

# Recursive Predictability Tests for Real-Time Data

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Work in progress  
Comments welcome!!

## Motivation

- Does money have a predictive content for output?
- Is there evidence of non-linearities?

Answers typically rely on:

1. simple Granger-causality tests (e.g. Stock and Watson (1989))
2. tests robust to time-varying parameters (e.g. Amato and Swanson (2001) and Stock and Watson (1999)).

### PROBLEM WE ADDRESS:

When time-varying parameters, it is tempting to use predictability tests recursively.

Tests applied recursively (“one shot tests”) have correct size  $\forall t$  but overall size  $\rightarrow 1$ .

## CONTRIBUTION OF THIS PAPER

1. Propose a procedure to recursively test for predictive ability with the correct size (“recursive tests”).
2. Our test statistics are calculated as usual, only their critical values are different, and depend upon the sample size.
3. Our test is more powerful than the fluctuation test proposed by Chu, Stinchcombe and White (1996).
4. We allow for a general GMM framework and possibly nonlinear regressions.
5. Link with forecasting in real time: repeated tests select overfitted models and deteriorate forecasting ability.
6. Difference with common tests for model selection or parameter instability. Latter are one-shot in nature and inapplicable to real-time data subject to revisions.

## GMM estimator

Want to test  $H_0$  :

$$H_0 : a(\theta_0) = 0,$$

by using sequential moments:

$$E[g(z_{Tt}, \theta_0)] = 0$$

Define: 1) *unconstrained GMM*:

$$\hat{\theta}_t = \hat{\theta}_T(\pi) = \arg \min_{\theta \in \Theta} \hat{Q}_T(\theta, \pi),$$

$$\hat{Q}_T(\theta, \pi) \equiv \frac{1}{2} \hat{g}_T(\theta, \pi)' \hat{S}_T(\pi) \hat{g}_T(\theta, \pi),$$

where:  $t = [T\pi] = T + 1, T + 2, \dots,$

$$\hat{g}_T(\theta, \pi) = (1/[T\pi]) \sum_{t=1}^{[T\pi]} g(z_{Tt}, \theta)$$

$S_T(\pi)$  = a sequence of consistent estimators of l.r. covariance of  $g(z_{Tt}, \theta_0)$ .

2) *constrained GMM*:

$$\bar{\theta}_t = \arg \min_{\theta \in \Theta} Q_T(\theta, \pi) \quad s.t. \quad a(\theta) = 0$$

## The Sequential Tests

Wald, Lagrange multiplier and LR-like test statistics:

$$W_T(\pi) = [T\pi] a(\hat{\theta}_T(\pi)) \hat{\Omega}^{-1} a(\hat{\theta}_T(\pi)),$$

$$LM_T(\pi) = [T\pi] \nabla Q_T(\bar{\theta}_T(\pi), \pi)' \bar{\Omega}^{-1} \nabla Q_T(\bar{\theta}_T(\pi), \pi),$$

$$LR_T(\pi) = -2[T\pi] (Q_T(\hat{\theta}_T(\pi), \pi) - Q_T(\bar{\theta}_T(\pi), \pi)),$$

$$\hat{\Omega} = [A(\hat{\theta}_T(\pi)) (\hat{G}'_T(\pi) \hat{S}_T(\pi)^{-1} \hat{G}'_T(\pi))^{-1} A(\hat{\theta}_T(\pi))$$

$$\bar{\Omega} = \bar{G}'_T(\pi) \bar{S}_T(\pi)^{-1} \bar{G}'_T(\pi))$$

$$\hat{G}_T(\pi) = (1/[T\pi]) \sum_{t=1}^{[T\pi]} \partial g(z_{Tt}, \hat{\theta}_T) / \partial \theta',$$

$\bar{G}_T(\pi)$ ,  $\bar{S}_T(\pi)$  are  $\hat{G}_T(\pi)$ ,  $\hat{S}_T(\pi)$  with  $\hat{\theta}_T(\pi)$  replaced by  $\bar{\theta}_T(\pi)$ .

## Assumption 1

- (a)  $\{z_{Tt}\}$  is a triangular array of random variables that is  $L^0$ -NED on a strong mixing base.
- (b) For some  $r > 2$ ,  $\{g(z_{Tt}, \theta_0)\}$  is a triangular array of random variables that is  $L^2$ -NED of size  $-1/2$  on a strong mixing base of size  $-r/(r-2)$  and  $\sup_{t \leq T, T \geq 1} E \|g(z_{Tt}, \theta_0)\|^r < \infty$ .
- (c)  $\lim_{T \rightarrow \infty} ((1/\sqrt{T}) \sum_{t=1}^{[T\pi]} g(z_{Tt}, \theta_0)) = \pi S$  for  $\forall \pi \in [1, \infty)$  for some positive definite  $q \times q$  matrix  $S$ .
- (d)  $\sup_{\pi \in [1, \infty)} \|\hat{\theta}(\pi) - \theta_0\| \xrightarrow{p} 0$  and  $\sup_{\pi \in [1, \infty)} \|\bar{\theta}(\pi) - \theta_0\| \xrightarrow{p} 0$  for some  $\theta_0$  in the interior of  $\Theta$ .
- (e)  $\sup_{\pi \in [1, \infty)} \|\hat{S}(\pi) - S\| \xrightarrow{p} 0$ .

