

# Modelling Time-Varying Exchange Rate Dependence using the Conditional Copula

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## A Quotation

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*Problems involving dependent pairs of variables have been studied most intensively in the case of bivariate Normal distributions... This is due primarily to the importance of [this case] but perhaps also to the fact that they exhibit only a particularly simple form of dependence...*

- E. L. Lehmann (1966).

# Overview

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## *1. Intro to / Review of Copulas*

2. Motivation

3. Evaluating copula models

4. Applying copulas to exchange rate modelling

a) The data

b) The models

c) Results

5. Conclusions and potential future work

# What is a copula?

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- It is a function (not a number)
- It links (or *couples*) marginal distributions together to form a joint distribution
- It contains all of the information in the joint distribution not captured in the marginal distributions
  - that is, all of the *dependence* information.

## Deriving a copula

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- Suppose  $X \sim F$  ,  $Y \sim G$  and  $(X,Y) \sim H$ .
- Let  $U \equiv F(X)$  and  $V \equiv G(Y)$  .
  - (Aside: Then (Pearson, 1933)  $U \sim \text{Unif}(0,1)$  and  $V \sim \text{Unif}(0,1)$ .  $U$  and  $V$  are the *probability integral transforms* of  $X$  and  $Y$ . )
- Let  $(U,V) \sim C$ . By standard theory on the distribution of transformations of random variables we obtain:

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- $c( F(X) , G(Y) ) = h(X,Y) \cdot \begin{vmatrix} dX/dU & dX/dV \\ dY/dU & dY/dV \end{vmatrix}$

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$$\begin{aligned} \text{■ } c( F(X) , G(Y) ) &= h(X,Y) \cdot \begin{vmatrix} dX/dU & dX/dV \\ dY/dU & dY/dV \end{vmatrix} \\ &= h(X,Y) \cdot \begin{vmatrix} f(X)^{-1} & 0 \\ 0 & g(Y)^{-1} \end{vmatrix} \end{aligned}$$

## Deriving a copula (cont'd)

---

- $c( F(X) , G(Y) ) = h(X,Y) \cdot \begin{vmatrix} f(X)^{-1} & 0 \\ 0 & g(Y)^{-1} \end{vmatrix}$
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- $h(X,Y) = f(X) \cdot g(Y) \cdot c( F(X) , G(Y) )$

Joint  
density

Marginal  
densities

Copula  
density

## Defining the copula

Definition: A 2-dimensional copula is a function  $C: [0,1] \times [0,1] \rightarrow [0,1]$  with the following properties:

1.  $C(0,w) = C(w,0) = 0$ , and  $C(w,1) = C(1,w) = w$ , for every  $w \in [0,1]$
  2.  $V_C([u_1, u_2] \times [v_1, v_2]) \equiv \Pr[u_1 \leq U \leq u_2 \cap v_1 \leq V \leq v_2]$   
 $= C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$   
 for all  $u_1, v_1, u_2, v_2 \in [0,1]$  s.t.  $u_1 \leq u_2$  and  $v_1 \leq v_2$ .
- Alternative definition: if  $U \sim \text{Unif}(0,1)$  and  $V \sim \text{Unif}(0,1)$ , then  $C$  is any function satisfying the properties of being a joint cdf of  $(U, V)$ ; it is a *multivariate cdf with  $\text{Unif}(0,1)$  margins*.

## Sklar's theorem

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- Sklar (1959) showed that we may decompose the distribution of  $(X, Y)$  into three parts:

$$H(x, y) = C(F(x), G(y)) \quad \forall x, y \in \bar{\mathcal{R}}$$

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$$H(x, y) \Leftrightarrow C(F(x), G(y)) \quad \forall x, y \in \bar{\mathcal{R}}$$

Joint dist'n  
of X and Y

*Copula* of  
X and Y

Marginal  
dist'n of X

Marginal  
dist'n of Y

# Why should economists care about copulas?

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- *Because economists care about dependence.*
- There are two main places in economics where knowledge of the copula would be useful...

# I. When the joint distribution is required

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- There are a number of situations where the first couple of moments are not enough:
  1. Pricing options with more than one underlying asset (Rosenberg, 2000)
  2. Calculating the Value-at-Risk of a portfolio of assets (Hull and White, 1998)
  3. Multivariate density forecasting (Diebold, Hahn and Tay, 1999)
  4. Quantile regression

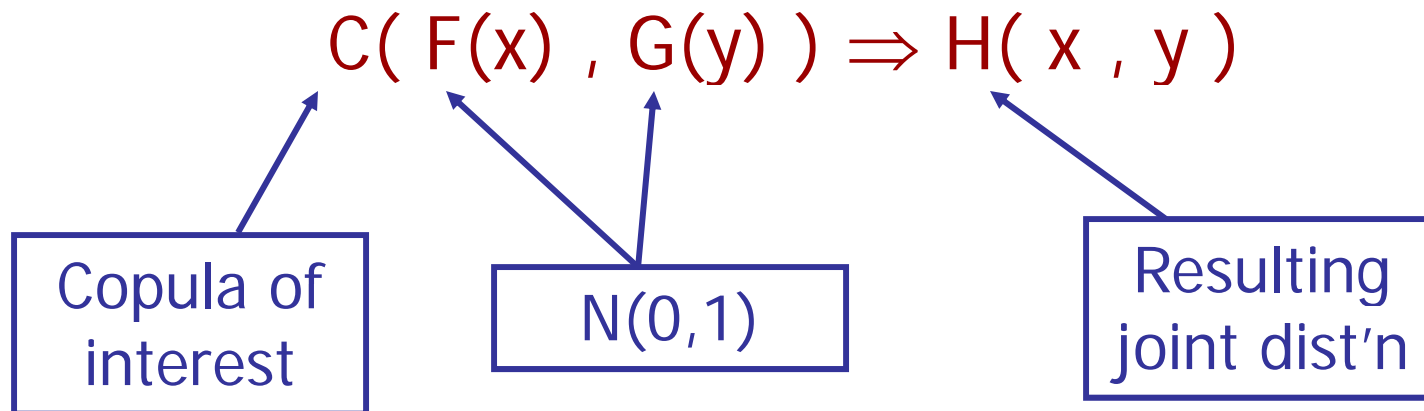
## II. When linear correlation isn't enough

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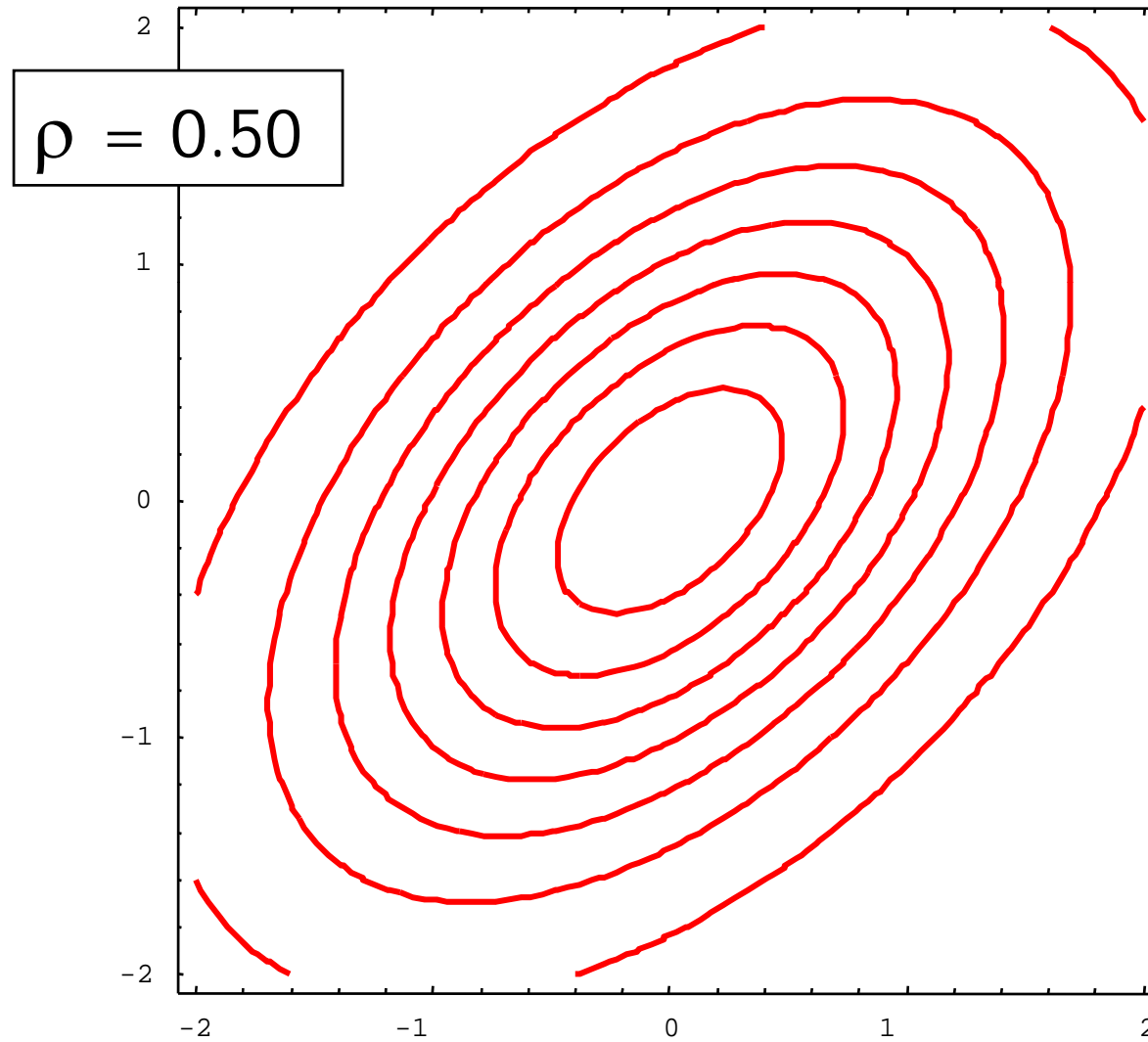
- Linear correlation is a nice, simple measure of dependence, but it's not always sufficient:
  1. Asymmetric equity correlations: stock returns seem more dependent on down days than on up days (Erb, Harvey and Viskanta, 1994)
  2. Financial contagion: international markets seem more dependent in crash states than during 'normal' states
  3. Asset pricing when you don't believe that investors have quadratic utility or that returns are multivariate normally distributed, as is assumed in the CAPM

# Visualizing copulas

- Copulas themselves aren't that nice to look at
- One way of looking at them is to use them to couple two standard normal margins, and then examine the resulting joint distribution.

