Testing for One-Factor Models versus Stochastic Volatility Models, in the Presence of Jumps and Microstructure Noise

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INTRODUCTION AND MOTIVATION

In the literature, different models for the short term rate. Some are one factor diffusion processes (e.g. Vasicek, 1977, Cox, Ingersoll and Ross, 1985). Others are multi-factor (e.g. Fong and Vasicek, 1991, Longstaff and Schwartz, 1992, Balduzzi et al., 1996).

Many models for the short rate with stochastic volatility, but nobody has formally checked whether the short term rate is driven by more than one factor.

It is well known, since Litterman and Scheinkman (1991), that three factors (level, steepness and curvature) drive most of the variation of the yield curve. Yet, it is not entirely clear what these factors are.

For example, Andersen, Benzoni and Lund (2003) suggest that short-term volatility could be one of these factors.
But, how sure are we that the short-rate has really stochastic volatility?

This paper tries to give an answer to the question above, through a test which discriminates between the classes of one factor and stochastic volatility models.

No assumptions are made on functional forms of either the drift or the diffusion term. Being able to choose between classes of models, our approach is nonparametric in nature.

Our tests based on comparing two estimators of quadratic variation:

1. a kernel estimator of the instantaneous variance, averaged over the sample realization of the asset trajectory;

2. realized volatility.
The two estimators have different behaviour under the null and alternative hypotheses; → our test as an Hausman type test.

We also propose a robust to jumps robust version of the test, through a slight modification of our estimators.

Finally, we suggest a two-scale modification of the test, which is robust to microstructure error.

Tests for the null hypothesis of one factor models versus stochastic volatility models have been already suggested in the financial literature. One factor models have important implications for option prices:

(i) monotonicity. Call (put) options are monotonically increasing (decreasing) in the prices of the underlying asset (but, see counterexamples in a jump-diffusion setting, by Bergman, Grundy and Wiener, 1996);
(ii) perfect correlation. As there is only one Brownian motion driving the behaviour of the asset price, option prices are perfectly correlated with the underlying asset prices;

(iii) redundancy. Option prices can be perfectly replicated with the risk free asset and the underlying asset, and so are redundant securities.

Bakshi, Cao and Chen (2000) have suggested statistical tests for monotonicity and perfect correlation, while Buraschi and Jackwerth (2001) have suggested a test for redundancy.

Both papers require the use of option data. Therefore, one has to choose both the moneyness and the maturity of the options, and the outcome of the test may be rather sensitive to that.

The test suggested here simply requires the availability of observations on the price (level) of the underlying asset.
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SET-UP

We consider the following class of one-factor diffusion models

\[ dX_t = \mu(X_t)dt + \sigma_t dW_{1,t} \]
\[ \sigma_t = \sigma(X_t) \]  

(1)

and the following class of stochastic volatility models

\[ dX_t = \mu_1(X_t)dt + \sigma_t \left((1-\rho^2)^{1/2} dW_{1,t} + \rho dW_{2,t}\right) \]
\[ d\sigma_t^2 = b_1(\sigma_t^2)dt + b_2(\sigma_t^2)dW_{2,t} \]

(2)

where \(W_{1,t}\) and \(W_{2,t}\) may be correlated, thus allowing for leverage effects.

We state the hypothesis of interest as

\[ H_0 : \sigma_t^2 = \sigma^2(X_t) \in \mathcal{F}_t^X \text{ a.s.} \]

versus the alternative

\[ H_A : \sigma_t^2 \text{ is a diffusion process not } \mathcal{F}_t^X \text{—measurable, a.s.} \]
Suppose we have data recorded at frequency $1/n$ over a fixed time span $[0, T]$, with $T = 1$ for notational convenience.