- (f)**  $g(z, \theta)$  is differentiable in  $\theta$  for  $\forall \theta \in \Theta$ ,  $\forall z \in Z$ ,  $g(z, \theta)$  is Borel measurable in  $z$  for  $\forall \theta \in \Theta$ ,  $\partial g(z, \theta)/\partial \theta$  is continuous in  $(z, \theta)$  on  $Z \times \Theta$ , and

$$\sup_{t \leq T, T \geq 1} E \left[ \sup_{\theta \in \Theta} \|\partial g(z_{Tt}, \theta)/\partial \theta'\|^{1+\varepsilon} \right] < \infty.$$

for some  $\varepsilon > 0$ .

- (g)**  $\lim_{T \rightarrow \infty} (1/T) \sum_{t=1}^{T\pi} E[\partial g(z_{Tt}, \theta)/\partial \theta']$  exists uniformly over  $\pi \in [1, \infty)$  and equals  $\pi G$  for  $\forall \pi \in [1, \infty)$ .

- (h)**  $G$  is of rank  $p$ .

- (i)** There is a function  $\mu : \mathfrak{R} \rightarrow \mathfrak{R}^q$  such that

$$\sup_{\pi \in [1, \infty)} \left\| T^{-1/2} \sum_{T=1}^{[T\pi]} g(z_{Tt}, \theta_0) - \mu(\pi) \right\| = o_p(1).$$

- (j)**  $a$  is continuously differentiable and  $A = \partial a(\theta_0)/\partial \theta'$  is of rank  $r$ .

**Theorem 1:** *Under Assumption 1 we have:*

$$W_T(\cdot) \Rightarrow \tilde{B}_q(\cdot)' \Upsilon \tilde{B}_q(\cdot),$$

$$LM_T(\cdot) \Rightarrow \tilde{B}_q(\cdot)' \Upsilon \tilde{B}_q(\cdot),$$

$$LR_T(\cdot) \Rightarrow \tilde{B}_q(\cdot)' \Upsilon \tilde{B}_q(\cdot)$$

where  $B_q(\cdot)$  is the  $q$ -dimensional standard Brownian motion,

$$\tilde{B}_q(\cdot) \equiv \frac{1}{\sqrt{\cdot}} B_q(\cdot) + S^{-\frac{1}{2}} \mu(\cdot)$$

$$\Upsilon \equiv C'(CC')^{-1}C$$

$$C \equiv A(G'S^{-1}G)^{-1}G'S^{-1/2},$$

$$M \equiv I_k - (G'S^{-1}G)^{-1/2}A'(A(G'S^{-1}G)^{-1}A')^{-1} \times \\ \times A(G'S^{-1}G)^{-1/2}.$$

**Theorem 2:** *Under Assumption 1 with  $\mu(\cdot) \equiv 0$ , as  $T \rightarrow \infty$ , we have:*

$$\begin{aligned}
& P(W_t \leq c^2 + \ln\left(\frac{t}{T}\right), \forall t \geq T) \rightarrow \\
& P(B_r(\pi)'B_r(\pi)/\pi \leq c^2 + \ln(\pi), \forall \pi \geq 1) \\
& = \int \prod_{i=1}^r \{1 - 2[1 - \Phi(a_i) + a_i\phi(a_i)]\} da,
\end{aligned}$$

$$\begin{aligned}
& P(LM_t \leq c^2 + \ln\left(\frac{t}{T}\right), \forall t \geq T) \rightarrow \\
& P(B_r(\pi)'B_r(\pi)/\pi \leq c^2 + \ln(\pi), \forall \pi \geq 1) \\
& = \int \prod_{i=1}^r \{1 - 2[1 - \Phi(a_i) + a_i\phi(a_i)]\} da,
\end{aligned}$$

$$\begin{aligned}
& P(LR_t \leq c^2 + \ln\left(\frac{t}{T}\right), \forall t \geq T) \rightarrow \\
& P(B_r(\pi)'B_r(\pi)/\pi \leq c^2 + \ln(\pi), \forall \pi \geq 1) \\
& = \int \prod_{i=1}^r \{1 - 2[1 - \Phi(a_i) + a_i\phi(a_i)]\} da,
\end{aligned}$$

show table 1

## A small Monte Carlo

Models are nested and linear:

$$M_0 : y_{t+1} = \epsilon_{t+1},$$

$$M_k : y_{t+1} = \beta' x_t^{(k)} + \epsilon_{t+1} \quad k = 1, 2$$

DGP is  $y_{t+1} = \beta' x_t + \epsilon_{t+1}$ ,

$t = 1, \dots, T^{\max}$  where  $T^{\max} = 500$ .