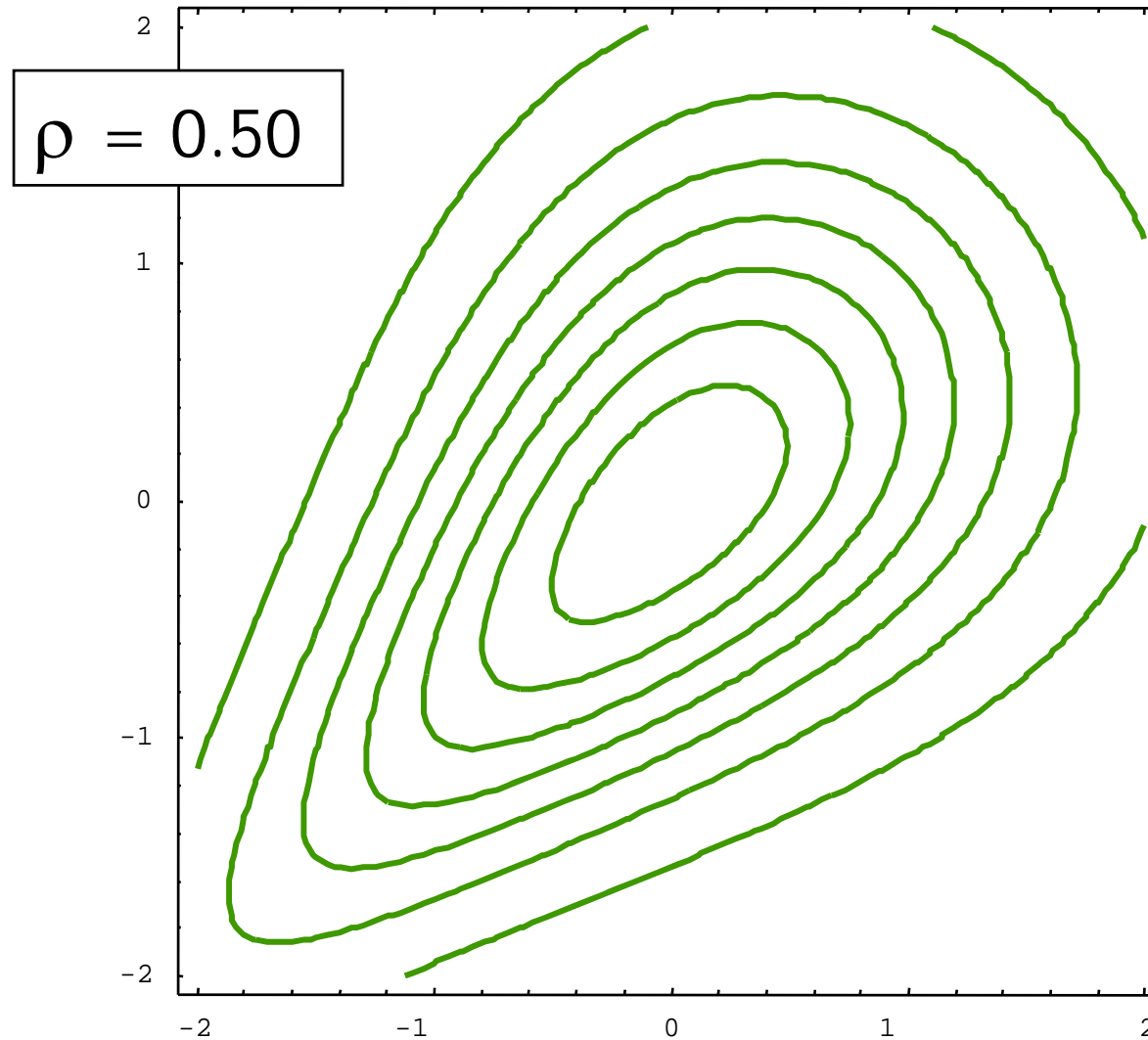


# Bivariate normal distribution



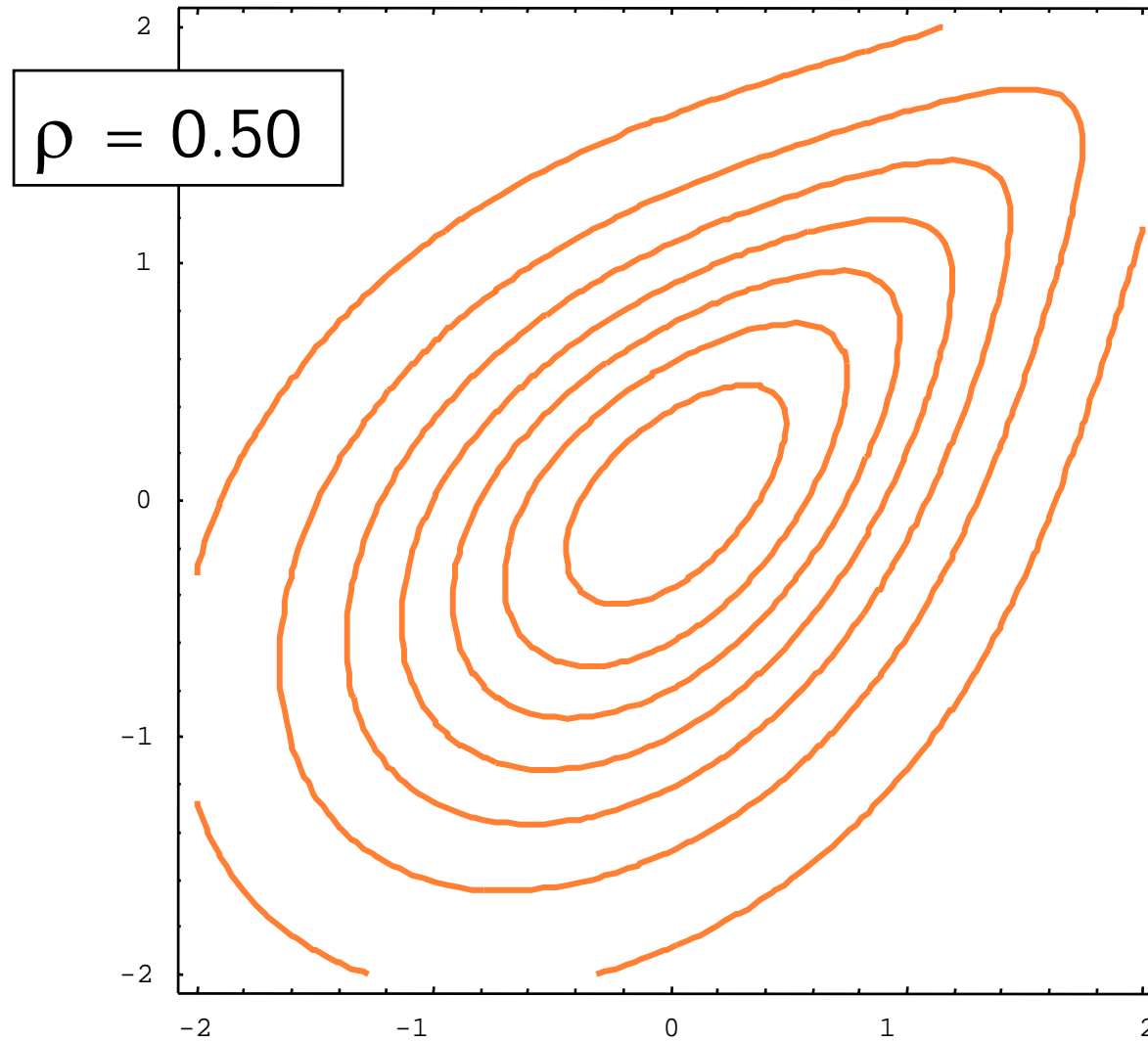
$\rho=0.5$

# Clayton's (1978) copula



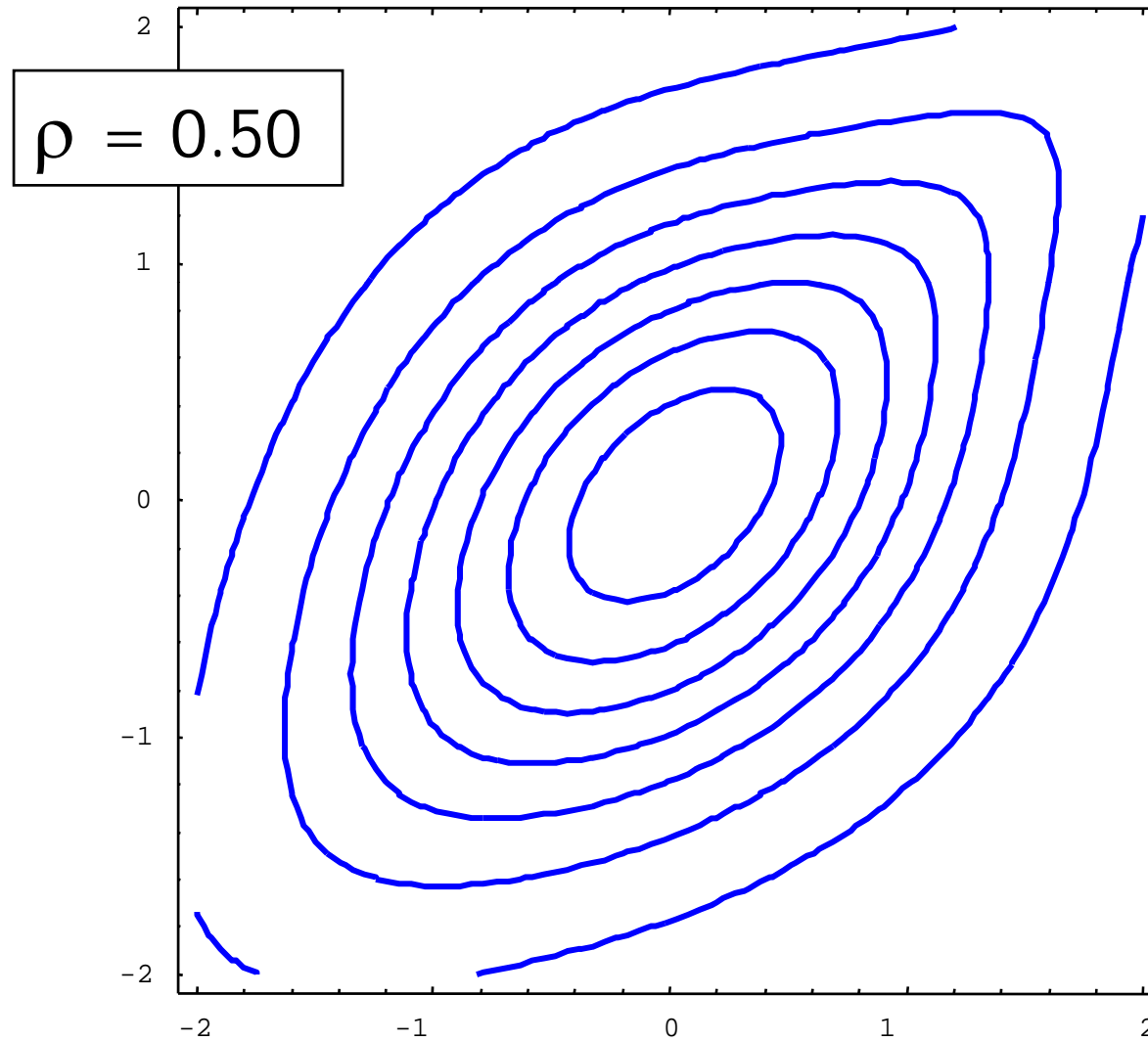
$\theta = 1$

# Gumbel's (1960) copula



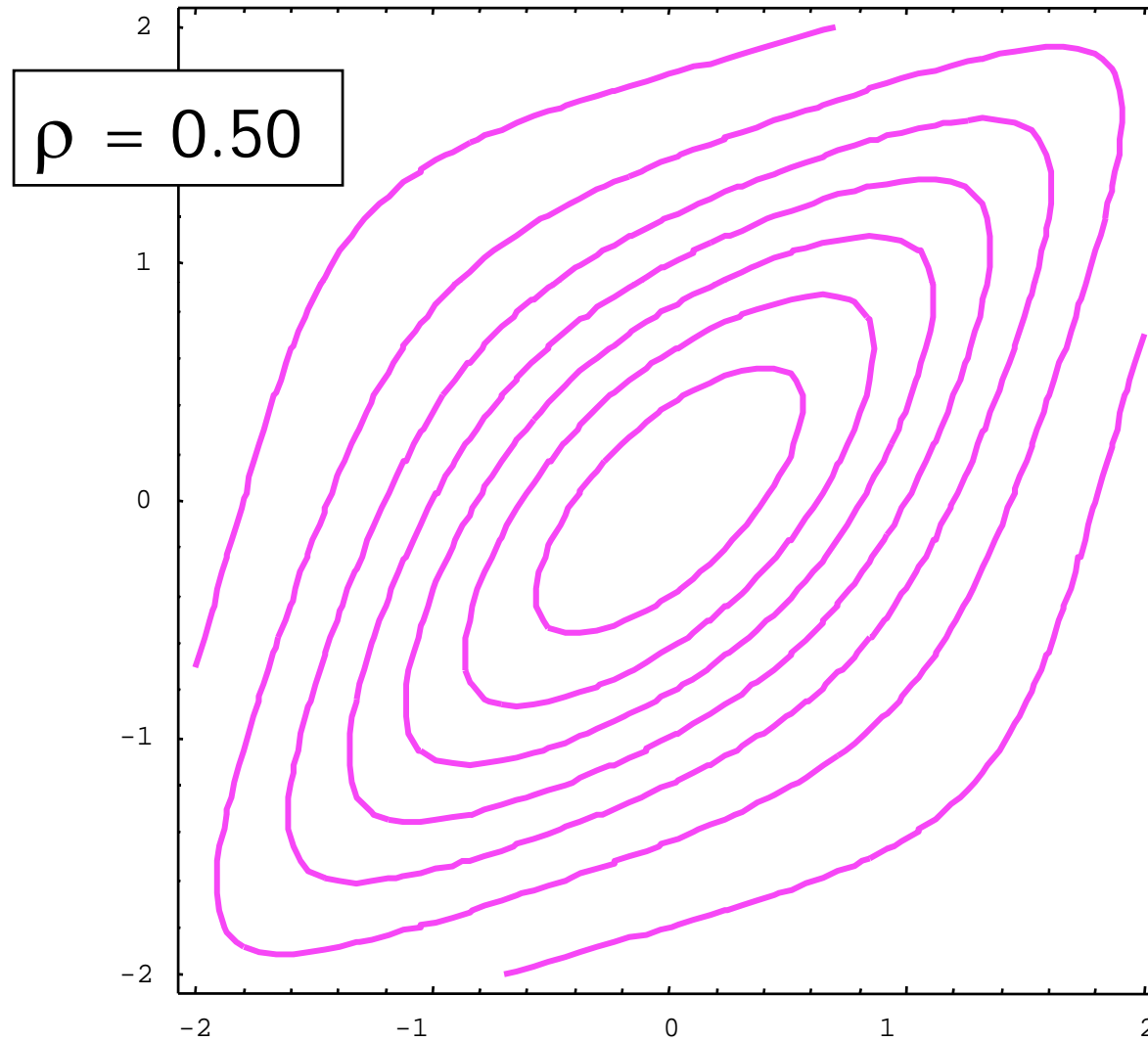
$\delta = 1.5$

# Joe Clayton (1997) copula



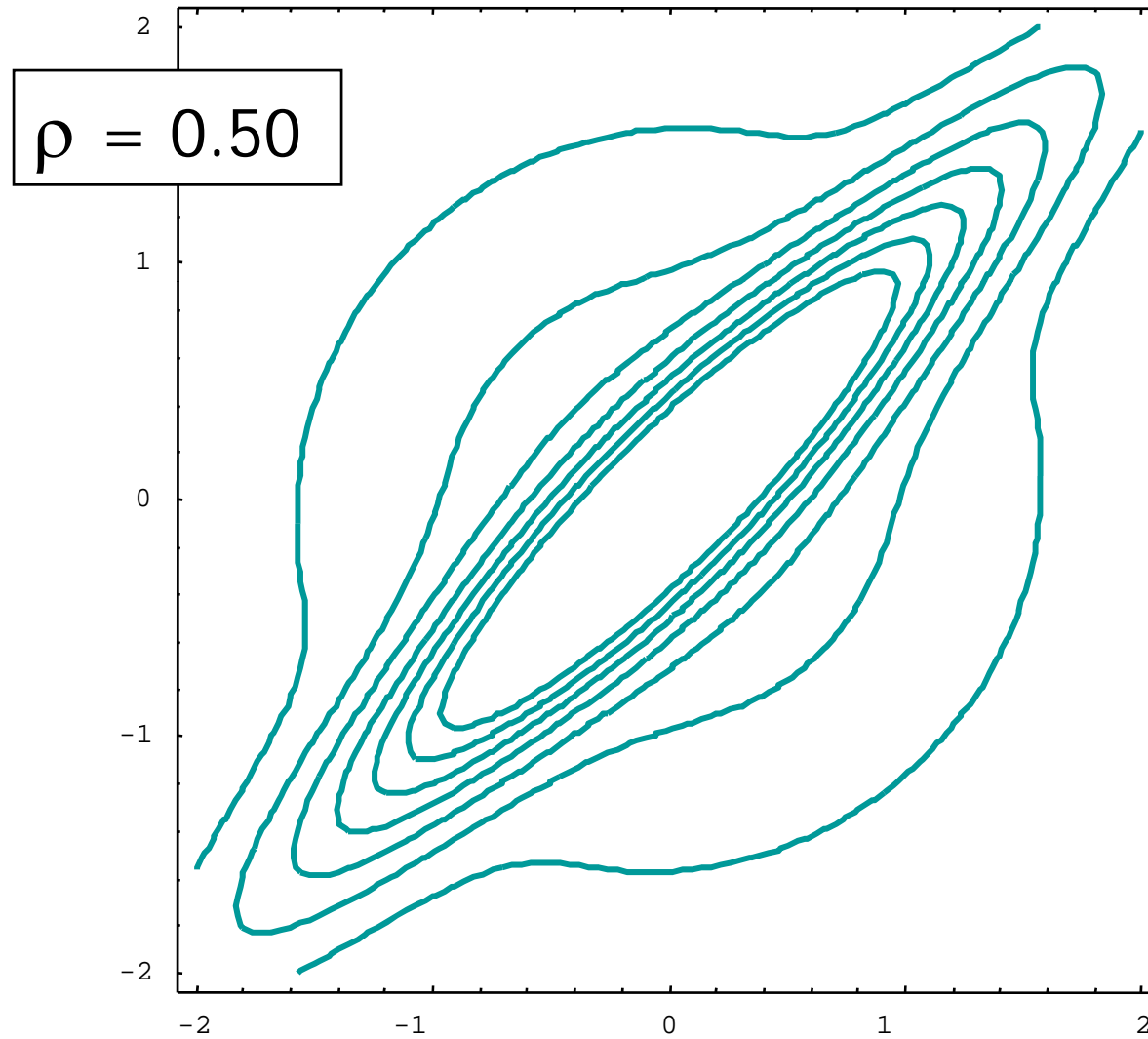
$\kappa=1.42,$   
 $\gamma=0.47$

# Student's $t_3$ copula



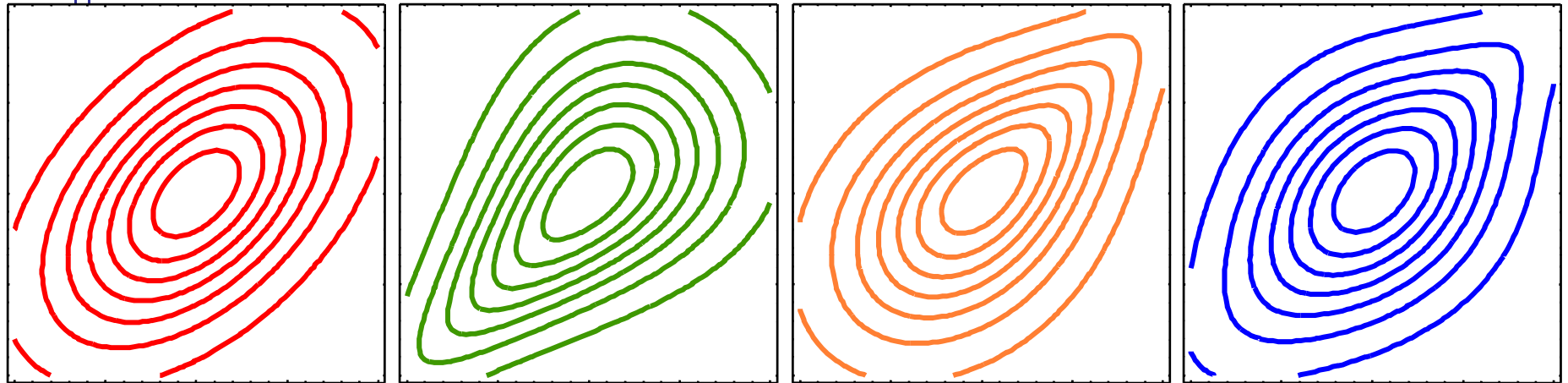
$\rho=0.50,$   
 $\nu=3$

# Mixture of Normal copulas



$\rho_1=0.95,$   
 $\rho_2=0.05,$   
 $\omega=0.5$

All of these distributions have  $N(0,1)$  marginal distributions and  $\rho=0.50$



	Gaussian	Clayton	Gumbel	Joe-Clayton
$\rho$	0.50	0.50	0.50	0.50
$\rho_s$	<i>0.4878</i>	0.4664	0.4755	0.4512
$\tau$	<i>0.3370</i>	0.3273	0.3272	<i>0.3151</i>
$\beta$	0.3333	0.3333	0.3311	0.2972
$\tau^U$	0.0000	0.0000	0.4126	0.3707
$\tau^L$	0.0000	0.5000	0.0000	0.2288

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# Goodness of fit testing

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$$(X,Y) \sim H = C( F(X) , G(Y) )$$

- We need to ensure that the marginal distribution models are well specified before modelling the copula
  - Evaluation of univariate density models
- Recall that the copula can be defined as the joint distribution of two *i.i.d.*  $\text{Unif}(0,1)$  random variables
  - So the evaluation of copula models is a special case of the more general problem of evaluating multivariate density models

# Goodness of fit testing

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- No single best method of evaluating density models has yet emerged – we will employ two different methods:
