Motivation

- Economic theory, or empirical conjecture, often yields a prediction that expected returns should be increasing (or decreasing) in some characteristic or feature.
  - Eg: firm size, ‘liquidity’, default risk, past performance, etc.

- Portfolio sorts are very widely-used in the literature, for a number of reasons:
  - Easy to handle stocks that drop out of the sample, or enter the sample late
  - Does not require assuming a linear relationship between expected returns and the factor
  - The difference between the expected returns on the top and bottom portfolios can (perhaps) be interpreted as the profits from a trading strategy
Portfolio sorts in the literature

- **One-way sorts:**
  - financial constraints: Lamont, Polk and Saa-Requejo (2001)
  - volatility: Ang, Hodrick, Xing and Zhang (2006)
  - ‘downside’ risk: Ang, Chen and Xing (2006)

- **Double sorts:** momentum and size (Rouwenhorst (1998)), financial constraints and R&D expenditures (Li (2007)), payout policy and leverage (Nielsen (2007))

- **Triple sorts:** Daniel, Grinblatt, Titman and Wermers (1997) and Vassalou and Xing (2004)
A test based on a portfolio sort is usually conducted as follows:

1. Individual stocks are sorted according to a given characteristic (e.g., size, past returns, etc.)
2. These stocks are then grouped into \( N \) portfolios (usually 3, 5 or 10)
3. Average returns on these portfolios over a subsequent period are then computed
4. The significance of the relationship is judged by whether the “top” and “bottom” portfolios have significantly different average returns.

However, such an approach does not exploit the rich prediction of the theory: that the expected returns of the sorted portfolios should be \textit{monotonically} increasing (or decreasing).
Is this relationship significantly positive?

Expected Returns and the Cash-Flow to Price Ratio

Value-weighted cashflow-to-price portfolio returns, 1963-2006

t-statistic = 2.404
t-test p-value = 0.008
Is this relationship significantly positive?

Expected Returns and the Cash-Flow to Price Ratio

Value-weighted cashflow-to-price portfolio returns, 1963-2006

t-statistic = 2.404
t-test p-value = 0.008
MR test p-value = 0.024
Is this relationship significantly negative?

Expected Returns and Past Short-Term Performance

Value-weighted short-term reversal portfolio returns, 1963-2006

t-statistic = 2.364

t-test p-value = 0.009
Is this relationship significantly negative?

Expected Returns and Past Short-Term Performance

Value-weighted short-term reversal portfolio returns, 1963-2006

t-statistic = 2.364

\( t \)-test p-value = 0.009

\( M^R \) test p-value = 0.258
This paper proposes a test of the **monotonic relationship** between expected returns on the sorted portfolios and the sorting characteristic.

Such a test is more directly related to the predictions of economic theories ($\partial \mu / \partial X > 0$, etc)

Our “MR” tests are nonparametric, powerful, and (relatively) easy to implement via the bootstrap.
Our MR test generalises to cover several interesting cases:

1. Sorts based on **multiple variables**: two-way sorts, three-way sorts, etc.
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2. **Piece-wise** monotonic relationships: a U-shaped or inverse-U shaped relationship, etc.
Our MR test generalises to cover several interesting cases:

1. **Sorts** based on *multiple variables*: two-way sorts, three-way sorts, etc.

2. **Piece-wise** monotonic relationships: a U-shaped or inverse-U shaped relationship, etc.

3. Monotonic relationships in *other parameters* of interest: risk-adjusted returns (alphas), or factor loadings (betas) etc.
Outline of the talk

1. Introduction and review of portfolio sorts

2. Theory for the test for a monotonic relationship
   1. Null and alternative hypotheses
   2. Two-way and D-way sorts
   3. Conducting the test via the bootstrap

3. Empirical findings
   1. One-way sorts
   2. Two-way sorts

4. Summary and conclusions
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If interest is limited to establishing such a trading strategy and it is possible to short the bottom-ranked stocks then the standard approach may suffice.
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If interest is focussed on testing the predictions of a theory that ranks stocks based on variables proxying for risk (or liquidity, or similar) then the complete cross-sectional pattern in expected returns should be used.
Portfolio sorts and the number of portfolios

- A key question when implementing these tests is ‘how many portfolios should I use?’
  - Not too many: grouping stocks averages out idiosyncratic effects, and allows the use of data from stocks with unequal histories of data
  - Not too few: including too many stocks in a portfolio will make it hard to find the effect of interest

- This paper, like most of the rest of the literature, takes the number of portfolios as given.