The proposed statistics are based on

$$Z_{n,r} = n^{1/2} \left( \frac{1}{n} \sum_{i=1}^{[(n-1)r]} S_n^2(X_{i/n}) - RV_{n,r} \right),$$

(3)

where $r \in (0, 1)$,

$$S_n^2(X_{i/n}) = \sum_{j=1}^{n-1} K \left( \frac{X_{j/n} - X_{i/n}}{\xi_n} \right) n \left( X_{(j+1)/n} - X_{j/n} \right)^2,$$

(4)

$$RV_{n,r} = \sum_{j=1}^{[(n-1)r]} \left( X_{(j+1)/n} - X_{j/n} \right)^2.$$

(5)

$S_n^2(X_{i/n})$ is a nonparametric estimator of the instantaneous volatility process evaluated at $X_{i/n}$ first studied Florens-Zmirou (1993).
$S^2_n(X_{i/n})$ is a consistent estimator of the instantaneous variance only under the null hypothesis. Therefore, also its average over the sample realization of the process on a finite time span, $1/n \sum_{i=1}^{[(n-1)r]} S^2_n(X_{i/n})$, is a consistent estimator of integrated volatility only under the null hypothesis.

$RV_{n,r}$, which is known as realized volatility. It is a “model free” estimator of the quadratic variation of the processes and is consistent for the integrated (daily) volatility under both hypotheses.

Assumptions: local Lipschitz and growth conditions on drift and variance to ensure the existence of strong solutions under both hypotheses. Kernel with bounded support.
ASYMPTOTICS

Theorem 1: Let Assumption A hold.

Under $H_0$,

(i) if, as $n, \xi_n^{-1} \to \infty, n\xi_n \to \infty$ and for any arbitrarily small $\varepsilon > 0, n^{1/2+\varepsilon}\xi_n \to 0$, then, pointwise in $r \in (0,1)$

$$Z_{n,r} \overset{d}{\to} Z_r$$

$$\equiv \text{MN} \left( 0, 2 \int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r,a)(L_X(1,a) - L_X(r,a))}{L_X(1,a)} da \right)$$

where

$$L_X(r,a) = \lim_{\psi \to 0} \frac{1}{\psi} \frac{1}{\sigma^2(a)} \int_{0}^{r} 1_{\{X_u \in [a, a+\psi]\}} \sigma^2(X_u) du$$

denotes the standardized local time of the process $X_t$.

(ii) Define $Z = \max_{j=1,\ldots,J} |Z_{r_j}|$. If, as $n, \xi_n^{-1} \to \infty, n\xi_n \to \infty$, and, for any $\varepsilon > 0$ arbitrarily small,
$n^{1/2+\varepsilon} \xi_n \to 0$, then

$Z_n \overset{d}{\to} Z$,

with

$$
\begin{pmatrix}
Z_{r_1} \\
Z_{r_2} \\
\vdots \\
Z_{r_J}
\end{pmatrix} 
\sim 
\text{MN} \left( 0, \begin{pmatrix}
V(r_1, r_1) & V(r_1, r_2) & \cdots & V(r_1, r_J) \\
V(r_2, r_1) & V(r_2, r_2) & \cdots & V(r_2, r_J) \\
\vdots & \vdots & \ddots & \vdots \\
V(r_J, r_1) & V(r_J, r_2) & \cdots & V(r_J, r_J)
\end{pmatrix} \right),
$$

where $\forall r < r'$

$$V(r, r') = V(r', r) = 2 \times$$

$$\int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r, a)(L_X(1, a) - L_X(r', a))}{L_X(1, a)} da.$$ 

(iii) Under $H_A$, if, as $n, \xi_n^{-1} \to \infty$, $n\xi_n \to \infty$ and $n\xi_n^2 \to 0$, then, pointwise in $r \in (0, 1]$, 

$$\text{Pr} \left( \omega : \frac{1}{n^{1/2}} |Z_n, r(\omega)| \geq \varsigma(\omega) \right) \to 1,$$
where \( \zeta(\omega) > 0 \) for all \( \omega \), but a subset with probability measure approaching zero.

The limiting distribution in (a) is a mixed normal with variance equal to the difference between the variances of the inefficient and efficient estimator. Part (b) follows from the joint limiting distribution of \((Z_{n,r_1}, \ldots, Z_{n,r_J})\) and the continuous mapping theorem.