  1. Diebold, *et al.* (1998) and (1999): Test (separately) whether  $U_t \equiv F_t(X_t | I_{t-1})$  and  $V_t \equiv G_t(Y_t | I_{t-1})$  are *i.i.d.* and Unif(0,1)
  2. Break the density model into multiple interval (or region) models and evaluate these
    - Extension of work by Christoffersen (1998) and Engle and Manganelli (1999).

# 1. Diebold, *et al.* (1998) and (1999)

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$$U_t \equiv F_t(X_t | I_{t-1}) \text{ and } V_t \equiv G_t(Y_t | I_{t-1}) \sim i.i.d. \text{ Unif}(0,1)$$

- Employ two tests: one for the *i.i.d.* property, and one for the Unif(0,1) property.
- ***i.i.d.*** : regress  $(u_t - \bar{u})^k$  and  $(v_t - \bar{v})^k$  on 20 lags of each, for  $k = 1, 2, 3, 4$ . Use LM test that all coefficients are zero.
- **Unif(0,1)**: Use Kolmogorov-Smirnov test that  $\{u_t\}$  and  $\{v_t\}$  are not significantly different from Unif(0,1).

## 2. 'Hit' tests

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- Christoffersen (1998) and Engle and Manganelli (1999) both proposed looking at the sequence of 'hits' to determine the adequacy of an interval forecast (like a VaR(q) forecast)

Let  $Hit_t \equiv \mathbf{1}\{x_t < VaR_t(0.05)\}$ ,  $t = 1, 2, \dots, T$ .

- If the forecast is good, the sequence of hits should be *i.i.d.* Bernoulli(0.05).
- We can thus test the accuracy of an interval forecast by testing

$H_0: Hit_t \sim i.i.d. \text{ Bernoulli}(0.05).$

## Multivariate Hit tests

- In the bivariate case, a hit would be defined as the observation  $(x_t, y_t)$  lying in some *region*,  $R$ , of the support of the distribution:

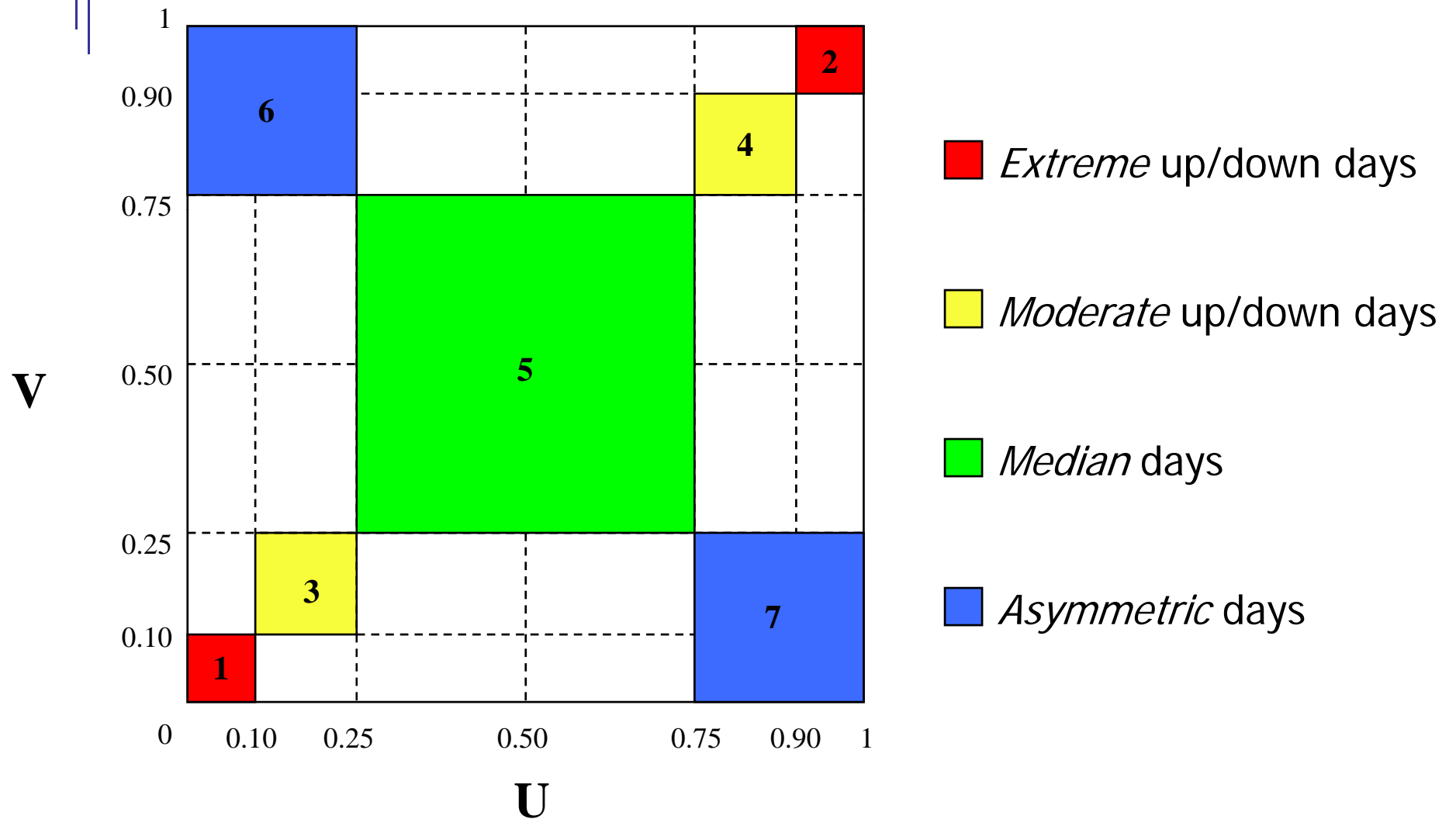
$$Hit_t \equiv \mathbf{1}\{(x_t, y_t) \in R\}, t = 1, 2, \dots, T.$$