- Fortunately, the profession has settled on a set of “reasonable” numbers of portfolios (3, 5, 10, perhaps 20), and so the potential to data-snoop is reduced.
Testing for a monotonic relationship in expected returns

- Let $\mu_i$, $i = 1, 2, \ldots, N$, be the expected return on the $i^{th}$ portfolio obtained from a ranking on some characteristic.

- Economic theory often suggests that an increasing ($\mu_{i-1} < \mu_i$) or decreasing ($\mu_{i-1} > \mu_i$) pattern in expected returns.

- We take as our null hypothesis the absence of any relationship, and seek to reject this in favour of the relationship predicted by the theory:

  \[ H_0 : \mu_1 = \mu_2 = \ldots = \mu_N \]
  \[ H_1 : \mu_1 < \mu_2 < \ldots < \mu_N \]

- This is parallel to standard practice: the theory is only endorsed if the data provides statistically significant evidence against the null in favour of the predicted relationship.
Testing for a monotonic relationship in expected returns

\[ H_0 : \mu_1 = \mu_2 = \ldots = \mu_N \]
\[ H_1 : \mu_1 < \mu_2 < \ldots < \mu_N \]

- Note that our alternative is a multivariate one-sided hypothesis: there are many possible violations of \( H_0 \) that are not consistent with \( H_1 \).

- Our test will only look for deviations of \( H_0 \) that are “in the direction” of \( H_1 \).

- We do not look for evidence against \( H_0 \) in the direction of a non-monotonic relationship, nor do we look for evidence of a monotonic relationship in the ‘wrong’ direction.

- This means that a rejection of the null is evidence of a relationship consistent with the theory.
Three types of patterns in expected returns

1. Reject $H_0$
2. Fail to reject $H_0$
3. Fail to reject $H_0$ (with a different pattern)
Wolak’s test for a monotonic relationship

- An alternative approach to test for (the absence of) a monotonic relationship was provided by Wolak (1989) and implemented by Richardson, Richardson and Smith (1992).

- In that test the null and alternative hypotheses are:

  $$H_0 : \mu_1 \leq \mu_2 \leq \ldots \leq \mu_N$$

  $$H_1 : \mu_i > \mu_j \text{ for some } i < j$$

- Here the weakly monotonic relationship is entertained under the null

  - Limited power (due to short samples or noisy data) may mean that a failure to reject the null of a monotonic relationship does not add much confidence to the conjectured relationship

  - Further, the null also includes the case of no relationship ($\mu_i = \mu_j$)

- We will present the results of both tests for comparison
Implementing the test

- Let

\[ \hat{\Delta}_i = \hat{\mu}_i - \hat{\mu}_{i-1}, \quad i = 2, \ldots, N \]

where \( \hat{\mu}_i \equiv \frac{1}{T} \sum_{t=1}^{T} r_{it} \)

- Then the null and the alternative can be rewritten as

\[ H_0 : \Delta_i = 0, \quad i = 2, \ldots, N \]
\[ H_1 : \min_{i=2,\ldots,N} \Delta_i > 0. \]

- To see this, note that if the smallest value of \( \Delta_i = \mu_i - \mu_{i-1} > 0 \), then we must have \( \mu_i > \mu_{i-1} \) for all portfolios \( i = 2, \ldots, N \). This motivates our choice of test statistic:

\[ J_T = \min_{i=2,\ldots,N} \hat{\Delta}_i \quad \text{or} \quad J_T = \min_{i=2,\ldots,N} \frac{\hat{\Delta}_i}{\hat{\sigma}_{\Delta_i}} \]
Two-way sorts and D-way sorts

For an $N \times K$ table, the number of inequalities implied by the alternative hypothesis is $2KN - N - K$, or $2N(N - 1)$ if $K = N$

- For a $5 \times 5$ table, 40 inequalities are implied
- For a $10 \times 10$ table 180 inequalities are implied

For a $D$-dimensional table with $N$ elements in each dimension the number of inequalities is $DN^{D-1}(N - 1)$

- For a $5 \times 5 \times 5$ table, 300 inequalities are implied
- For a $3 \times 3 \times 3 \times 3$ table, 216 inequalities are implied

This shows how complicated and how rich the full set of relations implied by theory can be when applied to D-way portfolio sorts.
Conducting the test for a monotonic relationship

