SMALL SAMPLE CONFIDENCE INTERVALS FOR MULTIVARIATE IRFs AT LONG HORIZONS

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MOTIVATION

Estimates of IRFs confidence bands are usually based on asymptotic normal approximation.

In practice, they depend on auxiliary hp on order of integration of the variables. So:

(1) economic results depend on whether series are $I(1)$, $I(0)$, CI.

(2) confidence bands coverage may be poor in small samples if high persistence.

GOAL: propose method to get confidence bands for IRFs that is robust to the presence of highly persistent processes.

HOW? Use local-to-unity asymptotic theory and allow IRF lead time to be fixed fraction of sample size.
EXAMPLE: Rogers (1999)
“Do monetary shocks influence real exchange rates?”

- Flexible price models  => NO
- Sticky price (Obstfeld-Rogoff)  => YES!

- Empirical evidence: VAR with
\[ \Delta(G/Y), \Delta Y, \Delta mm, \Delta rm, ?q \]

- Rogers (1999) runs both VAR in differences and in levels:
  \begin{align*}
  (\text{VAR1}) & \Delta(G/Y), \Delta Y, \Delta mm, \Delta rm, \Delta q \\
  (\text{VAR2}) & \Delta(G/Y), \Delta Y, \Delta mm, \Delta rm, q \\
  \end{align*}
He finds monetary shocks are important IN BOTH CASES

IS THIS APPROACH SATISFACTORY??
Neither VARs in levels nor VARs in differences (pre-test) are guaranteed to provide accurate IRF bands!

If data are truly I(1):
- VARs in differences (pre-test) work well
- VARs in levels work poorly

one minus coverage rate for $\rho = 1.00$
If data are persistent:

- VARs in differences (pre-test) work poorly
- VARs in levels work better, but still not great

one minus coverage rate for $\rho = 0.97$
If data are stationary:

- VARs in differences (pre-test) work poorly
- VARs in levels work better

Our method instead works quite well, especially at medium-long horizons.

How does it work? Remember...
GOAL: propose method to get confidence bands for IRFs that is robust to the presence of highly persistent processes

HOW?

- Use **local-to-unity** asymptotic theory:
  \[ y_t = \rho y_{t-1} + u_t, \quad \rho = 1 + \frac{c}{T} \]
- Allow IRF lead time to be **fixed fraction of sample size**: \( h = \delta T \)
Example: the half-life

- Use **local-to-unity** asymptotic theory:
  \[ y_t = \rho y_{t-1} + \epsilon_t, \quad \rho = 1 + \frac{c}{T} \text{ so that} \]
  \[ E_t y_{t+h} = \rho^h y_t \text{ and the IRF is: } \rho^h \]
- Allow IRF lead time to be **fixed fraction** of sample size: \( \frac{h}{T} = \delta \)

Combining the two, the IRF is:

\[ \rho^h = \rho^{T \frac{h}{T}} = \left[ (1 + \frac{c}{T})^T \right]^{\frac{h}{T}} \simeq (e^c)^{\frac{h}{T}} \simeq (e^c)^\delta \]
• Thus, the IRF at horizon \( h \) (IRF\(_h\)) is: \( e^{c\delta} \)

• The half-life is s.t.:

\[
e^{c\delta} = \frac{1}{2} \Rightarrow \delta = \frac{\log(1/2)}{c}
\]

\[
\Rightarrow h^* = \delta T = \frac{\log(1/2)}{c} T
\]
The IRF and the Half-life depend crucially on $c$:

$$IRF_h = e^{c\delta}$$

$$h^* = \frac{\log(1/2)}{c} T$$

To construct a confidence interval for these, by monotonicity, we can:

- first construct a confidence interval for “$c$”: $(c_L, c_U)$
- then use the formulas to construct confidence intervals for:
  - the IRF $h$: $(e^{c_L\delta}, e^{c_U\delta})$
  - the half-life $h^*$: $\left( \frac{\log(1/2)}{c_U} T, \frac{\log(1/2)}{c_L} T \right)$

but how do we get $(c_L, c_U)$?