- Unlike univariate interval forecasts, the probability of a hit will *not* be constant: it depends on the amount of copula probability mass in that region.
- In our case if the model is accurate we will find:

$$Hit_{it} \equiv \mathbf{1}\{(x_t, y_t) \in R^i\} \sim i.n.i.d. \text{ Bernoulli}(p_{it})$$

- where  $p_{it}$  is given by the copula at time  $t$ :  $C_t(\cdot, \cdot | I_{t-1})$

# Hit regions



# Individual region hit tests

- We model the hits in a logistic regression framework:

$H_0$ : the model is correctly specified

$$\Rightarrow Hit_{it} \sim \text{Bernoulli}(p_{it}), \text{ for } t = 1, 2, \dots, T.$$

$H_1$ : the model is not correctly specified

$$\Rightarrow Hit_{it} \sim \text{Bernoulli}(\pi_{it}), \text{ for } t = 1, 2, \dots, T, \text{ where}$$

$$\pi_{it} = \pi_i(Z_{it}, \beta_i, p_{it}) = \Lambda\left(Z_{it}' \cdot \beta_i - \ln \frac{1 - p_{it}}{p_{it}}\right)$$

- where  $\Lambda(x) = (1 + e^{-x})^{-1}$  is the logistic function,  $Z_{it}$  contains anything thought to influence the probability of a hit,  $\beta_i$  is the parameter vector to be estimated.
- Thus the test becomes  $H_0: \beta_i = 0$  vs.  $H_1: \beta_i \neq 0$ .

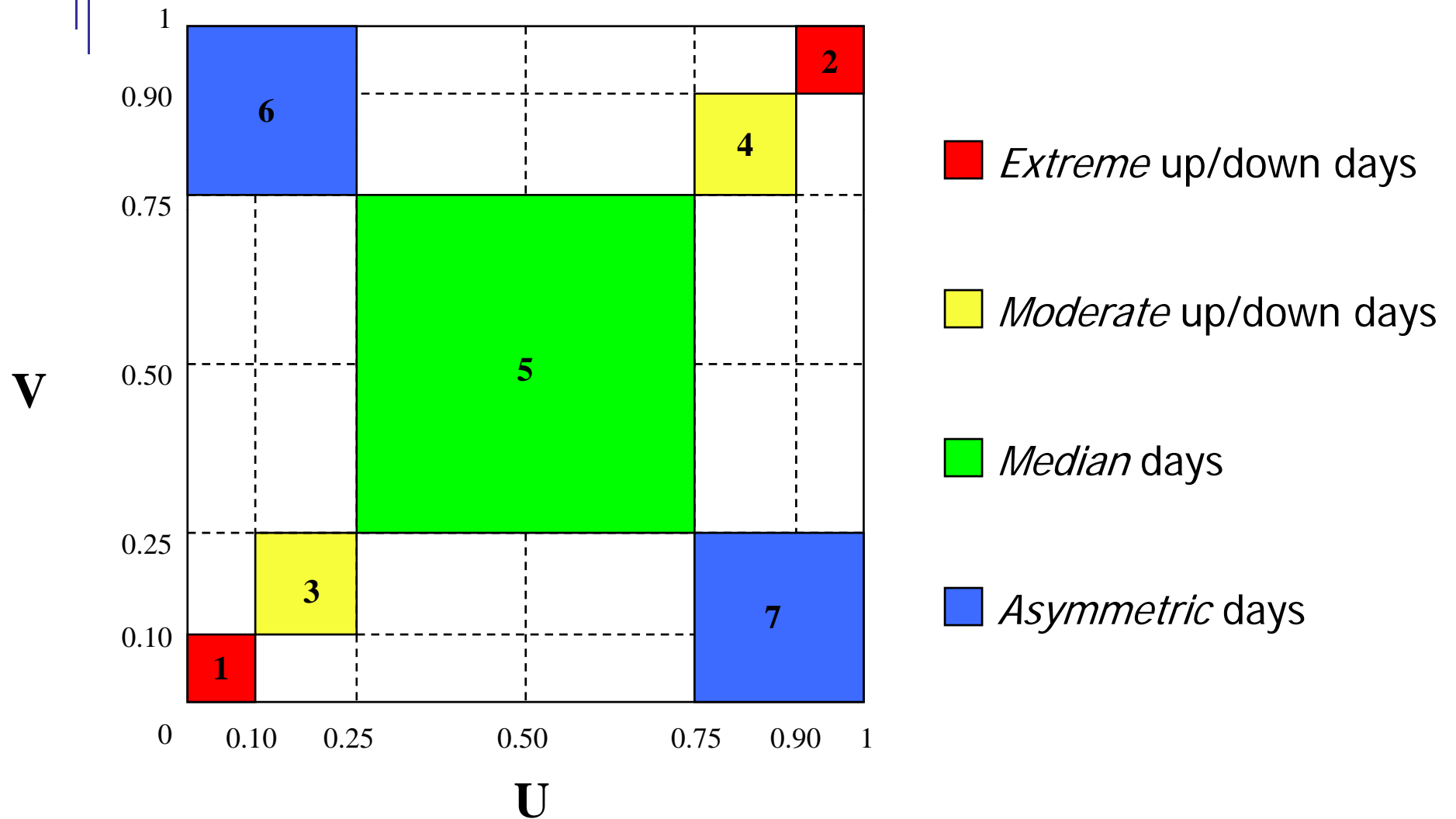
# Testing multiple region forecasts at once

- Define  $Hit_{it} \equiv \mathbf{1}\{(x_t, y_t) \in R^i\}$ , where  $R^i \cap R^j = \emptyset$  and  $\cup_{i=0}^K R^i = S$ , then define a new random variable:

$$M_t \equiv \sum_{i=0}^K i \cdot Hit_{it}$$

- $M_t$  is a multinomial random variable
- Let  $P_t = [p_{0t}, p_{1t}, \dots, p_{Kt}]$ , where  $p_{it}$  is  $\Pr[Hit_{it}=1]$  implied by model.
- Then  $M_t \sim \text{Multinomial}(P_t)$  if the model is correctly specified

# Hit regions



# Multiple region hit tests

$H_0$ : the model is correctly specified

$\Rightarrow M_t \sim \text{Multinomial}(P_{it})$ , for  $t = 1, 2, \dots, T$ .

$H_1$ : the model is not correctly specified

$\Rightarrow M_{it} \sim \text{Multinomial}(\Pi_t)$ , for  $t = 1, 2, \dots, T$ , where

$$\pi_{1t} = \pi_1(Z_t, \beta, P) = \Lambda \left( Z'_{1t} \cdot \beta_1 - \ln \frac{1 - P_{it}}{P_{it}} \right)$$

$$\pi_{jt} = \pi_j(Z_t, \beta, P) = \left( 1 - \sum_{i=1}^{j-1} \pi_{it} \right) \cdot \Lambda \left( Z'_{jt} \cdot \beta_j - \ln \frac{1 - \sum_{i=1}^j P_{jt}}{P_{jt}} \right), \text{ for } j = 2, \dots, K$$

$$\pi_{0t} = 1 - \sum_{j=1}^K \pi_{jt}$$

Again, the test reduces to  $H_0: \beta = 0$  vs.  $H_1: \beta \neq 0$ .

## Difference from previous 'hit' tests

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- Christoffersen (1998) proposed modelling the sequence of this as a first-order Markov chain, as a test of the *i.i.d.* assumption
  - Difficult to allow for higher order serial dependence
  - Doesn't include exogenous variables that may influence the probability of a hit
- Engle and Manganelli (1999) model the hits in a *linear probability model*.
  - Easy to include more lags or exogenous regressors
  - Simple to estimate and draw inference from
  - BUT normal distribution may not be a good approximation to a Bernoulli random variable...

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# Copulas in Economics

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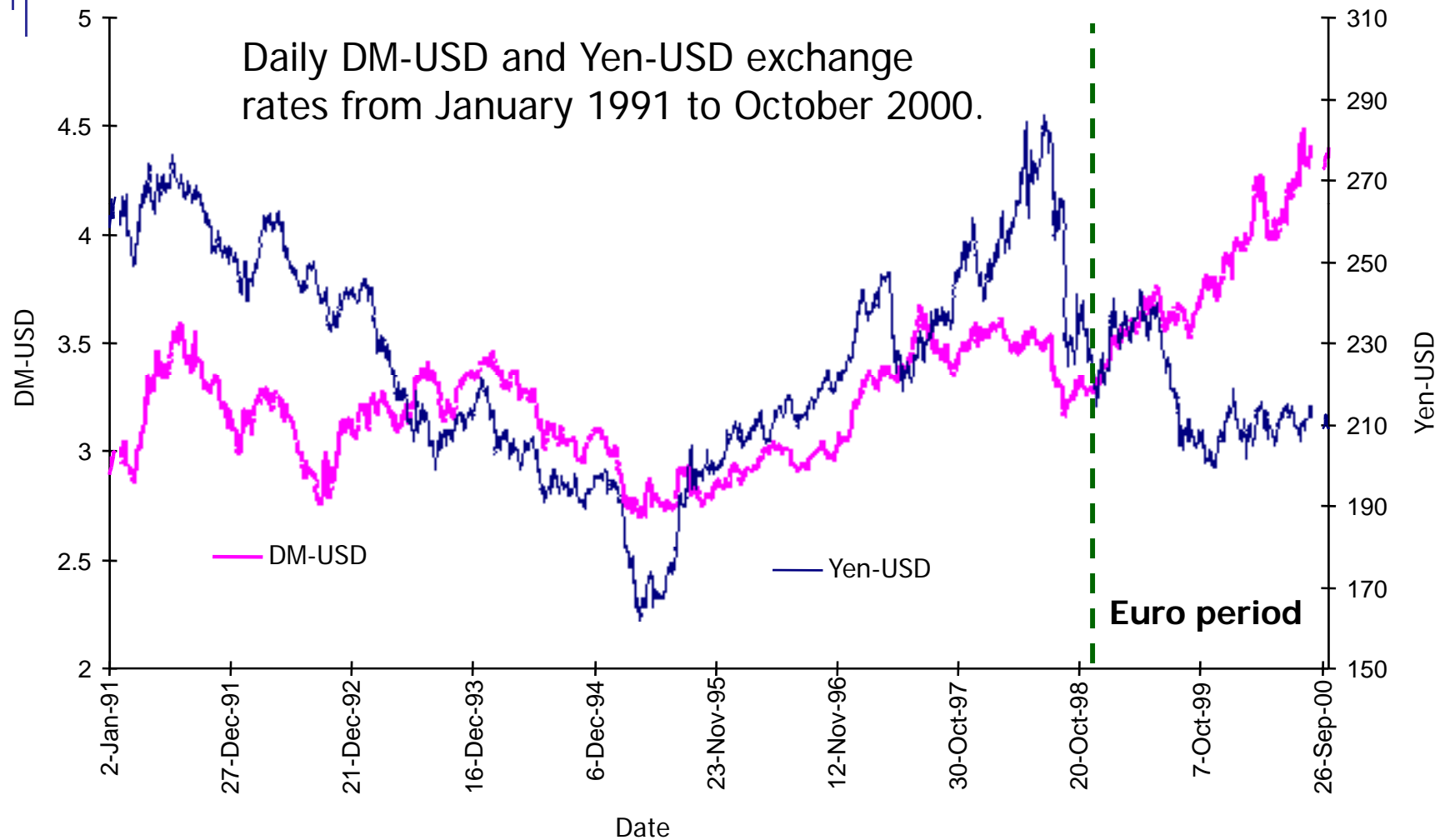
- The original theory of copulas was developed for the *i.i.d.* case. But we can extend it:
- $H(x_t, y_t) = C(F(x_t), G(y_t))$  - *i.i.d.*
- $H_t(x_t, y_t) = C_t(F_t(x_t), G_t(y_t))$  - *i.n.i.d.*
- $H_t(x_t, y_t | I_{t-1}) = C_t(F_t(x_t | I_{t-1}), G_t(y_t | I_{t-1}) | I_{t-1})$   
where  $I_t = \sigma(\mathbf{z}_{t+1}, x_t, y_t, \mathbf{z}_t, x_{t-1}, y_{t-1}, \dots)$ . - *d.n.i.d.*