- Under standard conditions we know that

\[ \sqrt{T} \left( [\hat{\mu}_1, \ldots, \hat{\mu}_N]' - [\mu_1, \ldots, \mu_N]' \right) \xrightarrow{d} N(0, \Omega) \]

- This is not so useful in our case as:
  1. Requires estimating \( \Omega \), which is large if the number of individual portfolios is even moderately-sized.
  2. We are interested in the distribution of

\[ \min_{i=2,\ldots,N} (\hat{\mu}_i - \hat{\mu}_{i-1}) \]

which is a non-standard test statistic, and requires simulation from the asymptotic distribution.
We instead draw on the theory in White (2000, Econometrica), developed for controlling for ‘data snooping’, who justifies the use of the bootstrap to obtain critical values.

We use the vector ‘stationary bootstrap’ of Politis and Romano (1994) to generate new samples of returns from the true sample.

- This preserves any cross-sectional correlation
- Accounts for autocorrelation and heteroskedasticity
- Accounts for non-normality of returns

This approach easily handles many inequality tests and thus two-way or D-way sorts are manageable.

White’s paper has 3654 constraints in total.
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   2. Two-way sorts

4. Summary and conclusions
One-way portfolio sorts

- Our data was taken from Ken French’s web site: we wanted to re-examine some widely-studied portfolio sorts.

- We consider five portfolios sorted on firm characteristics, and three on past performance:
  - (1) market equity (size), (2) book to market ratio, (3) cashflow to price, (4) earnings to price, (5) dividend yield, (6) short-term performance: past 1 month, (7) “momentum”: past 12 months’ performance, (8) long-term performance: past 5 years.

- Portfolios comprise stocks from the NYSE, NASDAQ and AMEX and are value-weighed (equal-weighted yielded similar results).

- We use returns from as far back as 1926 (here I focus on post-1963).
One-way portfolio sorts (from Table 3)

Average returns on sorted portfolios

<table>
<thead>
<tr>
<th></th>
<th>Market Equity</th>
<th>Book-Market</th>
<th>Cashflow-Price</th>
<th>Momentum</th>
<th>Short-term Reversal</th>
</tr>
</thead>
<tbody>
<tr>
<td>top</td>
<td>1.27</td>
<td>0.82</td>
<td>0.85</td>
<td>1.65</td>
<td>0.68</td>
</tr>
<tr>
<td>2</td>
<td>1.21</td>
<td>0.95</td>
<td>0.90</td>
<td>1.24</td>
<td>0.75</td>
</tr>
<tr>
<td>3</td>
<td>1.24</td>
<td>0.99</td>
<td>0.97</td>
<td>1.14</td>
<td>0.94</td>
</tr>
<tr>
<td>4</td>
<td>1.19</td>
<td>1.01</td>
<td>0.96</td>
<td>0.94</td>
<td>0.89</td>
</tr>
<tr>
<td>5</td>
<td>1.21</td>
<td>1.01</td>
<td>1.07</td>
<td>0.90</td>
<td>0.94</td>
</tr>
<tr>
<td>6</td>
<td>1.10</td>
<td>1.11</td>
<td>1.03</td>
<td>0.80</td>
<td>1.02</td>
</tr>
<tr>
<td>7</td>
<td>1.15</td>
<td>1.19</td>
<td>1.09</td>
<td>0.90</td>
<td>1.06</td>
</tr>
<tr>
<td>8</td>
<td>1.10</td>
<td>1.22</td>
<td>1.13</td>
<td>0.86</td>
<td>1.26</td>
</tr>
<tr>
<td>9</td>
<td>1.03</td>
<td>1.27</td>
<td>1.33</td>
<td>0.74</td>
<td>1.28</td>
</tr>
<tr>
<td>bottom</td>
<td>0.89</td>
<td>1.40</td>
<td>1.33</td>
<td>0.18</td>
<td>1.15</td>
</tr>
</tbody>
</table>
## One-way portfolio sorts (from Table 4)