Under the alternative, there exists a (almost surely) strictly positive random variable \( \zeta \), such that \((1/n^{1/2})|Z_{n,r}| \geq \zeta\), with probability approaching one. Intuitively, what drives the power is the fact that there are subsets of strictly positive probability measure in which \( X_{i/n} \) and \( X_{j/n} \) are very close each other, while \( \sigma_{i/n}^2 \) and \( \sigma_{j/n}^2 \) are instead far away each other.
COMPUTING VALID ASYMPTOTIC CRITICAL VALUES

The variance components have to be estimated. In order to conduct inference based on the statistic $Z_n$, we first need a consistent estimator of the matrix in Theorem 1(b). Furthermore, the maximum over a $J$-dimensional mixed normal random variable is no longer a mixed normal.

Need a simulation-based procedures which allows to compute asymptotic valid critical values for $Z_n$. For $j = 1, \ldots, J$, define,

$$
\hat{\Omega}_n(r_j, r_j) = \frac{2}{n} \sum_{i=1}^{[(n-1)r_j]} \left( \sum_{h=1}^{[(n-1)r_j]} K \left( \frac{X_{h/n} - X_{i/n}}{\xi_n} \right) \right) \sigma_n^4(X_{i/n})
$$
with
\[ \hat{\sigma}_n^4(X_i/n) = \frac{1}{3} \sum_{j=1}^{n-1} \frac{K \left( \frac{X_j/n - X_i/n}{\xi_n} \right)}{\sum_{l=1}^{n-1} K \left( \frac{X_l/n - X_i/n}{\xi_n} \right)} n^2 \left( \frac{X_{(j+1)/n} - X_{j/n}}{\xi_n} \right)^4 \]

and also define,
\[ \hat{C}_n(r, r) = \frac{1}{n} \sum_{i=1}^{[(n-1)r]} \hat{\sigma}_n^4(X_i/n) \]

and construct
\[ \hat{V}_n(r_j, r_j) = \hat{C}_n(r_j, r_j) - \hat{\Omega}_n(r_j, r_j) \]

As for the cross terms, for all \( r_j \neq r_j' \), define
\[ \hat{\Omega}_n(r_j, r_j') = \frac{2}{n} \sum_{i=1}^{[(n-1)r_j]} \left( \sum_{h=1}^{[(n-1)r_j]} \frac{K \left( \frac{X_h/n - X_i/n}{\xi_n} \right)}{\sum_{l=1}^{n-1} K \left( \frac{X_h/n - X_l/n}{\xi_n} \right)} \right) \hat{\sigma}_n^4(X_i/n) \]
and
\[ \hat{V}_n(r_j, r'_j) = \hat{C}_n \left( \min(r_j, r'_j), \min(r_j, r'_j) \right) - \hat{\Omega}_n(r_j, r'_j) \] (6)

We can now proceed as follows. For \( s = 1, ..., S \), where \( S \) denotes the number of replications, let

\[
\hat{d}^{(s)}_{n, r} = \begin{pmatrix}
\hat{d}^{(s)}_{n, r_1} \\
\hat{d}^{(s)}_{n, r_2} \\
\vdots \\
\hat{d}^{(s)}_{n, r_J}
\end{pmatrix}
\]

\[ = \begin{pmatrix}
\hat{V}_n(r_1, r_1) & \hat{V}_n(r_1, r_2) & \cdots & \hat{V}_n(r_1, r_J) \\
\hat{V}_n(r_1, r_2) & \hat{V}_n(r_2, r_2) & \cdots & \hat{V}_n(r_2, r_J) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{V}_n(r_1, r_J) & \hat{V}_n(r_2, r_J) & \cdots & \hat{V}_n(r_J, r_J)
\end{pmatrix}^{1/2}
\]

\[ \times \begin{pmatrix}
\eta^{(s)}_1 \\
\eta^{(s)}_2 \\
\vdots \\
\eta^{(s)}_J
\end{pmatrix}, \]
where for each \( s \), \( \left( \eta_1^{(s)} \eta_2^{(s)} \ldots \eta_J^{(s)} \right)' \) is drawn from a \( N(0, I_J) \).