# Applying copulas to FX modelling

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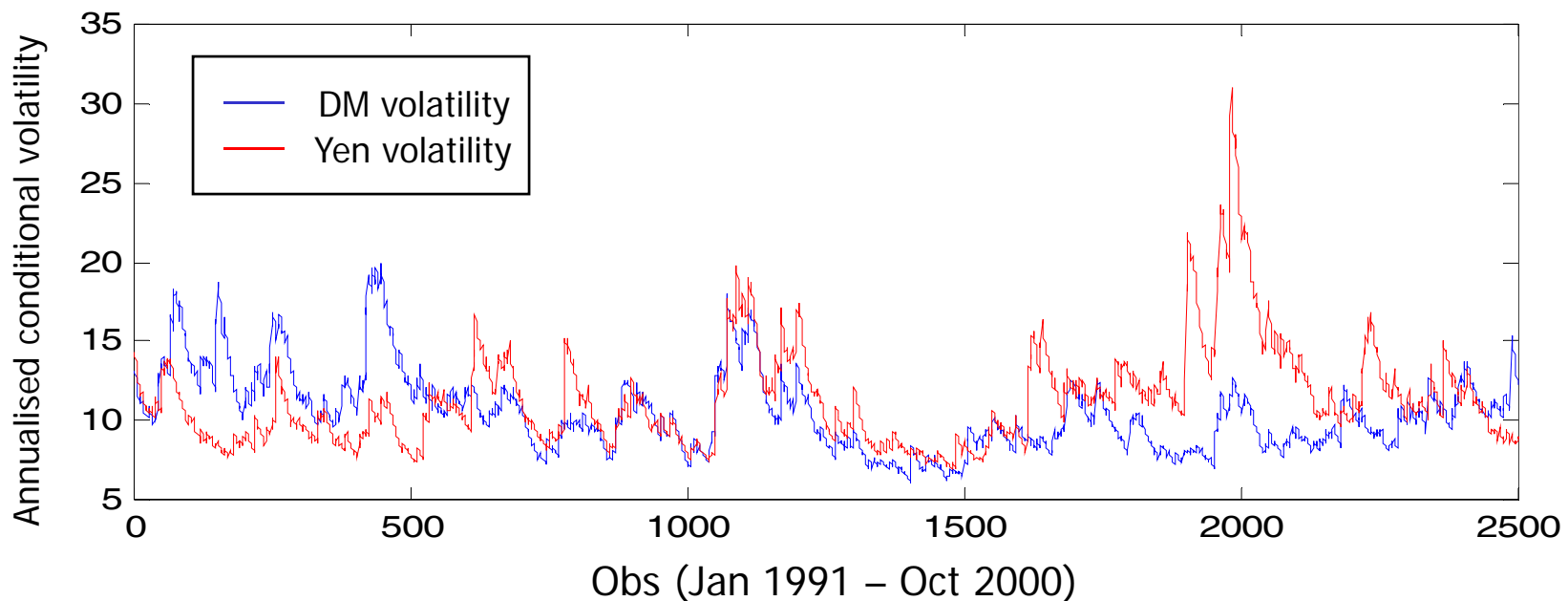
- As an application we will model the joint distribution of the DM-USD and Yen-USD exchange rates, allowing for time-variation in the conditional distribution.
  - The two most heavily traded exchange rates
    - The DM-USD and Yen-USD exchange rates made up 29% and 18% of total foreign exchange trading in 1996 (Melvin, 2000). Next largest were DM-FFr and GBP-USD with 6% and 5% respectively. (AUD-USD was 8<sup>th</sup> largest at 2%).
  - Provide an application where time variation in dependence is likely to be significant

# The data



# Why a time varying copula?

- We have substantial evidence of time variation in the conditional volatility of exchange rates (Bollerslev, 1987, Engle *et al.* 1990, Andersen *et al.* 2000, amongst others)



## Why a time varying copula?

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- It then seems natural to allow for time variation in the conditional dependence also...

# The model

- We will exploit the knowledge we have on modelling individual exchange rates, and then work on modelling the copula...

- Let  $I_t = \sigma(x_t, y_t, x_{t-1}, y_{t-1}, \dots)$

- $\hat{C}_t(\hat{F}_t(x_t | I_{t-1}), \hat{G}_t(y_t | I_{t-1}) | I_{t-1}) \Rightarrow \hat{H}_t(x_t, y_t | I_{t-1})$

Normal copula  
vs.  
Joe-Clayton copula

AR- $t$  GARCH

Model for the  
joint  
distribution

## The model (cont'd)

- The marginal distributions:

$$\mathbf{X}_t = \boldsymbol{\mu}_x + \phi_x \mathbf{X}_{t-1} + \boldsymbol{\varepsilon}_t$$

$$\mathbf{h}_t^x = \boldsymbol{\omega}_x + \alpha_x \boldsymbol{\varepsilon}_{t-1}^2 + \beta_x \mathbf{h}_{t-1}^x$$

$$\sqrt{\nu_x / [\mathbf{h}_t^x (\nu_x - 2)]} \cdot \boldsymbol{\varepsilon}_t \sim t_{\nu_x}$$

$$F_t(\cdot \mid \mathbf{I}_{t-1})$$

$$\mathbf{Y}_t = \boldsymbol{\mu}_y + \phi_{1y} \mathbf{Y}_{t-1} + \phi_{10y} \mathbf{Y}_{t-10} + \boldsymbol{\eta}_t$$

$$\mathbf{h}_t^y = \boldsymbol{\omega}_y + \alpha_y \boldsymbol{\eta}_{t-1}^2 + \beta_y \mathbf{h}_{t-1}^y$$

$$\sqrt{\nu_y / [\mathbf{h}_t^y (\nu_y - 2)]} \cdot \boldsymbol{\eta}_t \sim t_{\nu_y}$$

$$G_t(\cdot \mid \mathbf{I}_{t-1})$$

## Two models for the copula

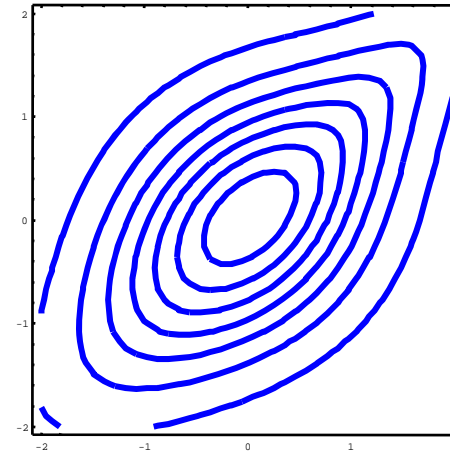
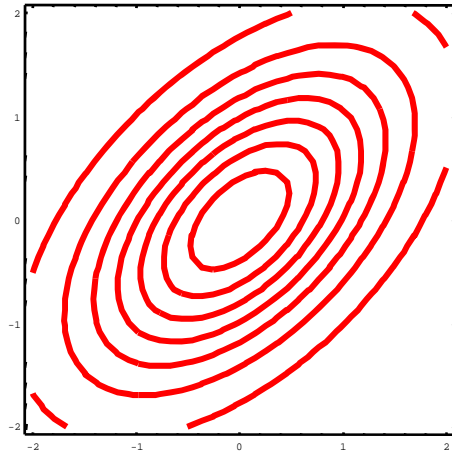
- We consider two possible models for the copula:
  - the Normal copula, and the Joe-Clayton copula.

$$C(u, v | \rho) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{\frac{-(s^2 - 2\rho st + t^2)}{2(1-\rho^2)}\right\} ds dt$$

$$C(u, v | \kappa, \gamma) = 1 - (1 - ((1 - (1 - u)^\kappa)^{-\gamma} + (1 - (1 - v)^\kappa)^{-\gamma} - 1)^{-1/\gamma})^{1/\kappa}$$

where  $U_t \equiv F_t(X_t | I_{t-1})$  and  $V_t \equiv G_t(Y_t | I_{t-1})$

# Comparing the two copula models



<i>Normal</i> copula	<i>Joe-Clayton</i> copula
One parameter: $\rho$	Two parameters: $\kappa, \gamma$
Symmetric tail behaviour	Asymmetric tail behaviour
Parameter maps to linear correlation	Parameters map to upper and lower tail dependence
Positive and negative dependence	Positive dependence only
Zero tail dependence	Non-zero tail dependence

## Upper and lower tail dependence

- As the name suggests, it measures dependence in the *tails*, or the extreme values, of the dist'n. [Recall  $U_t \equiv F_t(X_t | I_{t-1})$  and  $V_t \equiv G_t(Y_t | I_{t-1})$  .]

$$\begin{aligned}\tau^L &= \lim_{q \rightarrow 0^+} \Pr[U \leq q \mid V \leq q] \\ &= \lim_{q \rightarrow 0^+} \frac{\Pr[U \leq q \cap V \leq q]}{\Pr[V \leq q]} \\ &= \lim_{q \rightarrow 0^+} \frac{C(q, q)}{q}\end{aligned}$$

$$\begin{aligned}\tau^U &= \lim_{q \rightarrow 1^-} \Pr[U > q \mid V > q] \\ &= \lim_{q \rightarrow 1^-} \frac{\Pr[U > q \cap V > q]}{\Pr[V > q]} \\ &= \lim_{q \rightarrow 1^-} \frac{1 - 2q + C(q, q)}{1 - q}\end{aligned}$$

# Time varying copulas and dependence

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- We attempt to capture time-varying dependence by allowing the parameters of the copulas to evolve over time, in a similar fashion to a GARCH model for heteroscedasticity.

## Time varying Joe Clayton copula

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- The parameters of the Joe-Clayton copula are strictly increasing functions of the upper and lower tail dependence measures (Joe, 1997):

$$\begin{aligned}\tau^L(\gamma) &= 2^{-1/\gamma} & \Leftrightarrow & \gamma(\tau^L) = -[\log_2(\tau^L)]^{-1} \\ \tau^U(\kappa) &= 2-2^{1/\kappa} & \Leftrightarrow & \kappa(\tau^U) = [\log_2(2-\tau^U)]^{-1}\end{aligned}$$

- As  $\kappa$  and  $\gamma$  are not easily interpretable, we will instead model  $\tau_t^U$  and  $\tau_t^L$ , and then obtain  $\kappa_t$  and  $\gamma_t$  by the above expressions.