There are various ways, the easiest (and fastest) is Stock’s (1991) method:
Figure 1

Confidence belt for local-to-unlity parameter $c$
based on demeaned ADF statistic

Bands in order of decreasing width: 95%, 90%, 80%, 70%; central line: median
In practice, the confidence interval for 'c' is obtained by using his table:

(source: Stock, 1991)

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<th>$c_0$</th>
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In this example, ADF test statistic = -3, thus $c_L = -26$, $c_U = 3.11$, $h = 20$
Since $c_L = -26$, $c_U = 3.11$, and let $h = 20$, $T = 100$

thus: $\delta = \frac{h}{T} = 0.2$

- the CI for IRF$_{h=12}$: $(e^{c_L\delta}, e^{c_U\delta}) = (e^{(-26)0.2}, e^{(-3.11)0.2})$
- the CI for half-life $h^*$:
  $\left(\frac{\log(1/2)}{c_U} T, \frac{\log(1/2)}{c_L} T\right) = \left(\frac{\log(1/2)}{-26} 100, \infty\right) = (2.6, \infty)$

Advantages of our method:

- easy and fast
- good coverage at medium/long horizons
- not pointwise
- no need to choose between I(0)/I(1)
- median unbiased
- robust to time trends and other deterministic components
PLAN

INTUITION

THE MODEL

DGP

Economic measures of interest

MONTE CARLO EVIDENCE

(1) Univariate IRFs
(2) Multivariate IRFs with one root $\approx 1$
(3) Multivariate IRFs with roots $\approx 1$

EMPIRICAL APPLICATIONS
THE MODEL

DGP:

\[(I - \Phi L)w_t = u_t\]

\[u_t = \Theta(L)\epsilon_t ,\]

\[\Theta(L) = \sum_{i=0}^{\infty} \Theta_i L^i , \Theta_0 = I\]

\[\eta_t = A_0\epsilon_t\]

\[\Phi = V^{-1} \Lambda V\]

Assumptions:

\[\Lambda = I + \frac{1}{T} C\]

\[\frac{h}{T} \rightarrow \delta\]
\[
\begin{align*}
    w_{t+h} &= \sum_{i=0}^{h-1} \Phi^i (I + \Phi^{-1} \Theta_1 + \ldots + \Phi^{-q} \Theta_q) \epsilon_{t+h-i} + \Phi^hw_t \\
    \approx & \Theta(I) \\
    \Rightarrow \quad w_{t+h} &= \sum_{i=0}^{h-1} \Phi^i \Theta(I)A_0 \eta_{t+h-i} + \Phi^hw_t + op(T^{1/2}) \\
    IRF_{h}^{k,j} &\equiv \frac{\partial \omega_{t+h}^{(k)}}{\partial \eta_{t}^{(j)}} = \Phi^h l'_k \Theta(I)A_0 l_j \\
    LR \text{ identification} &= \text{triangular } \Theta(I)A_0 \\
    \text{Wold ordering identification} &= \text{triangular } A_0.
\end{align*}
\]

\[
\begin{align*}
    FEV_h &\equiv E(w_{t+h} - \Lambda^h w_t)(w_{t+h} - \Lambda^h w_t)' \\
    &= \sum_{i=0}^{h-1} \Lambda^i V\Theta(I)A_0 A'_0 \Theta(I)' V' \Lambda^i \\
    \frac{1}{h} FEV_h &\to FEV_{\delta} = \int_0^{\delta} e^{Cs} V\Theta(I)A_0 A'_0 \Theta(I)' V' e^{Cs} ds \\
    FEV_{h}^{jk} &= \frac{FEV_{\delta}^{jk}}{\sum_{k=1}^{n} FEV_{\delta}^{jk}}
\end{align*}
\]
UNIVARIATE IRFs (Rossi, 2001)
Write DGP as:

\[(1 - \lambda_1 L) (1 - \lambda_2 L) \ldots (1 - \lambda_p L) (y_t - d_t) = \epsilon_t \]

\[= e^{c/T} \]

\[b(L) \text{ has STABLE roots} \]

Effect of the shock \(\epsilon_t\) on \(y_t\) after \(h\) periods.

\[
\frac{(y_{t+h} - \mu_0)}{(y_t - \mu_0)} = \frac{\lambda_1^{h+p-1}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \ldots (\lambda_1 - \lambda_p)} + \sum_{i=2}^{p} \frac{\lambda_i^{h+p-1}}{\prod_{k=1, k \neq i}^{p} (\lambda_i - \lambda_k)}
\]

As \(h \to \infty\), \(\lambda_i^{h+p-1} \to 0 \quad \forall i \neq 1\).

\[
\lambda_1^{h+p-1} = \left(1 + \frac{c}{T}\right)^{\frac{1}{T}(h+p-1)} = \left(1 + \frac{c}{T}\right)^{\frac{(h+p-1)}{T}} \to
\]

\[
\lambda_1 - \lambda_i = 1 + \frac{c}{T} - \lambda_i \approx 1 - \lambda_i \quad \forall i \neq 1
\]

\[
(1 - \lambda_2) (1 - \lambda_3) \ldots (1 - \lambda_p) = b(1). \quad \text{Thus:}
\]

\[
\frac{(y_{t+h} - \mu_0)}{(y_t - \mu_0)} \to e^{c\delta} b(1)^{-1} \quad \text{as} \quad T \to \infty
\]
PLAN

INTUITION

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Economic measures of interest

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(1) Univariate IRFs
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EMPIRICAL APPLICATIONS
UNIVARIATE IRFs

DGP:

\[
\prod_{j=1}^{p} (1 - \lambda_j L) y_t = \epsilon_t
\]

\(\lambda_j = \text{possible roots of the process,}\)

\(\lambda_1 = 1 + c/T.\)

Nominal coverage = 0.90, T=100, MC=1000.