Compute \( \max_{j=1,\ldots,J} |\hat{d}_{n,r}^{(s)}| \), repeat this step \( S \) times, and construct the empirical distribution. Finally, let \( CV_{\alpha}^{S,n} \), the \((1 - \alpha)\) quantile of \( \max_{j=1,\ldots,J} |\hat{d}_{n,r}^{(s)}| \). We have:

**Proposition 1:** Let Assumption A hold and suppose that as \( n, \xi_n^{-1} \to \infty, n\xi_n \to \infty \), and, for any \( \varepsilon > 0 \) arbitrarily small, \( n^{1/2 + \varepsilon} \xi_n \to 0 \). Then, as \( n \to \infty \) and \( S \to \infty \),

(i) under the null,

\[
\lim_{n \to \infty} \Pr \left( Z_n > CV_{\alpha}^{S,n} \right) = \alpha
\]

and

(ii) under the alternative,

\[
\lim_{n \to \infty} \Pr \left( Z_n > CV_{\alpha}^{S,n} \right) = 1.
\]
ALLOWING FOR JUMPS

We now provide a test for discriminating between the following class of one-factor jump diffusions

\[ \begin{align*}
    dX_t &= \mu(X_{t-})dt + \sigma_t dW_{1,t} + dJ_t \\
    \sigma_t^2 &= \sigma^2(X_{t-})
\end{align*} \]

and the following class of stochastic volatility jump diffusions

\[ \begin{align*}
    dX_t &= \mu_1(X_{t-})dt + \sigma_t \left( (1 - \rho^2)^{1/2} dW_{1,t} + \rho dW_{2,t} \right) \\
    + dJ_t' \\
    d\sigma_t^2 &= b_1(\sigma_t^2)dt + b_2(\sigma_t^2)dW_{2,t} + dJ_t''
\end{align*} \]

where \( J_t = \sum_{i=1}^{N_t} c_i \), \( N_t \) is a finite activity counting process, the \( c_i \) are i.i.d. random variables independent of \( N_t \) and \( W_t \).

Why can’t we use the previous results? Because the effect of a jump at time \( i/n \) to \( \frac{1}{n} \sum_{i=1}^{[nr]-1} S_n^2(X_{i/n}) \) depends on the local time at \( X_{i/n} \), while the effect to \( RV_{n,r} \) does not.
Literature on estimating quadratic variation of jump diffusion processes. Bandi and Nguyen (2003) provide consistent estimation of the conditional moments of the diffusion in (7). They show that the conditional second moment captures both the continuous and the jump component of volatility.

We state the hypotheses of interest as

$$H'_0 : \sigma^2_t \left( X_t^- \right) \in \mathcal{F}_t^X, \text{ a.s.},$$

versus the alternative

$$H'_A : \sigma^2_t \text{ is a diffusion process not } \mathcal{F}_t^X \text{-measurable, a.s.},$$

where $\mathcal{F}_t^X = \sigma \left( X_{s^-}, s \leq t \right)$.

The statistic is based on

$$Z_{J_n,r} = \mu_{2/3}^{-3} n^{1/2} \left( \frac{1}{n} \sum_{i=1}^{\lfloor nr \rfloor} -3 \left( SJ_n^2(X_i/n) - TV_{n,r} \right) \right)$$
where $\mu_k = E \left( |Z|^k \right)$, with $Z$ a standard normal, $r \in (0, 1)$,

$$SJ_n^2(X_{i/n})$$

$$= \frac{1}{\sum_{j=1}^{n-3} K^+ \left( \frac{X_{i/n} - X_{j/n}}{\xi_n} \right)} \left( \sum_{j=1}^{n-3} K^+ \left( \frac{X_{i/n} - X_{j/n}}{\xi_n} \right) \right)$$

$$n \left( \left| \Delta X_{(j+3)/n} \right|^{2/3} \left| \Delta X_{(j+2)/n} \right|^{2/3} \left| \Delta X_{(j+1)/n} \right|^{2/3} \right)$$

where $K^+$ is a right kernel function (i.e. zero for negative values) and

$$TV_{n,r}$$

$$= \sum_{j=1}^{[nr]-3} \left| \Delta X_{(j+3)/n} \right|^{2/3} \left| \Delta X_{(j+2)/n} \right|^{2/3} \left| \Delta X_{(j+1)/n} \right|^{2/3} .$$