# Time varying Joe Clayton copula

- Joe-Clayton copula:

$$\tau_t^U = \Lambda \left( \omega_U + \beta_U \tau_{t-1}^U + \alpha_U \frac{1}{10} \sum_{j=1}^{10} |u_{t-j} - v_{t-j}| \right) \quad \text{Upper tail dependence}$$

$$\tau_t^L = \Lambda \left( \omega_L + \beta_L \tau_{t-1}^L + \alpha_L \frac{1}{10} \sum_{j=1}^{10} |u_{t-j} - v_{t-j}| \right) \quad \text{Lower tail dependence}$$

$$\Lambda(x) = \frac{1}{1 + e^{-x}} \quad \text{The logistic function}$$

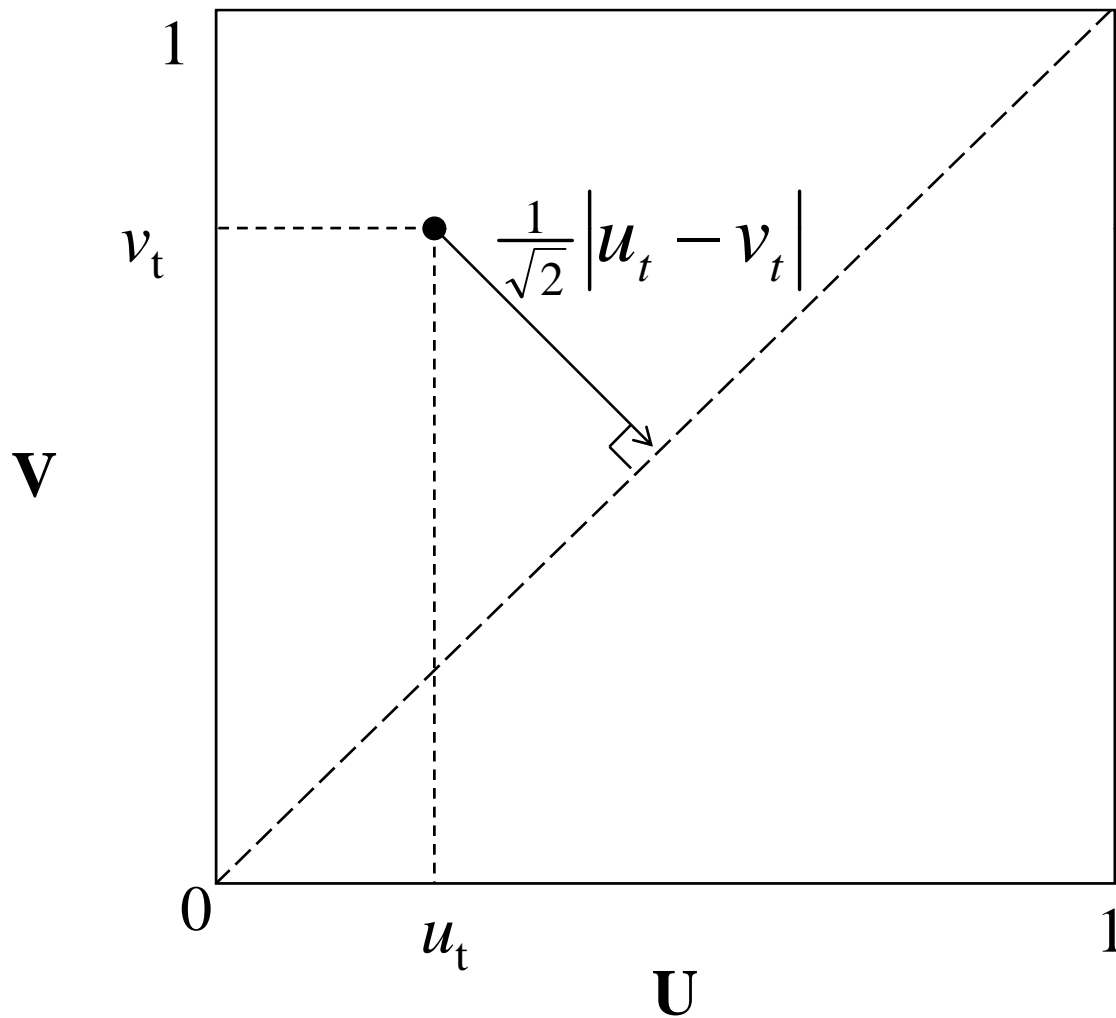
## Time varying Joe Clayton copula

- On the choice of forcing variable in the Joe-

Clayton copula: 
$$\frac{1}{10} \sum_{j=1}^{10} |u_{t-j} - v_{t-j}|$$

- Under perfect positive dependence, all points  $(u_t, v_t)$  lie on the main diagonal of the copula support, while under independence they are scattered all over the support
- Thus, average distance from the point to the main diagonal is an approximate measure of how 'close' the last ten observations were to being perfectly dependent.

# Time varying Joe Clayton copula



# Time varying normal copula

- Normal copula:

$$\rho_t = \tilde{\Lambda} \left( \omega_\rho + \beta_\rho \rho_{t-1} + \alpha_\rho \frac{1}{10} \sum_{j=1}^{10} \Phi^{-1}(u_{t-j}) \cdot \Phi^{-1}(v_{t-j}) \right)$$

where  $\Phi^{-1}$  is the inverse of the standard normal c.d.f.

$$\tilde{\Lambda}(x) = \frac{1 - e^{-x}}{1 + e^{-x}} \quad \text{The modified logistic function}$$

# Overview

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1. Intro to / Review of Copulas
2. Motivation
3. Evaluating copula models
4. Applying copulas to exchange rate modelling
  - a) The data
  - b) The models*
  - c) Results
5. Conclusions and potential future work

# Overview

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1. Intro to / Review of Copulas
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## The best existing alternatives

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- An alternative to the model just presented may be an AR-BEKK model for the mean and variance, with some residual distribution assumption:
  1. AR-BEKK with bivariate Normal residuals:
    1. Passes LM tests of dynamic specification
    2. Fails (miserably!) K-S and hit tests of density specification of margins and copula
  2. AR-BEKK with bivariate Student's  $t$  residuals:
    1. Passes LM tests
    2. Passes K-S tests
    3. Fails Yen marginal density hit test, and fails copula hit test

# Estimation

- Two-stage maximum likelihood was employed:
- $(X, Y) \sim h(\theta) = f(x; \varphi) \cdot g(y; \gamma) \cdot c(F(x; \varphi), G(y; \gamma); \kappa)$
- $LL_H(\theta) = LL_F(\varphi) + LL_G(\gamma) + LL_C(\varphi, \gamma, \kappa)$

$$\hat{\varphi}_n = \arg \max_{\varphi \in \Phi} n^{-1} \sum_{t=1}^n \log f(x_t; \varphi)$$

$$\hat{\gamma}_n = \arg \max_{\gamma \in \Gamma} n^{-1} \sum_{t=1}^n \log g(y_t; \gamma)$$

$$\hat{\kappa}_n = \arg \max_{\kappa \in K} n^{-1} \sum_{t=1}^n \log c(F(x_t; \hat{\varphi}_n), G(y_t; \hat{\gamma}_n); \kappa)$$

## Results for the marginal distributions

	DM Margin		Yen Margin	
	Coeff	Std Error	Coeff	Std Error
$\mu$	0.0276*	0.0111	0.0144	0.0111
$\phi_1$	0.0142	0.0200	-0.0043	0.0195
$\phi_{10}$			0.0664*	0.0183
$\omega$	0.0039	0.0030	0.0059	0.0034
$\beta$	0.9485*	0.0161	0.9453*	0.0161
$\alpha$	0.0448*	0.0126	0.0458*	0.0125
$\nu$	5.8073*	0.6383	4.3817*	0.3800

## Specification tests of marginal distributions

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- Both margins pass the LM and K-S tests for correct dynamic specification and correct density specification
  
- Hit tests:
  - Break unit interval into five regions:  
0 , 10% , 25% , 75% , 90% and 1.
  - Include as regressors: a constant (to capture bias in probability of a hit) and three variables counting number of hits in past day, week and month (to capture serial dependence in hits)
  - Results: DM margin passes all regions and the joint test, Yen margin fails lower tail region but passes the joint test

# Constant dependence copulas

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## Constant dependence copulas

---

1. Normal copula:  $\hat{\rho} = 0.4560$  (0.0167)
  - a) Implied residual correlation: 0.4141

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---

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a) Implied residual correlation: 0.4141

2. Joe-Clayton copula:

$$\hat{\kappa} = 1.3356 \text{ (0.0348)} \Rightarrow \tau^U(\hat{\kappa}) = 0.3197$$

$$\hat{\gamma} = 0.4202 \text{ (0.0384)} \Rightarrow \tau^L(\hat{\gamma}) = 0.1921$$

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a) Implied residual correlation: 0.4267

b) Test for significance of asymmetry:

$$H_0: \tau^U(\hat{\kappa}) = \tau^L(\hat{\gamma}) \quad \text{vs.} \quad H_1: \tau^U(\hat{\kappa}) \neq \tau^L(\hat{\gamma})$$

$$\text{Test stat (} p\text{-value)} = 2.9197 \text{ (0.0035)}$$

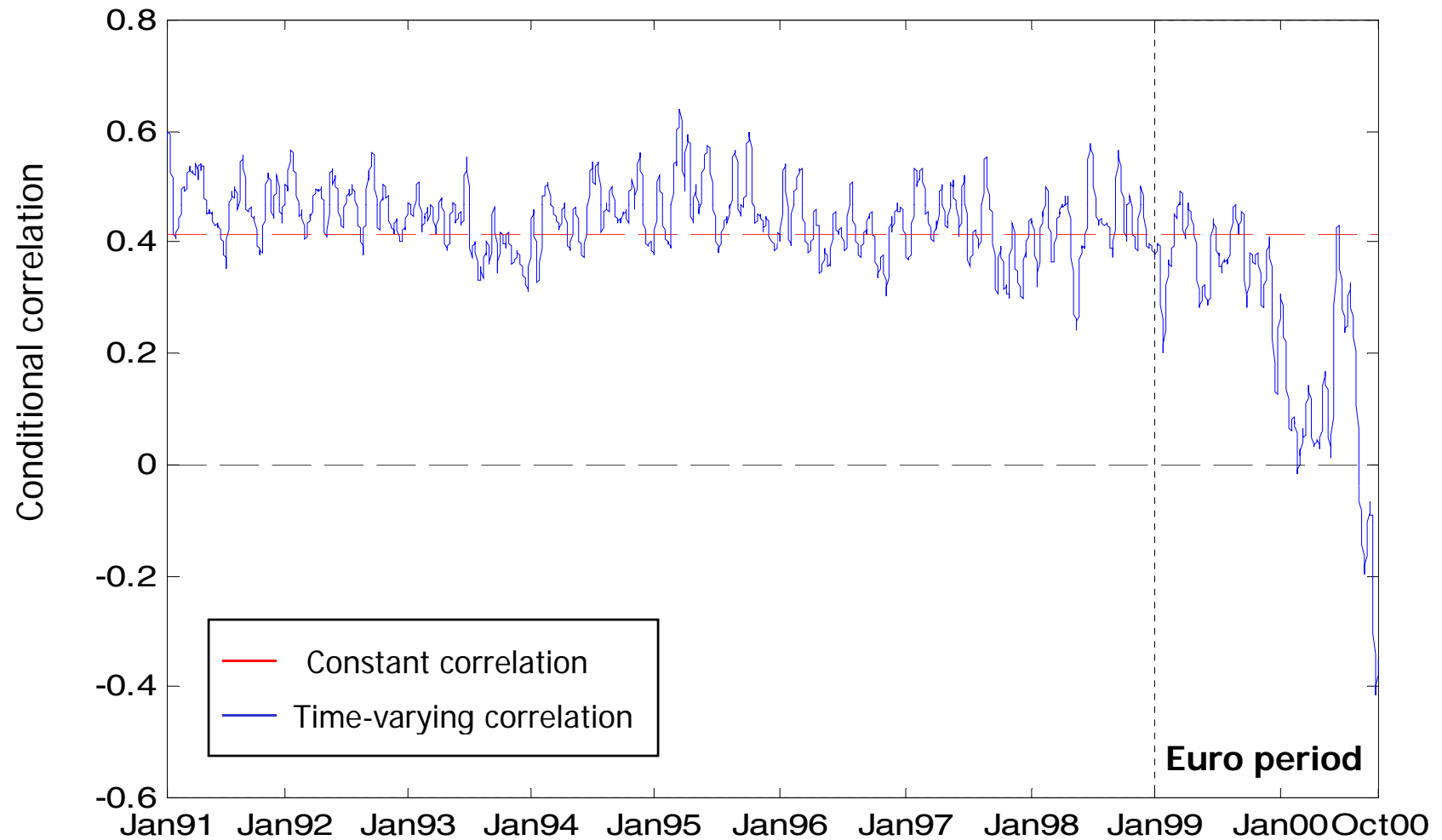
$\Rightarrow$  Reject  $H_0$ , i.e. *asymmetry is significant*.

## Time varying normal copula

	$\omega_\rho$	$\beta_\rho$	$\alpha_\rho$
<i>Coeff</i>	0.0015	2.0684*	0.1212
<i>Std Error</i>	0.0052	0.0160	0.0250

- These parameters are difficult to interpret on their own, as they enter into the logistic transformation.
- Can more easily see the results in a graph...