$$\beta = 0,$$

$$\epsilon_t \sim N_k(0, I)$$

$$x_t^{(k)} \sim N_k(0, \Sigma_X)$$

where:

$$\Sigma_X^{(1)} = 1 \text{ for } k = 1$$

$$\Sigma_X^{(2)} = \begin{pmatrix} 1 & \rho_X \\ \rho_X & 1 \end{pmatrix} \text{ for } k = 2$$

## TEST WE COMPARE

### 1) “Fluctuation test”:

$$F_t = \sqrt{t} \hat{\Sigma}_t^{-1/2} \hat{\beta}_t$$

where  $\hat{\Sigma}_t \equiv \hat{\sigma}^2 \left( \frac{1}{t} \sum_{s=1}^t x_s x_s' \right)^{-1}$ . For  $t = Tr$ ,  
 $F_t \rightarrow B_k(r)$  = std  $k$ -dim Brownian Motion

From CSW, reject if

$$\max_k |F_t| > k_\alpha^F,$$

$$k_\alpha^F \text{ s.t. } \alpha = (1 - \Phi(k_\alpha^F) - a\phi(k_\alpha^F))$$

### 2) “Recursive Wald test”:

$$W_t = t \hat{\beta}_t' \hat{\Sigma}_t^{-1} \hat{\beta}_t$$

$W_t = F_t' F_t \rightarrow R_k(r)$  = std Bessel process

### 3) “One-shot GC test”: $\forall t$ ,

$$W_t = t \hat{\beta}_t' \hat{\Sigma}_t^{-1} \hat{\beta}_t \sim \chi_k^2$$

## **Local power comparison.**

show FIGURE 1 AND TABLE 2

# **Relationship between recursive model selection and forecasting.**

show FIGURE 2 and TABLE 3

## **Empirical applications**

### **1) The real-time predictive content of money for output.**

#### **DATA:**

Quarterly real-time data for M1 and M2, and industrial production from the Federal Reserve Bank of Philadelphia.

Data on 3-month Treasury Bills (never revised).

Data for CPI (not s.a.) from N. Swanson.

Estimation period = 1959:01 to 1978:01

#### **RESULTS:**

We recursively test whether a subset of the parameters are equal to zero at each point in time between 1978:1 and 2002:11.

Based upon the results of the test, we select the forecasting model.

INSERT TABLES 4 AND 5, AND FIGURE  
3  
INSERT TABLE 6

## **Empirical applications**

### **2) Linear versus non-linear models, and forecasting**

Non-linear models fit the data better than linear, but forecasts are worse (e.g. Stock and Watson (1999), Swanson and White (1997)).

#### **DATA**

##### **Unemployment:** Civilian U.S.

Unemployment Rate, s.a., monthly, percentage units, 1948:01 -2003:07. From U.S. Department of Labor, Bureau of Labor Statistics.

**Real and Nominal GDP:** Nominal U.S. GDP, s.a., annual, quarterly, billions \$ units, 1947:1- 2003:4. Real U.S. GDP, s.a., quarterly, billions of chained 1996 \$, from 1947:1-2003:4. From U.S. Department of Commerce, Bureau of Economic Analysis.

##### **Consumption:** Real Personal

Consumption expenditures, s.a., annual rate, monthly, in billions of chained 1996 Dollars, 1967:1-2003:6. From U.S. Department of Commerce, Bureau of Economic Analysis.

All variables except unemployment are first differences of logs; unemployment is in levels.

Recursively choose between nested linear and non-linear (ANN) models

ANN (single layer with 1 hidden unit):

$$y_t = \beta_0' \zeta_t + \gamma_1 g(\beta_1' \zeta_t) + \epsilon_t, \quad g(z) = (1 + e^z)^{-1}$$

where  $\zeta_t = [y_{t-1}, y_{t-2}, \dots, y_{t-p}]$ .

Test for non-linearities (Teräsvirta et al. (1993)).  $H_0 : \varphi_{ij} = 0$  in:

$$\hat{u}_t = \beta_0' \zeta_t + \sum_{i=1}^p \sum_{j=1}^{p-1} \varphi_{ij} \zeta_t^{(i)} \zeta_t^{(j)}$$

where  $\hat{u}_t$ , is the residual of a linear regression of  $y_t$  onto  $\zeta_t$ , and  $\zeta_t^{(i)}$  is the  $i$ -th column of  $\zeta_t$  (constant excluded).

If reject linear, estimate model by Stock and Watson (1999) or NLLS.

## **Conclusions**

This paper proposes a test for recursive predictive ability.

The test statistics are the same as those commonly used in the literature, only their critical values are different.

These critical values are reported and can be easily used.

Our test has good power and can significantly improve forecasts based on recursive model selection procedures.

Find weaker empirical evidence of predictive ability of money for output, and of non-linear relationships between some representative macroeconomic variables.