Limit theory for $TV_{n,r}$ by Barndorff-Nielsen, Shephard and Winkel (2006).
Theorem 0.1 Under $H'_0$,

(i) a pointwise in $r \in (0, 1)$

\[ Z J_{n,r} \xrightarrow{d} Z J_r \]
\[ \sim \text{MN} \left( 0, \beta \int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r, a) (L_X(1, a) - L_X(r, a))}{L_X(1, a)} da \right) \]

where

\[ \beta = \frac{\mu_{4/3}^3 - 5\mu_{2/3}^6 + 2 \left( \mu_{2/3}^2 \mu_{4/3}^2 + \mu_{2/3}^4 \mu_{4/3} \right)}{\mu_{2/3}^6}. \]

(i)b Define $Z J = \max_{j=1,\ldots,J} |Z J_{r,j}|$. Then

\[ Z J_{n} \xrightarrow{d} Z J, \]
with
\[
\left( \begin{array}{c}
Z J_{r_1} \\
Z J_{r_2} \\
\vdots \\
Z J_{r_J}
\end{array} \right)
\sim \text{MN} \left( 0, \begin{pmatrix}
V(r_1, r_1) & V(r_1, r_2) & \cdots & V(r_1, r_J) \\
V(r_2, r_1) & V(r_2, r_2) & \cdots & V(r_2, r_J) \\
\vdots & \vdots & \ddots & \vdots \\
V(r_J, r_1) & V(r_J, r_2) & \cdots & V(r_J, r_J)
\end{pmatrix} \right),
\]
where $\forall r, r'$,
\[V(r, r') = \beta\]
\[
\int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(\min(r, r'), a) L_X(1, a) - L_X(r, a) L_X(r', a)}{L_X(1, a)} da
\]
Under $H'_A$,

(ii) Pointwise in $r \in (0, 1)$,
\[
\Pr \left( \omega : \frac{1}{n^{1/2}} |Z J_{n,r}(\omega)| \geq \varsigma(\omega) \right) \to 1,
\]
where $\varsigma(\omega) > 0$ for all $\omega$, apart from a subset with probability measure approaching zero.
The price paid for robustness to jumps is a loss of efficiency. The coefficient in variance terms moves from 2 to $\beta \approx 3.7$.

Valid asymptotic critical values are obtained as before. However, need new “robust” estimators of variances.
ALLOWING FOR MICROSTRUCTURE NOISE

Transaction data occurring in financial markets are contaminated by market microstructure effects, such as bid-ask spreads, liquidity ratios, turnover, and asymmetric information.

Robust counterparts to $RV_{n,r}$ are already available in the literature (see, e.g., Zhang, 2004, Aït-Sahalia, Mykland and Zhang, 2006, Barndorff-Nielsen, Hansen, Lunde, and Shephard, 2006 and Zhang, Mykland and Aït-Sahalia, 2005).

Need to derive a robust counterpart of $n^{-1} \sum_{i=1}^{[nr]} - 1 S_n^2(X_{i/n})$.

Kernel estimators for regression functions in the presence of measurement error Fan and Troung (1993) for the case in which the density of the measurement...
error is known and Schennach (2004) for the case in unknown.

Their results do not apply to our context. Here, the evaluation point is \( X_{i/n} \) is also measured with error. Therefore, the fact that \( Y_{j/n} \) is close to \( Y_{i/n} \) does not imply that \( X_{j/n} \) is close to \( X_{i/n} \).

When density of error has tails declining exponentially, logarithmic rate of convergence.

Different route, based on two-scale version of the Florens-Zmirou type estimator. We assume that the error term approaches zero as \( n \to \infty \), though this can occur at a very slow rate. It is an empirical fact that the microstructure noise is typically very small. Zhang, Mykland and Aït-Sahalia (2006) argue that the microstructure noise is indeed "too small" to be considered \( O_p(1) \).
We observe

\[ Y_{j/n} = X_{j/n} + \epsilon_{j/n}, \]

with \( \epsilon_{j/n} \) iid, \( E(X_{j/n}\epsilon_{j/n}) = 0 \), and \( \epsilon_{j/n} = O_p(a_n) \), uniformly in \( j \), with \( a_n \to 0 \) as \( n \to \infty \).