# Time varying normal copula

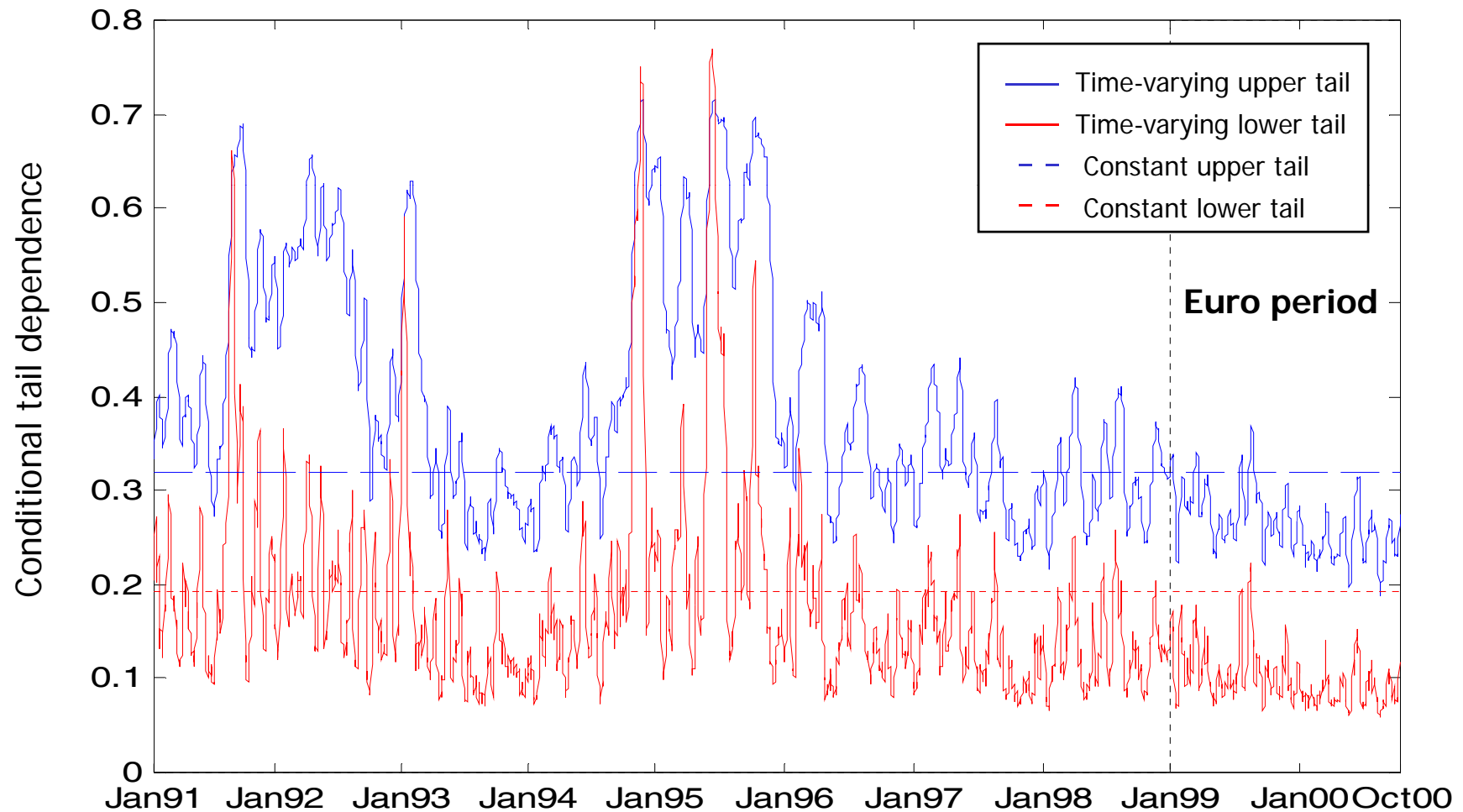


## Time varying Joe Clayton copula

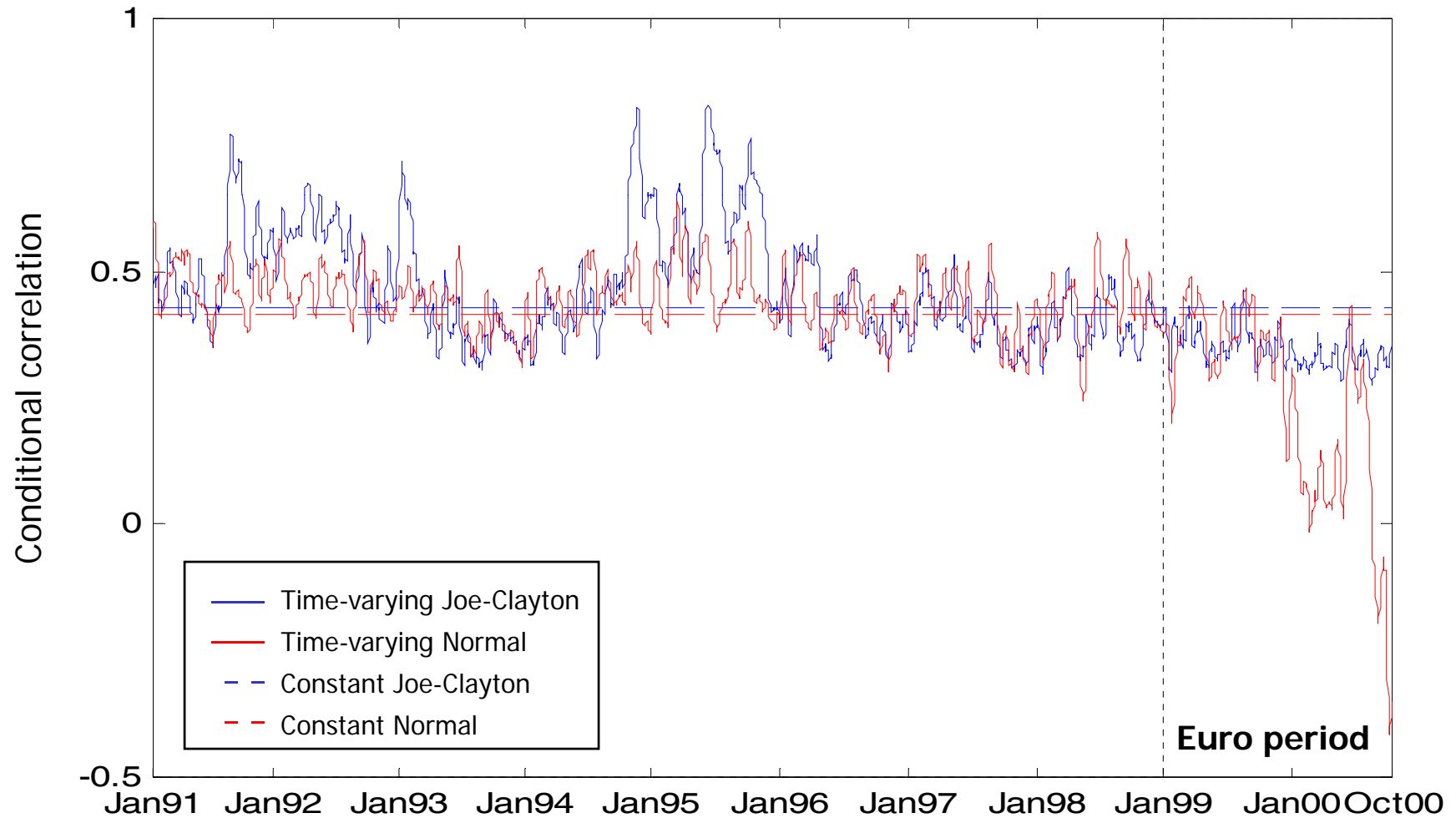
	$\omega_U$	$\beta_U$	$\alpha_U$	$\omega_L$	$\beta_L$	$\alpha_L$
Coeff	-2.0621	4.4548	-.9192	-1.3444	4.1406	-6.5119
s.e.	0.2056	0.2938	0.8966	0.5876	0.5880	3.1394

- Again interpretation of the parameter estimates directly is difficult – look at graph of time path of parameters

# Tail dependence in the Joe Clayton copula



# Conditional correlation estimates



# Evaluation and comparison of models

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- The estimation results clearly imply quite different dependence structures - we need some way of comparing the different models' fit.
- Unfortunately, many standard means of comparing fit are not applicable/relevant here:
  - a)  $R^2$  has little or no meaning
  - b) LR-type tests difficult as the models are non-nested
- We will use the tests introduced earlier (LM, K-S and Hit tests)

## LM and K S tests

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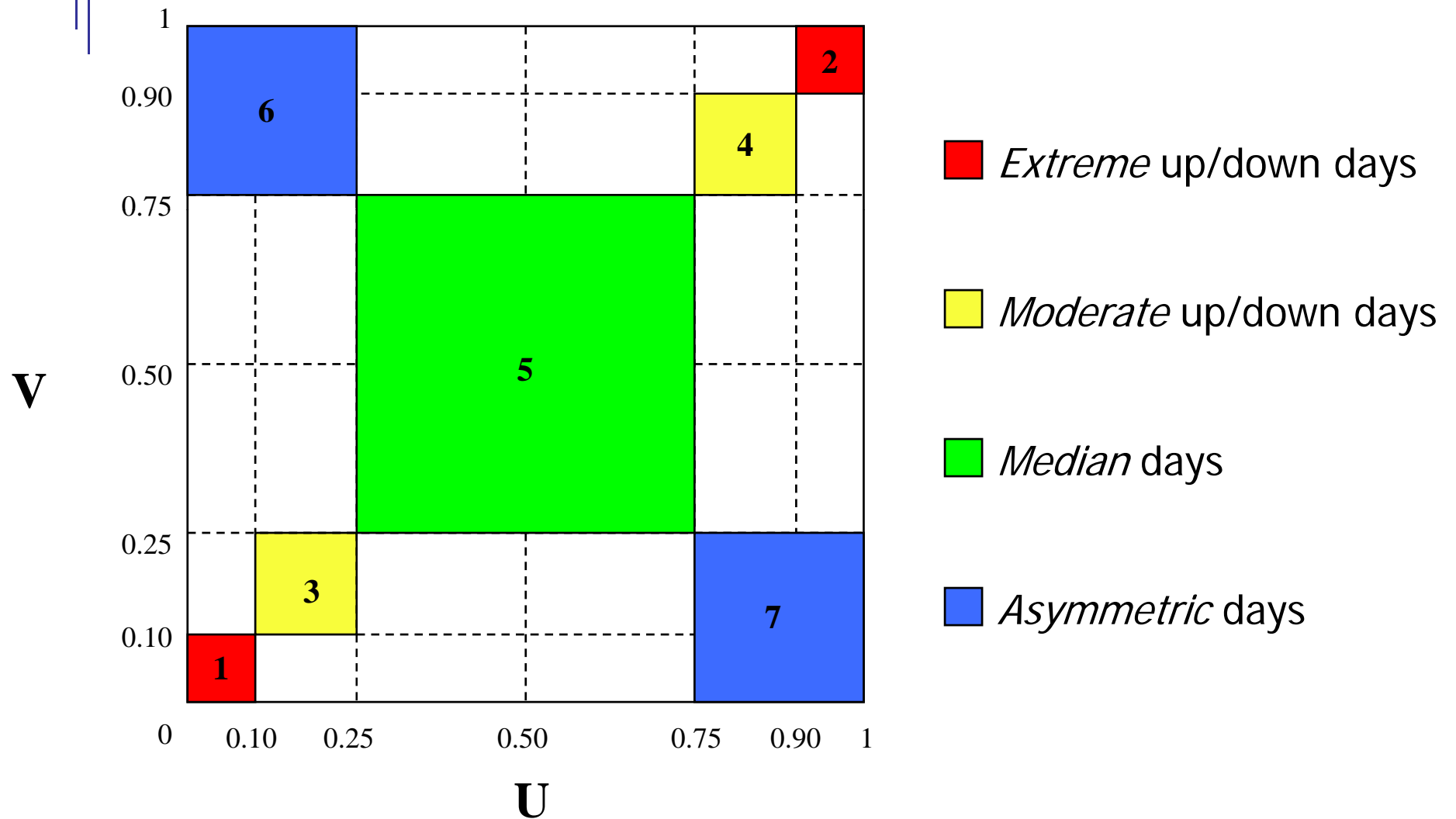
- All 4 copula models pass the LM and K-S tests for goodness-of-fit.
- We turn now to the 'hit' tests for hopefully a more powerful test.

## Hit test specification

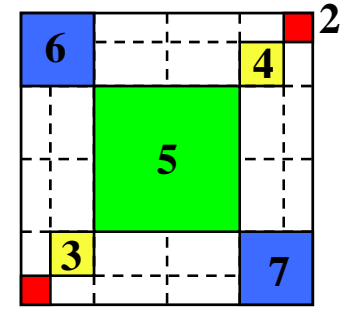
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- $Z_t$  contains a constant, and a count of the number of hits in the last day, one week and one month.
- We use a standard LR test to test the null hypothesis that  $\beta = 0$ .