We study confidence bands for IRFs ("are the effects of a shock still significant after \(h\) periods?")
We compare:

- our method;
- Andrews and Chen (1994);
- Wright (2000);
- standard (Efron) bootstrap;
- IRF in levels
- IRF in differences
- pre-test based IRF
$\rho = 0.99, \text{ } 1\text{-nominal coverage } = 0.90$

*Figure 1(a)* One minus coverage rates of various methods for AR(2) DGPs.
\[ \rho = 0.99, \text{ 1-nominal coverage} = 0.90 \]

*Figure 2(a)* One minus coverage rates of various methods for AR(2) DGPs.
\[ \rho = 0.95, \text{ 1-nominal coverage} = 0.90 \]

*Figure 1(b)* One minus coverage rates of various methods for AR(2) DGPs.
\[ \rho = 0.95, \text{ } 1\text{-nominal coverage} = 0.90 \]

*Figure 2(b)* One minus coverage rates of various methods for AR(2) DGPs.
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$\rho = 0.99$  \quad $\rho = 0.97$
Conclusions:
1. Our method has coverage close to nominal except when there is a second root $\approx 1$ or stationary/h small
2. Andrews-Chen method quickly worsens if more serial correlation
Bootstrap not median unbiased
Wright is conservative
3. Length of our method big. If upper bound for comparability, compares favorably

Our confidence bands contain the WHOLE trajectory of the IRF with a certain confidence level, i.e. they are NOT POINTWISE!
EMPIRICAL APPLICATION:
IRFs for real GDP
(Campbell and Mankiw, 1987)

Data are obtained from the Bureau of Economic Analysis, U.S. Department of Commerce
We allow for a deterministic time trend. Our method uncovers that shocks to real GDP are more persistent than commonly found (shocks to GDP have a half-life of 20 quarters, common methods estimate 7 quarters only).
PLAN

INTUITION

THE MODEL
   DGP
       Economic measures of interest

MONTE CARLO EVIDENCE
   (1) Univariate IRFs
   (2) Multivariate IRFs, one root \( \approx 1 \)
   (3) Multivariate IRFs, roots \( \approx 1 \)

EMPIRICAL APPLICATIONS
MULTIVARIATE IRFs WITH A ROOT CLOSE TO UNITY

DGP:

\[ w_{1t} = \mu + \rho w_{1t-1} + u_{1t} \]
\[ w_{2t} = w_{2t-1} + u_{2t} \]
\[ \rho = 1 + \frac{c}{T}, \quad \Psi(L)u_t = \varepsilon_t, \Psi(L) = 1 - \Psi L, \]
\[ \Psi = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix} \quad \text{Note that: } V = I, \]
\[ \Lambda = \begin{bmatrix} 1 + \frac{c}{T} & 0 \\ 0 & 1 \end{bmatrix}. \]

Let \( \Omega = \Psi(1)^{-1}\Sigma\Psi(1)^{-1}' = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \)

Assume \( \Omega^{1/2} = \Psi(1)^{-1}\Sigma^{1/2} \) is lower triangular so that shocks to \( w_{2t} \) do not affect \( w_{1t} \) in the long run. In this simple model, since \( \Psi \) is diagonal, \( \Sigma^{1/2} \) is also lower triangular.
METHODS

We invert:

- ADF as in Stock (1994)
- ERS (=Elliott, Rothemberg, Stock) as in Elliott and Stock (2001)
- Elliott and Jansson (2000), who exploit info from $\Delta w_{2t}$
  (captured by $R^2 = \omega_{11}^{-1} \omega_{21} \omega_{21} \omega_{22}^{-1}$)
- VAR in first difference
- VAR in level
- Pre-test (by either ADF or ERS)
For this model, the structural IRFs at long horizons are:

\[
\frac{\partial w_{t+h}}{\partial \eta_t} = \Lambda^h \Theta(I) \Sigma^{1/2} = \Lambda^h \Omega^{1/2}
\]

\[
= \begin{bmatrix}
    e^{c\delta} & 0 \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    \omega_{11}^{1/2} & 0 \\
    \omega_{21} \omega_{11}^{-1/2} & \omega_{2,1}^{1/2}
\end{bmatrix}
\]

