

$$\begin{vmatrix} C_Y - 1 - \lambda \end{vmatrix} = (C_Y - 1) - \lambda = 0,$$
$$\lambda = C_Y - 1 < 0,$$

or that *the marginal propensity to consume must be less than one*. But this is weaker than the previous condition in view of the fact that the marginal propensity to invest is assumed positive.

I have prepared a 3×3 classification indicating the signs of the nine terms. All but four have definite signs. Of these four, one is essentially ambiguous as indicated by a question mark. The remaining three show under question marks their normal, presumptive signs.

	i	Y	I
Increase in propensity to consume $\quad\alpha$	$+$	$+$	$?$
Increase in marginal efficiency of capital $\quad\beta$	$+$	$+$	$\genfrac{}{}{0pt}{}{?}{+}$
Increase in amount of money $\quad M$	$-$	$\genfrac{}{}{0pt}{}{?}{+}$	$\genfrac{}{}{0pt}{}{?}{+}$

Case 2. I now turn to a system based on a difference equation. It is founded upon considerations similar to these underlying the Kahn-Clark multiplier block diagrams, and for this reason alone is worth consideration. In addition, the analytical contrasts between differential and difference systems is brought out. Reversing the order of the previous exposition, let us take investment as an independent parameter and the interest rate as a constant. Let consumption be a given function of income during the preceding period of time:

$$(72) \qquad C_t = C(\bar{\imath}, Y_{t-1}) = C(Y_{t-1}).$$

What properties must this function satisfy if the equilibrium is to be stable? Income clearly equals consumption plus investment:

$$(73) \qquad Y_t = C_t + I_t.$$

Recalling that investment is treated as a constant, $\bar{I}$, and using the consumption relation, we find

$$(74) \qquad Y_t = C(Y_{t-1}) + \bar{I},$$

or, to a first approximation,

$$(75) \qquad (Y_t - Y^o) = C_{Y^o}(Y_{t-1} - Y^o),$$

where

$$(76) \qquad Y^o = C(Y^o) + \bar{I}$$

is the equilibrium level of income for investment equal to $\bar{I}$.

The solution of this difference equation takes the form

$$(77) \qquad Y_t = Y^o + K(C_{Y^o})^t$$

and is stable only if

$$(78) \qquad |C_{Y^o}| < 1$$

or

(79) $$-1 < C_{Y^o} < 1.^{21}$$

While the marginal propensity to consume is usually assumed to be positive, it need not be so, and still the equilibrium can be a stable one. Even if it lies between zero and minus one, it is interesting to observe that the "multiplier" is positive since

(80) $$\frac{dY^o}{d\bar{I}} = \frac{1}{1 - C_{Y^o}} > 0,$$

but less than unity because of negative "secondary" effects.

Let us now drop the assumption that investment is a datum, although keeping the interest rate constant. Our dynamic system is of the form

(81) $$C(\bar{\imath}, Y_{t-1}) - Y_t + I_t = 0,$$

(82) $$F(\bar{\imath}, Y_t) - I_t = 0,$$

and the equilibrium is stable only if

(83) $$|\lambda| = \left| \frac{C_Y}{1 - F_Y} \right| < 1,$$

or

(84) $$-|1 - F_Y| < C_Y < |1 - F_Y|.$$

Now if the marginal propensity to invest is less than unity $(1 - F_Y > 0),^{22}$ this leads to essentially the same stability conditions as before, namely the marginal propensity to consume plus the marginal propensity to invest must be less than unity $(C_Y + F_Y < 1)$. But, and this is paradoxical, if the marginal propensity to invest is sufficiently large, i.e., greater than $+2$, the marginal propensity to consume may exceed unity, and yet the equilibrium will be stable! Moreover, beyond a certain critical value the larger the marginal propensity to invest, the more stable is the system.

If we now consider the system in which none of the variables are taken as given, namely

[21] This inequality is in effect the formal justification of Keynes' reply to those criticizing his fundamental law, that the burden of proof lay upon them to explain why, if their allegations were correct, the economic system was not hopelessly unstable. See the passages quoted from a letter of Keynes in E. W. Gilboy, "The Propensity to Consume: Reply," *Quarterly Journal of Economics*, August, 1939, p. 634. While fundamentally correct, Keynes does overlook the possibility of other stabilizers such as the marginal propensity to invest, interest rate, etc.

[22] In the marginal-efficiency relation I have made investment depend upon income, which itself includes investment. Other writers, notably Lange (*op. cit.*), have made it depend only upon consumption. The result is indifferent since they can be shown to be equivalent. If, however, it is assumed that $dI/dC > 0$, the marginal propensity to invest, $dI/dY = (dI/dC)/[1 + (dI/dC)]$, cannot exceed unity.

$$C(i_t,\, Y_{t-1}) - Y_t + I_t = 0,$$
$$(85) \qquad\qquad F(i_t,\, Y_t) - I_t = 0,$$
$$L(i_t,\, Y_t) - M = 0,$$

stability requires that

$$(86) \qquad\qquad |\lambda| = \left| \frac{L_i C_Y}{\Delta + L_i C_Y} \right| < 1.$$

In what may be termed the normal case, where the marginal propensity to invest is less than unity, this requires as before that

$$(87) \qquad\qquad \Delta < 0,$$

and immediately all the eight determinations of sign of Case 1 become correct.

In the unusual, but possible, case where

$$(88) \qquad 1 - F_Y < 0 < C_Y < (F_Y - 1) - \frac{L_Y}{L_i}(F_i + C_i)$$

the equilibrium will be stable, but the signs of our 3×3 table now are as follows:

	i	Y	I
α	−	−	?
β	−	−	?
M	−	?	?

In words, the only theorem which remains true under all circumstances is that an increase in the amount of money must lower interest rates if the equilibrium is stable.

This example illustrates the additional complexities which systems based upon difference equations involve. In a later analytic treatment some of the reasons for this will be explained.

The examples here adduced serve, I hope, to illustrate the light which dynamical analysis sheds upon comparative statics. Problems in theory and business cycles of any complexity will almost surely require similar analytic treatment if useful and meaningful theorems are to be derived. In a later paper I examine in more detail formal mathematical aspects of the problem.

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