# Hit regions



# Hit test results



Region	Constant Normal	Constant Joe-Clayton	Time-varying Normal	Time-varying Joe-Clayton
1	9.4482	7.3304	7.4209	6.6815
2	4.1361	9.8002*	2.2563	16.0750*
3	6.4374	6.1412	6.4108	6.5137
4	1.9063	1.8627	1.4813	1.4595
5	8.2495	6.7740	7.9588	3.3560
6	2.8800	6.9902	0.9284	3.1363
7	6.5945	8.4697	0.6086	3.2932
<b>ALL</b>	<b>37.8395</b>	<b>46.1668*</b>	<b>26.9924</b>	<b>41.4195*</b>

Individual 5% critical value: 9.49, Joint 5% critical value: 41.3371.

## Hit test results

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- The two normal copulas pass all tests
- The two Joe-Clayton copulas both fail in the extreme upper tail region
  - An unexpected result, as the Joe-Clayton copula is more flexible in the tails than the Normal
  - Casts some doubt on the previous finding of asymmetric dependence

## A structural break?    The euro

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- Did the introduction of the euro on Jan 1, 1999 significantly change the conditional joint distribution of these exchange rates?
- We expand the information set now to include an indicator variable as to whether the data came from the pre-euro or post-euro period
- The model parameters are allowed to change over the two periods, though the functional forms are assumed to be the same

## Results for the marginal distributions, with structural break

	DM Margin		Yen Margin	
	Coeff	Std Error	Coeff	Std Error
$\mu^1$	<b>0.0128</b>	0.0120	0.0144	0.0111
$\mu^2$	<b>0.0982*</b>	0.0268	—	—
$\phi_1$	0.0111	0.0200	-0.0043	0.0195
$\phi_{10}$			0.0664*	0.0183
$\omega^1$	<b>0.0037</b>	0.0029	0.0059	0.0034
$\omega^2$	<b>0.0053</b>	0.0038	—	—
$\beta$	0.9485*	0.0160	0.9453*	0.0161
$\alpha$	0.0446*	0.0125	0.0458*	0.0125
$\nu$	5.6860*	0.6180	4.3817*	0.3800

# Specification tests of marginal distributions, with structural break

- Multinomial test of marginal models with break:

	Full Sample		Pre-Euro		Post-Euro	
	DM	Yen	DM	Yen	DM	Yen
Test stat	14.29	35.05	16.26	25.89	13.10	21.43
p-value	0.58	0.17	0.43	0.06	0.67	0.17

# Constant copulas with structural break

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1. Normal copula:  $\rho^1 = 0.5435$  (0.0146)  
 $\rho^2 = 0.0855$  (0.0508)

2. Joe-Clayton copula:

$$\tau_1^U(\kappa) = 0.3964 \quad \tau_1^L(\gamma) = 0.2542$$

$$\tau_2^U(\kappa) = 0.0001 \quad \tau_2^L(\gamma) = 0.0018$$

a) Implied residual correlation: 0.4983 down to 0.0781

# Constant copulas with structural break

1. Normal copula:  $\rho^1 = 0.5435$  (0.0146)  
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a) Implied residual correlation: 0.4983 down to 0.0781

b) Test for significance of asymmetry:

$$H_0: \tau_i^U(\kappa) = \tau_i^L(\gamma) \quad \text{vs.} \quad H_1: \tau_i^U(\kappa) \neq \tau_i^L(\gamma) \quad \text{for } i=1,2$$

Test stats (*p-values*) = 3.13 (0.0017) and -0.03 (0.9785)

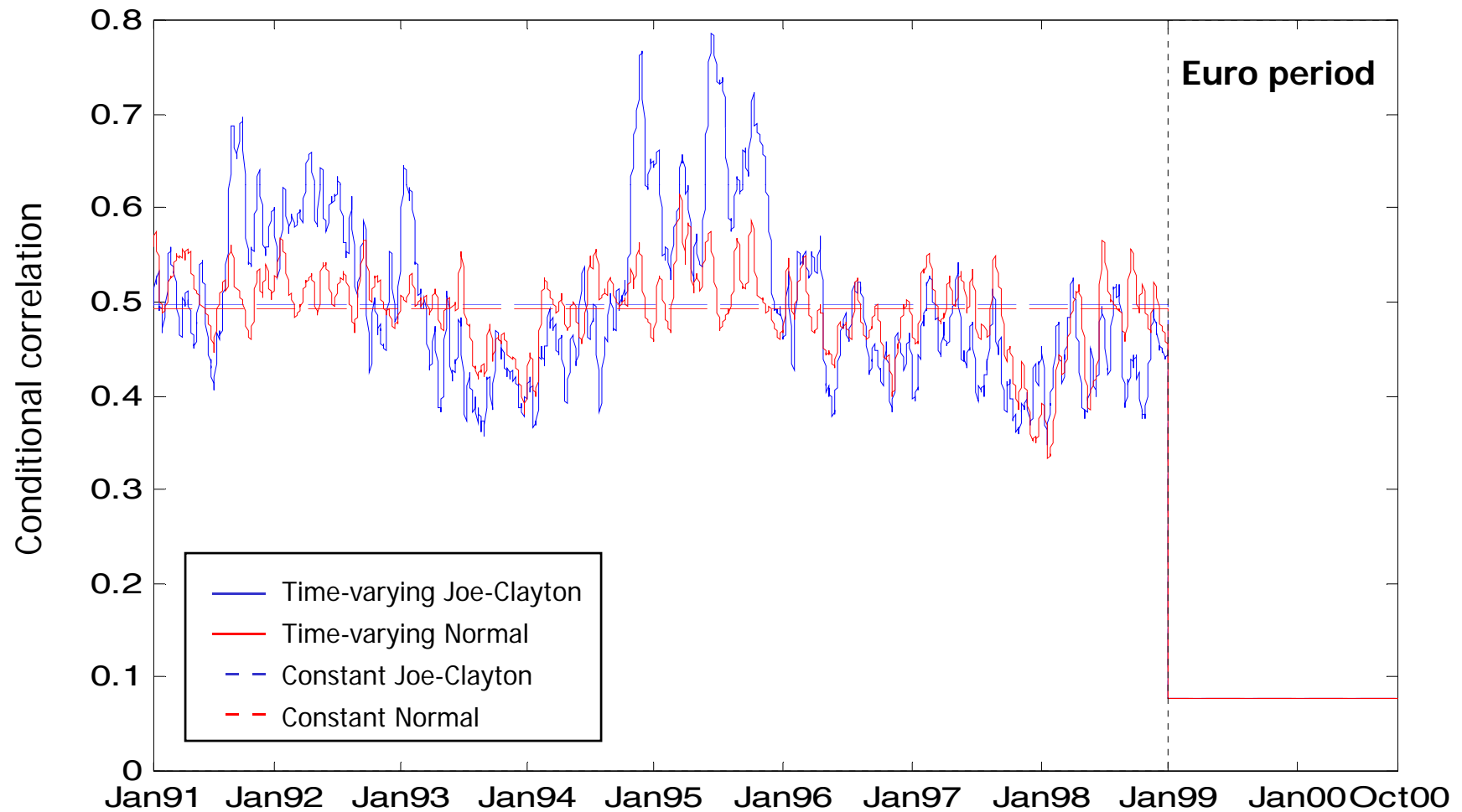
⇒ *Asymmetry is significant for DM/Yen, but not for Euro/Yen*

# Time varying copulas with structural break

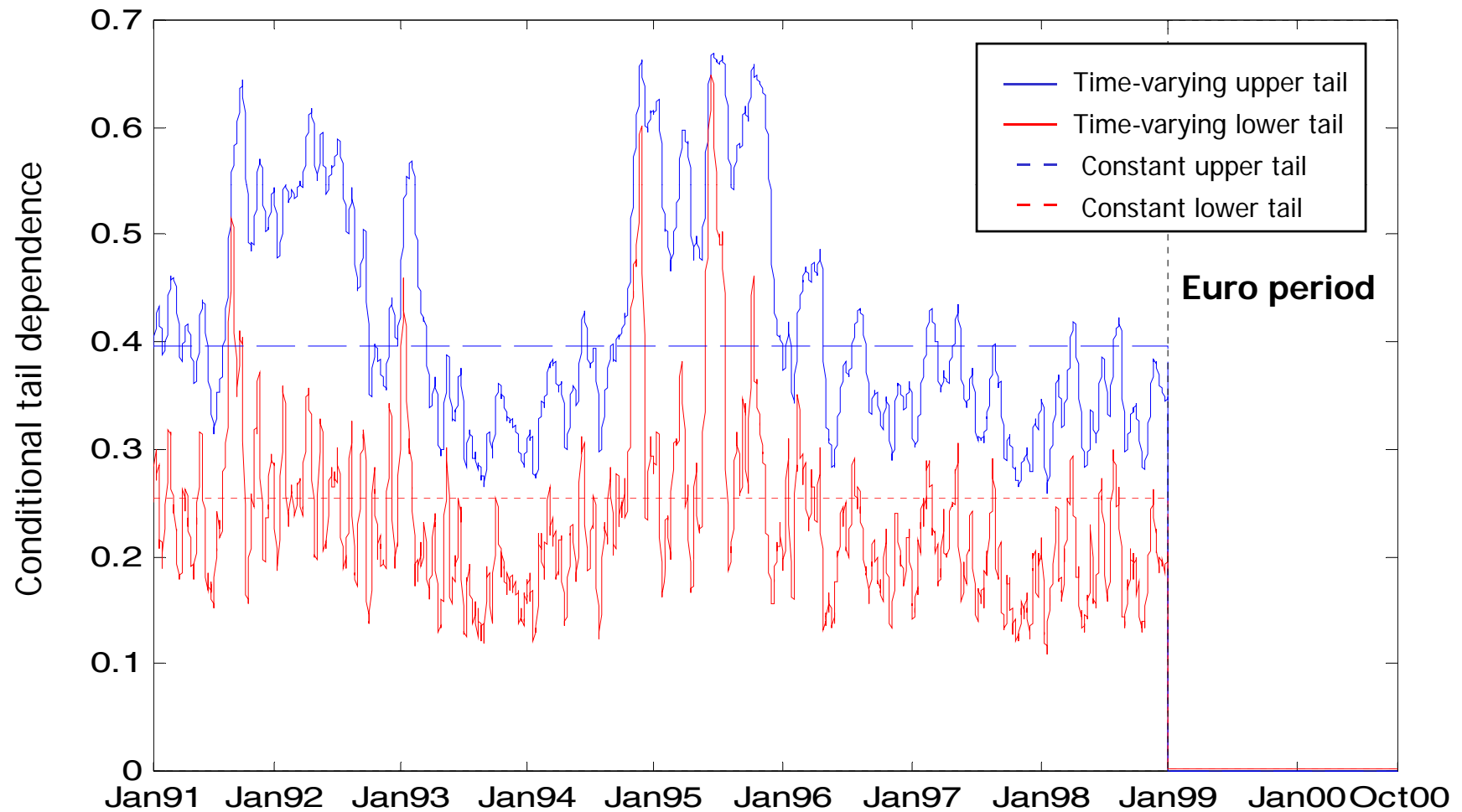
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- Allowed all parameters of copula to change and then inspected resulting parameter time paths.
- Dependence appeared very near constant in the post-euro sample, and so constancy was imposed to reduce number of parameters estimated

# Time varying conditional correlation



# Time varying conditional tail dependence



# Conclusions

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## METHODS:

- We've shown how copula theory may be used to develop flexible models for time-varying conditional joint distributions
- We've also discussed a means of evaluating multivariate density models – the logistic 'hit' test and multinomial test

# Conclusions

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## EMPIRICAL RESULTS:

- Time variation in the dependence structure between the DM-USD and Yen-USD seems significant
- Asymmetry in the dependence structure was found to be important – dependence was greater during appreciations of the USD (depreciations of both the Yen and DM) than during depreciations of the USD (appreciations of both the Yen and DM)

# Conclusions

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## EMPIRICAL RESULTS (cont'd):

- Substantial evidence of a structural break in the joint distribution was found:
  - i. DM-USD unconditional drift and variance increased
  - ii. Yen-USD margin did not change
  - iii. Large decrease in conditional dependence following the introduction of the euro
  
- Conditional dependence measures (like correlation) implied by the models differ depending on the copula used

# Potential future work

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- For me:
  1. Two-stage estimation
  2. The case of unequal amounts of data (Yen-USD and Euro-USD exchange rates)
  3. Applications to economic models (portfolio allocation decisions, and possibly contagion)
  
- For someone else:
  1. Different forms of time variation in dependence – eg Markov switching
  2. Extensions to higher dimensions
  3. The search for the 'best fitting' FX copula, stock market copula, etc...