Notice that

\[
\omega_{2,1} = \omega_{22} - \omega_{21} \omega_{11}^{-1} \omega_{12} = \omega_{22} (1 - R^2).
\]

\[
\frac{\partial w_{1t+h}}{\partial \eta_{1t}} = \omega_{11}^{1/2} e^{c\delta}
\]

\[
\frac{\partial w_{1t+h}}{\partial \eta_{2t}} = 0
\]

\[
\frac{\partial w_{2t+h}}{\partial \eta_{1t}} = \omega_{21} \omega_{11}^{-1/2} = \omega_{22}^{1/2} (R^2)^{1/2}
\]

\[
\frac{\partial w_{2t+h}}{\partial \eta_{2t}} = \omega_{2,1}^{1/2} = \omega_{22}^{1/2} (1 - R^2)^{1/2}
\]
Figure 4(a): one minus coverage rate for various values of $\rho$, $R^2 = 0.5$
Figure 4(b): one minus coverage rate for various values of $\rho$, $R^2 = 0.5$
CONCLUSIONS

- Pre-test and VARs in levels/difference have big coverage distortions
- Elliott and Jansson (2000) delivers confidence bands with smallest length:

Table 4: Median confidence interval length

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<th>$\delta$</th>
<th>ADF</th>
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<th>EJ</th>
<th>Pret_ADF</th>
<th>Pret_ERS</th>
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MULTIVARIATE WITH 2 ROOTS LOCAL TO UNITY

DGP:

\[ w_t = \Phi w_{t-1} + \Sigma^{1/2} \eta_t, \eta_t \sim N(0, I) \]

\[ \Phi = \begin{pmatrix} 0.9975 & 0 \\ 0 & 0.9889 \end{pmatrix}; \Sigma^{1/2} = \begin{pmatrix} 1 & 0.4 \\ 0 & 1 \end{pmatrix} \]

The following figure shows that coverage properties remain pretty good.
Figure 5: one minus coverage rate for various values of \( \rho = 0.90 \), \( R^2 = 0.5 \) for different values of the second root.
PLAN

INTUITION

THE MODEL
  DGP
  Economic measures of interest

MONTE CARLO EVIDENCE
  (1) Univariate IRFs
  (2) Multivariate IRFs with one root \( \approx 1 \)
  (3) Multivariate IRFs with roots \( \approx 1 \)

EMPIRICAL APPLICATIONS
EMPIRICAL APPLICATION TO EXCHANGE RATES
Eichenbaum and Evans (1995)

Wold ordering identification:

- U.S. industrial production
- U.S. CPI
- U.S. non borrowed reserves / total reserves (measure of mon. policy – we focus on contractionary moves)
- short term interest rate differential (home - foreign)
- real exchange rate (an increase in the real exchange rate means that the rate depreciates)
  (thus, a contractionary monetary policy move is expected to drive the real exchange rate down)
Eichenbaum and Evans (1995) used a VAR in levels to calculate the IRFs. Was that appropriate?

Table 7: Unit root tests for real exchange rates

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<td>0.192</td>
<td>0.126</td>
</tr>
<tr>
<td>EJ 5% c.v.</td>
<td>3.454</td>
<td>3.545</td>
<td>3.530</td>
<td>3.444</td>
</tr>
<tr>
<td>N. lags</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

(†) The 5% and 2.5% critical values for ADF are, respectively, -2.890 and -3.170.
(‡) The 5% and 2.5% critical values for ERS are, respectively, 3.11 and 2.47.

So we cannot reject a unit root – but we cannot reject many persistent specifications as well...
Figure 6(a): Confidence Intervals for a response of $q_t$ to a monetary shock

†The confidence bands are the following: VAR levels (solid line with diamonds), ERS (solid line - the central, thickest line is the median unbiased IRF estimate), VAR first differences (dotted line with stars).
Figure 6(b): Confidence Intervals for a response of $q_t$ to a monetary shock†

†The confidence bands are the following: VAR levels (solid line with diamonds), ERS (solid line - the central, thickest line is the median unbiased IRF estimate), VAR first difference (dotted line with stars).
EXTENSIONS AND RELATED WORK:

- Similar method applicable to CI for half-life for PPP (Rossi, Confidence intervals for half-life deviations from PPP, 2001)
- Apply this method to Francis-Ramey/Christiano-Eichenbaum-Vigfusson debate (Pesavento and Rossi, Do technology shocks drive hours up or down? Evidence from an agnostic procedure, 2004)
- Application to Expectations Hypotheses and Predictive Ability tests