<table>
<thead>
<tr>
<th></th>
<th>top-bottom</th>
<th>t-test</th>
<th>Wolak p-value</th>
<th>MR test p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ME</td>
<td>0.39</td>
<td>1.54</td>
<td>0.06</td>
<td>0.72</td>
</tr>
<tr>
<td>BE-ME</td>
<td>0.57</td>
<td>2.54</td>
<td><strong>0.01</strong></td>
<td>1.00</td>
</tr>
<tr>
<td>CF-P</td>
<td>0.48</td>
<td>2.40</td>
<td><strong>0.01</strong></td>
<td>0.99</td>
</tr>
<tr>
<td>E-P</td>
<td>0.60</td>
<td>2.68</td>
<td><strong>0.00</strong></td>
<td>0.99</td>
</tr>
<tr>
<td>D-P</td>
<td>0.07</td>
<td>0.29</td>
<td>0.38</td>
<td>0.82</td>
</tr>
<tr>
<td>Momentum</td>
<td>1.47</td>
<td>5.67</td>
<td><strong>0.00</strong></td>
<td>0.87</td>
</tr>
<tr>
<td>ST reversal</td>
<td>0.46</td>
<td>2.36</td>
<td><strong>0.01</strong></td>
<td>0.87</td>
</tr>
<tr>
<td>LT reversal</td>
<td>0.51</td>
<td>2.21</td>
<td><strong>0.01</strong></td>
<td>1.00</td>
</tr>
</tbody>
</table>
Summary of results from one-way portfolio sorts

- Not surprisingly, most of the relationships between these factors and expected returns are significant, using both tests.

- Only two contradictions were found: momentum and short-term reversal were significant features of returns using the \( t \)-test, but are not significant monotonic relationships according to our MR test.
  
  - In both cases there were “reversals” against the monotonic pattern, and these were significant.
  
  - *Important*: not all “reversals” against the monotonic pattern lead to a failure to reject. The cashflow-price, earnings-price, and long-term reversal factors all have some non-monotonicity, yet are still significant according to our test.
We next examine some two-way portfolio sorts, again taken from Ken French’s web site.

We look at $5 \times 5$ portfolios sorted on size and four other factors: book-to-market, momentum, short-term reversal and long-term reversal.

- These sorts are “independent” double sorts
- Our tests apply equally well to “independent” or “conditional” double sorts.
## Two-way portfolio sorts (from Table 6)

<table>
<thead>
<tr>
<th></th>
<th>top-bottom</th>
<th>t-test</th>
<th>Wolak p-value</th>
<th>MR test p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ME × BE/ME</td>
<td>0.86</td>
<td>3.63</td>
<td><strong>0.00</strong></td>
<td><strong>0.01</strong></td>
</tr>
<tr>
<td>ME × mom</td>
<td>1.36</td>
<td>5.37</td>
<td><strong>0.00</strong></td>
<td><strong>0.00</strong></td>
</tr>
<tr>
<td>ME × STR</td>
<td>1.84</td>
<td>5.67</td>
<td><strong>0.00</strong></td>
<td>0.82</td>
</tr>
<tr>
<td>ME × LTR</td>
<td>0.74</td>
<td>3.38</td>
<td><strong>0.00</strong></td>
<td>0.10</td>
</tr>
</tbody>
</table>
To examine in more detail the cause of the difference in the $t$-test and the MR test results, consider the following tables.
Two-way portfolio sorts (from Table 7, Panel A)

<table>
<thead>
<tr>
<th>Market equity</th>
<th>value</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>growth</th>
<th>MR pval</th>
<th>Joint pval</th>
</tr>
</thead>
<tbody>
<tr>
<td>small</td>
<td>1.80</td>
<td>1.58</td>
<td>1.42</td>
<td>1.17</td>
<td>0.87</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.59</td>
<td>1.49</td>
<td>1.40</td>
<td>1.30</td>
<td>0.93</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.50</td>
<td>1.36</td>
<td>1.33</td>
<td>1.26</td>
<td>1.02</td>
<td>0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>4</td>
<td>1.45</td>
<td>1.34</td>
<td>1.25</td>
<td>1.08</td>
<td>1.02</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>big</td>
<td>1.29</td>
<td>1.10</td>
<td>1.06</td>
<td>0.96</td>
<td>0.94</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>MR pval</td>
<td>0.00</td>
<td>0.01</td>
<td>0.06</td>
<td>0.38</td>
<td>0.43</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Joint MR pval</td>
<td>0.09</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.01</td>
<td></td>
</tr>
</tbody>
</table>

MR joint value 2 3 4 growth pval pval
## Two-way portfolio sorts (from Table 7, Panel B)