\[ ZM_{l,r} = l^{1/2} \left( \frac{1}{l} \sum_{i=1}^{\lfloor lr \rfloor - 1} SM_l^2(Y_i/l) - MRV_{l,r} \right), \]

where \( l^2 a_n^2 B^{-1} \to 0 \), \( Bl = n \), \( r \in (0, 1) \).

\[ SM_l^2(Y_i/l) = \]

\[ \frac{1}{B} \sum_{b=1}^{B} \sum_{j=1}^{l-1} K \left( \frac{Y_i - Y_{(j-1)B+b}}{\xi n} \right) l \left( \frac{Y_{jB+b}}{n} - \frac{Y_{(j-1)B+b}}{n} \right)^2 \]

\[ = \frac{1}{B} \sum_{b=1}^{B} \sum_{j=1}^{l-1} K \left( \frac{Y_i - Y_{(j-1)B+b}}{\xi n} \right) \]

where \( \xi \to 0 \) as \( n \to \infty \), and \( a_n \xi^{-1} \to 0 \) and \( \xi^2 l \to 0 \).
Also,

\[ \text{MRV}_{l,r} = \frac{1}{B} \sum_{b=1}^{B} \sum_{j=1}^{[lr]-1} \left( \frac{Y_{jB+b} - Y_{(j-1)B+b}}{n} \right)^2. \]

**Theorem 0.2** Under \( H_0 \),

(i) if, as \( n \to \infty, \xi^{-1}, a_n^{-1}, l, B \to \infty, l\xi \to \infty, l\xi^2 \to 0, \)
\( a_n\xi^{-1} \to 0, l^2a_n^2B^{-1} \to 0, \) then, pointwise in \( r \in (0, 1) \)

\[ ZM_{l,r} \xrightarrow{d} ZM_r \]
\[ \sim \text{MN} \left( 0, \frac{2}{3} \int_{-\infty}^{\infty} \sigma^4(a) \frac{L_X(r, a)(L_X(1, a) - L_X(r, a))}{L_X(1, a)} da \right) \]

where

\[ L_X(r, a) = \lim_{\psi \to 0} \frac{1}{\psi \sigma^2(a)} \int_{0}^{r} 1\{a-\psi \leq X_u \leq a+\psi\} \sigma^2(X_u) du \]

(8)

denotes the standardized local time of the process \( X_t \).
(i)b Define $ZM_l = \max_{j=1,\ldots,J} |ZM_{l,r_j}|$ and

$$ZM = \max_{j=1,\ldots,J} |ZM_{r_j}|.$$ 

If as $n \to \infty$, $\xi^{-1}, a_n^{-1}, l, B \to \infty$, $l\xi \to \infty$, $l\xi^2 \to 0$, $a_n \xi^{-1} \to 0$, $l^2 a_n^2 B^{-1} \to 0$, then

$$ZM_l \xrightarrow{d} ZM$$

where $ZM$ is defined as $Z$ but for the constant term.

But for the constant term, we have the same limiting distribution as in Theorem 1 and 2, though convergence occurs at rate $l^{1/2}$ rather than $n^{1/2}$.

If we set $l = n^{1/3}$, recalling that $\xi$ approaches zero at a rate faster than $l^{1/2}$, and $a_n$ is of a smaller order than $\xi$, then we simply need that $a_n = n^{-(1/6-\varepsilon)}$, for some $\varepsilon > 0$ arbitrarily small. Such a slow rate of convergence to zero is indeed fully consistent with the order of magnitude of the noise found empirically for most assets.
EMPIRICAL ILLUSTRATION

US: Bank of America Seven-day Eurodollar (bid-ask midpoint)

daily 1/6/75-25/2/95

\[ Z_n \] : Reject at 1%, 5%, 10% (as Ait-Sahalia 1996, Bandi 2002)

\[ ZJ_n \] : Reject at 1%, 5%, 10% (as Ait-Sahalia 1996, Bandi 2002)