<table>
<thead>
<tr>
<th>Market equity</th>
<th>Momentum</th>
<th></th>
<th></th>
<th></th>
<th>loser</th>
<th>MR pval</th>
<th>Joint pval</th>
</tr>
</thead>
<tbody>
<tr>
<td>small</td>
<td>2.01</td>
<td>1.79</td>
<td>1.67</td>
<td>1.48</td>
<td>0.97</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.78</td>
<td>1.51</td>
<td>1.28</td>
<td>1.20</td>
<td>0.61</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.66</td>
<td>1.26</td>
<td>1.13</td>
<td>1.00</td>
<td>0.61</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>4</td>
<td>1.61</td>
<td>1.21</td>
<td>1.03</td>
<td>0.88</td>
<td>0.66</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>big</td>
<td>1.25</td>
<td>1.03</td>
<td>0.84</td>
<td>0.76</td>
<td>0.65</td>
<td>0.00</td>
<td></td>
</tr>
</tbody>
</table>

| MR pval       | 0.00     | 0.02 | 0.00 | 0.01 |       | 0.24    |            |
| Joint MR pval |          | 0.05 |      |      |       | 0.00    |            |
Two-way portfolio sorts (from Table 7, Panel C)

<table>
<thead>
<tr>
<th>Market equity</th>
<th>loser</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>winner</th>
<th>MR pval</th>
<th>Joint pval</th>
</tr>
</thead>
<tbody>
<tr>
<td>small</td>
<td>2.57</td>
<td>1.63</td>
<td>1.42</td>
<td>1.04</td>
<td>0.13</td>
<td><strong>0.00</strong></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.95</td>
<td>1.53</td>
<td>1.34</td>
<td>1.05</td>
<td>0.41</td>
<td><strong>0.00</strong></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.74</td>
<td>1.40</td>
<td>1.29</td>
<td>0.98</td>
<td>0.53</td>
<td><strong>0.00</strong></td>
<td><strong>0.00</strong></td>
</tr>
<tr>
<td>4</td>
<td>1.48</td>
<td>1.32</td>
<td>1.19</td>
<td>0.97</td>
<td>0.69</td>
<td><strong>0.00</strong></td>
<td></td>
</tr>
<tr>
<td>big</td>
<td>1.11</td>
<td>1.03</td>
<td>0.96</td>
<td>0.94</td>
<td>0.73</td>
<td><strong>0.04</strong></td>
<td></td>
</tr>
<tr>
<td>MR pval</td>
<td><strong>0.00</strong></td>
<td><strong>0.02</strong></td>
<td><strong>0.02</strong></td>
<td>0.10</td>
<td>0.94</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Joint MR pval</td>
<td>0.89</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.82</td>
</tr>
</tbody>
</table>
Two-way portfolio sorts (from Table 7, Panel D)

<table>
<thead>
<tr>
<th>Market equity</th>
<th>loser</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>winner</th>
<th>MR pval</th>
<th>Joint pval</th>
</tr>
</thead>
<tbody>
<tr>
<td>small</td>
<td>1.57</td>
<td>1.35</td>
<td>1.44</td>
<td>1.29</td>
<td>0.87</td>
<td>0.46</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.34</td>
<td>1.25</td>
<td>1.28</td>
<td>1.23</td>
<td>1.03</td>
<td>0.18</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.23</td>
<td>1.20</td>
<td>1.09</td>
<td>1.19</td>
<td>1.00</td>
<td>0.70</td>
<td>0.23</td>
</tr>
<tr>
<td>4</td>
<td>1.20</td>
<td>1.08</td>
<td>1.02</td>
<td>1.06</td>
<td>0.96</td>
<td>0.33</td>
<td></td>
</tr>
<tr>
<td>big</td>
<td>1.05</td>
<td>0.99</td>
<td>0.89</td>
<td>0.87</td>
<td>0.76</td>
<td>0.01</td>
<td></td>
</tr>
</tbody>
</table>

| MR pval       | 0.03  | 0.00  | 0.00  | 0.02  | 0.70   |         |            |
| Joint MR pval | 0.29  |       |       |       |        | 0.10    |            |
Summary and conclusions

- Theoretical research in financial economics often generates a prediction about the sign of the relationship between expected returns and some characteristic or feature.

- This paper presents a new, nonparametric, direct test of such a prediction.

- In our empirical work we find evidence in favour of some existing results, but in contradiction with others:
  - Sorts on past performance (Short-term Reversal, Momentum, and Long-term Reversal) yield significant differences in “top” vs. “bottom” portfolios, however these relationships are not monotonic.

- Matlab code to replicate all results in this paper is available at:
  www.econ.ox.ac.uk/members/andrew.patton/code.html