UK: Bank of England, 1 month yield

daily: 3/3/97-14/11/05

\[ Z_n \] : Reject at 1%, 5%, 10%

\[ ZJ_n \] : Reject at 1%, 5%, 10%
Japan: Bank of Japan, average interest rate on certificates of Deposit, less than 30 days

weekly: 4/4/88-14/8/06

$Z_n$: Do not reject at 1%, 5%, 10%

$ZJ_n$: Do not reject at 1%, 5%, 10%

After 1998, rate very close to zero. Same findings by repeating the test over first subsample.
COST OF MISSPECIFICATION: BOND PRICING

The risk neutral diffusion for the short rate is given by
\[
    dr_t = \alpha (r - r_t) \, dt + \nu_t^{1/2} \, dW_{1,t}
\]
\[
    dv_t = \gamma (v - v_t) \, dt + \nu_t^{1/2} \, dW_{2,t},
\]
where \( W_{1,t} \) and \( W_{2,t} \) are uncorrelated BMs. Then, the price of a zero-coupon bond at time \( t \) paying one dollar at time \( T \) is given by
\[
    P_{FV}(t, T, r_t, v_t) = \exp \left( -r_t D(\tau) + v_t F(\tau) + G(\tau) \right),
\]
where \( \tau = T - t \),
\[
    D(\tau) = \frac{1 - e^{-\alpha \tau}}{\alpha},
\]
\[
    F(\tau) = \frac{i e^{-\alpha \tau}}{\alpha} + 2\alpha \sum_{j=1}^{2} K_j e^{-\beta_j \alpha \tau}
\]
\[
\left[ \beta_j M \left( d_j, 2d_j, i\kappa e^{-\alpha \tau} \right) + \frac{i\kappa e^{-\alpha \tau}}{2} M \left( d_j + 1, 2d_j + 1, i\kappa e^{-\alpha \tau} \right) \right]
\]
\[
\sum_{j=1}^{2} K_j e^{-\beta_j \alpha \tau} M \left( d_j, 2d_j, i\kappa e^{-\alpha \tau} \right)
\]

\[
\begin{align*}
\kappa &= \frac{1}{\alpha^2}, \quad d_1 = -\frac{i}{2\alpha^2} + \frac{1}{2}, \quad d_2 \\
&= \frac{i}{2\alpha^2} + \frac{1}{2}, \quad \beta_1 = -\frac{i}{2\alpha^2}, \quad \beta_2 = \frac{i}{2\alpha^2}
\end{align*}
\]

and \( K_1, K_2 \) are chosen to satisfy the boundary condition \( F(0) = 0 \). Finally,

\[
G(\tau) = -\alpha r \int_t^T D(u) du + \gamma v \int_t^T F(u) du.
\]

\( M \) is the confluent hypergeometric function, defined as

\[
M(d, e, z) = 1 + \sum_{n=1}^{\infty} \frac{d(d+1) \ldots (d+n) z^n}{e(e+1) \ldots (e+n) n!}.
\]

Suppose that a naive fund manager (wrongly) thinks that the short rate evolves (under the risk neutral measure) as a Cox-Ingersoll-Ross process

\[
dr_t = \alpha (r - r_t) dt + r_t^{1/2} dW_{1,t}.
\]
Then he would price the zero coupon bond according to the formula

\[ P_{CIR}(t, T, r_t) = A(t, T) \exp(-B(t, T)r_t), \]

where

\[
A(t, T) = \left[ \frac{2\delta e^{(\alpha + \delta)\tau/2}}{(\alpha + \delta)(e^{\delta \tau} - 1) + 2\delta} \right]^{2\alpha r},
\]

\[
B(t, T) = \frac{2(e^{\delta \tau} - 1)}{(\alpha + \delta)(e^{\delta \tau} - 1) + 2\delta},
\]

\[
\delta = (\alpha^2 + 2)^{1/2}
\]

Our objective is to compare \( P_{FV}(t, T, r_t, v_t) \) with the mispriced \( P_{CIR}(t, T, r_t) \), for different states of the world (different values of \( r_t, v_t \), which are simulated under (??)) and different maturities \( \tau \).
<table>
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<tr>
<th>( r = .01, v = .01 )</th>
<th>( P_{FV}(t, T, r, v) )</th>
<th>( P_{CIR}(t, T, r) )</th>
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